

Mathematical Methods
UNITS 3&4 Exam 1
2020 Written Trial Examination

Reading time: 15 minutes

Writing time: 1 hour

SOLUTIONS

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Question 1 (4 marks)

a.

$$y = \sqrt{3e^{2x} - e^{4x}}$$

$$y = (3e^{2x} - e^{4x})^{\frac{1}{2}}$$

M1

$$\frac{dy}{dx} = \frac{1}{2}(3e^{2x} - e^{4x})^{-\frac{1}{2}}(6e^{2x} - 4e^{4x})$$

$$\frac{dy}{dx} = \frac{6e^{2x} - 4e^{4x}}{2\sqrt{3e^{2x} - e^{4x}}}$$

A1

$$\frac{dy}{dx} = \frac{\cancel{2}(3e^{2x} - 2e^{4x})}{\cancel{2}\sqrt{3e^{2x} - e^{4x}}}$$

$$\therefore \frac{dy}{dx} = \frac{e^{2x}(3 - 2e^{2x})}{\sqrt{3e^{2x} - e^{4x}}}$$

b.

$$f(x) = x^2 \tan(3x)$$

$$f'(x) = \tan(3x)(2x) + x^2(3\sec^2(3x))$$

$$f'(x) = 2x \tan(3x) + 3x^2 \sec^2(3x)$$

$$\therefore f'(x) = 2x \tan(3x) + \frac{3x^2}{\cos^2(3x)}$$

M1

$$f'(\pi) = 2\pi \tan(3\pi) + \frac{3\pi^2}{(\cos(3\pi))^2}$$

$$f'(\pi) = 0 + \frac{3\pi^2}{(-1)^2}$$

$$\therefore f'(\pi) = 3\pi^2$$

A1

Question 2 (4 marks)

a.

$$f(g(x)) = 3(2x-1)^2 + (2x-1)$$

$$f(g(x)) = 3(4x^2 - 4x + 1) + (2x - 1)$$

$$f(g(x)) = 12x^2 - 12x + 3 + 2x - 1$$

$$\therefore f(g(x)) = 12x^2 - 10x + 2$$

A1

b.

$$h(x) = 12x^2 - 10x + 2$$

$$h'(x) = 24x - 10$$

$$h'\left(\frac{1}{2}\right) = 12 - 10 = 2$$

$$\therefore m_{\perp} = -\frac{1}{2}$$

M1

$$h\left(\frac{1}{2}\right) = 12\left(\frac{1}{2}\right)^2 - 10\left(\frac{1}{2}\right) + 2$$

$$h\left(\frac{1}{2}\right) = 3 - 5 + 2 = 0 \quad \left(\frac{1}{2}, 0\right)$$

M1

$$y - 0 = -\frac{1}{2}\left(x - \frac{1}{2}\right)$$

$$y = -\frac{1}{2}x + \frac{1}{4}$$

$$4y = -2x + 1$$

$$\therefore 2x + 4y = 1$$

A1

Question 3 (4 marks)

a.

$$y = 3x \sin(2x)$$

$$\frac{dy}{dx} = \sin(2x) \times 3 + 3x \times 2 \cos(2x)$$

$$\therefore \frac{dy}{dx} = 3 \sin(2x) + 6x \cos(2x)$$

A1

b.

$$\int 3 \sin(2x) + 6x \cos(2x) \, dx = 3x \sin(2x)$$

$$\int 3 \sin(2x) \, dx + \int 6x \cos(2x) \, dx = 3x \sin(2x)$$

$$\int 6x \cos(2x) \, dx = 3x \sin(2x) - \int 3 \sin(2x) \, dx$$

$$\int 6x \cos(2x) \, dx = 3x \sin(2x) + \frac{3}{2} \cos(2x)$$

M1

$$\int_0^{\pi/4} 3x \cos(2x) \, dx = \left[\frac{1}{2} \left(3x \sin(2x) + \frac{3}{2} \cos(2x) \right) \right]_0^{\pi/4}$$

$$\int_0^{\pi/4} 3x \cos(2x) \, dx = \left[\frac{3x}{2} \sin(2x) + \frac{3}{4} \cos(2x) \right]_0^{\pi/4}$$

$$\int_0^{\pi/4} 3x \cos(2x) \, dx = \left[\left(\frac{3\pi}{8} \sin\left(\frac{\pi}{2}\right) + \frac{3}{4} \cos\left(\frac{\pi}{2}\right) \right) - \left(0 + \frac{3}{4} \cos(0) \right) \right]$$

M1

$$\int_0^{\pi/4} 3x \cos(2x) \, dx = \frac{3\pi}{8} + 0 - \frac{3}{4}$$

$$\therefore \int_0^{\pi/4} 3x \cos(2x) \, dx = \frac{3\pi - 6}{8} = \frac{3(\pi - 2)}{8}$$

A1

Question 4 (5 marks)

a.

$$2 \sin \left(2 \left(x - \frac{\pi}{12} \right) \right) - 1 = 0$$

$$0 \leq x \leq 2\pi$$

$$\sin \left(2 \left(x - \frac{\pi}{12} \right) \right) = \frac{1}{2}$$

$$\frac{-\pi}{12} \leq x - \frac{\pi}{12} \leq \frac{23\pi}{12}$$

$$\text{Basic Angle} = \frac{\pi}{6}, \text{1st \& 2nd quadrants}$$

$$\frac{-\pi}{6} \leq 2 \left(x - \frac{\pi}{12} \right) \leq \frac{23\pi}{6}$$

M1

$$2 \left(x - \frac{\pi}{12} \right) = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}$$

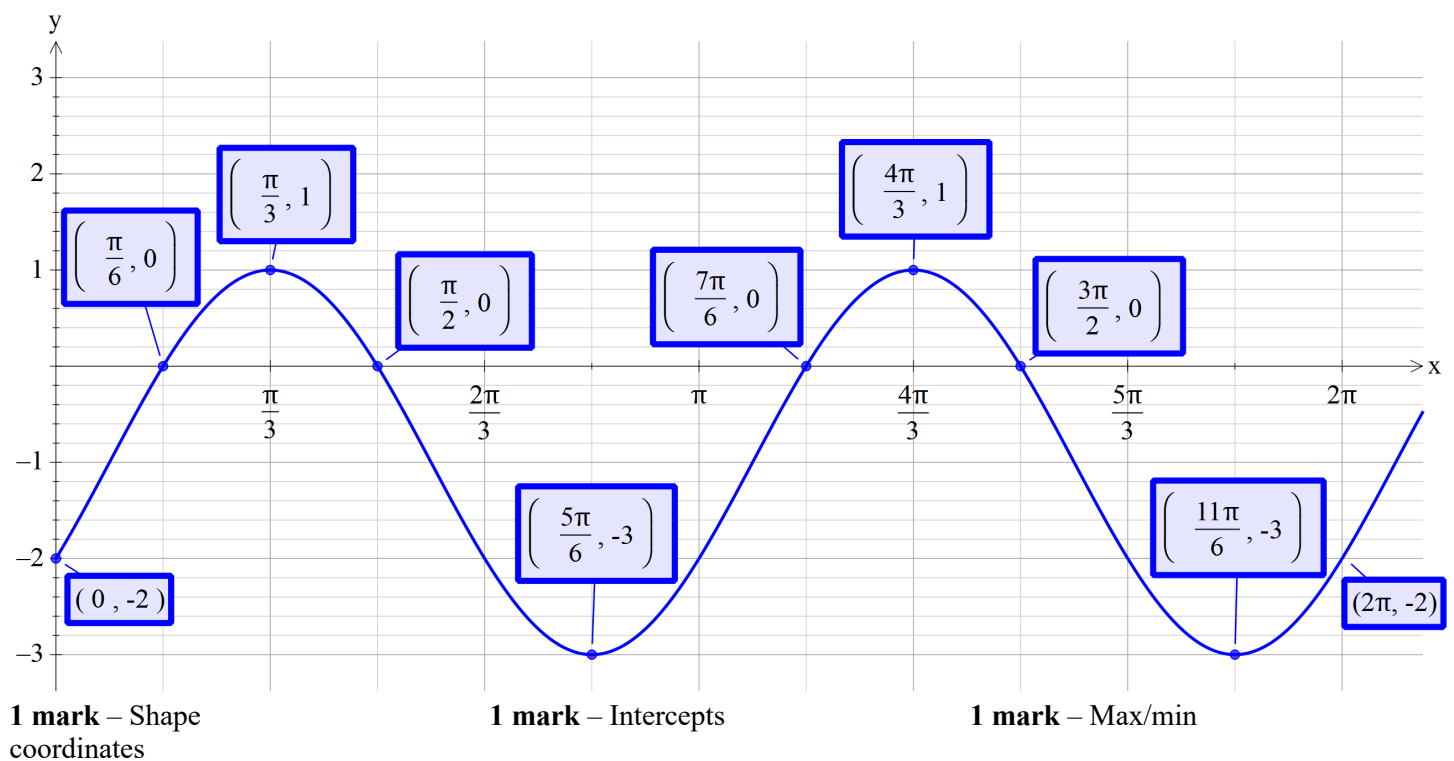
$$x - \frac{\pi}{12} = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}$$

$$x = \frac{2\pi}{12}, \frac{6\pi}{12}, \frac{14\pi}{12}, \frac{18\pi}{12}$$

$$\therefore x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{7\pi}{6}, \frac{3\pi}{2}$$

A1

b.



Question 5 (7 marks)

a.

$$f(x) = \frac{5-3x}{1-x}$$

Let $y = f(x)$, swap x & y

$$x = \frac{5-3y}{1-y}$$

$$x - xy = 5 - 3y$$

$$3y - xy = 5 - x$$

$$y(3-x) = 5-x$$

M1

$$y = \frac{5-x}{3-x}$$

$$\therefore f^{-1} : R \setminus \{3\} \rightarrow R, f^{-1}(x) = \frac{5-x}{3-x}$$

A1

OR

$$f(x) = \frac{5-3x}{1-x} = \frac{2}{1-x} + 3$$

Let $y = f(x)$, swap x & y

$$x = \frac{2}{1-y} + 3$$

$$x - 3 = \frac{2}{1-y}$$

M1

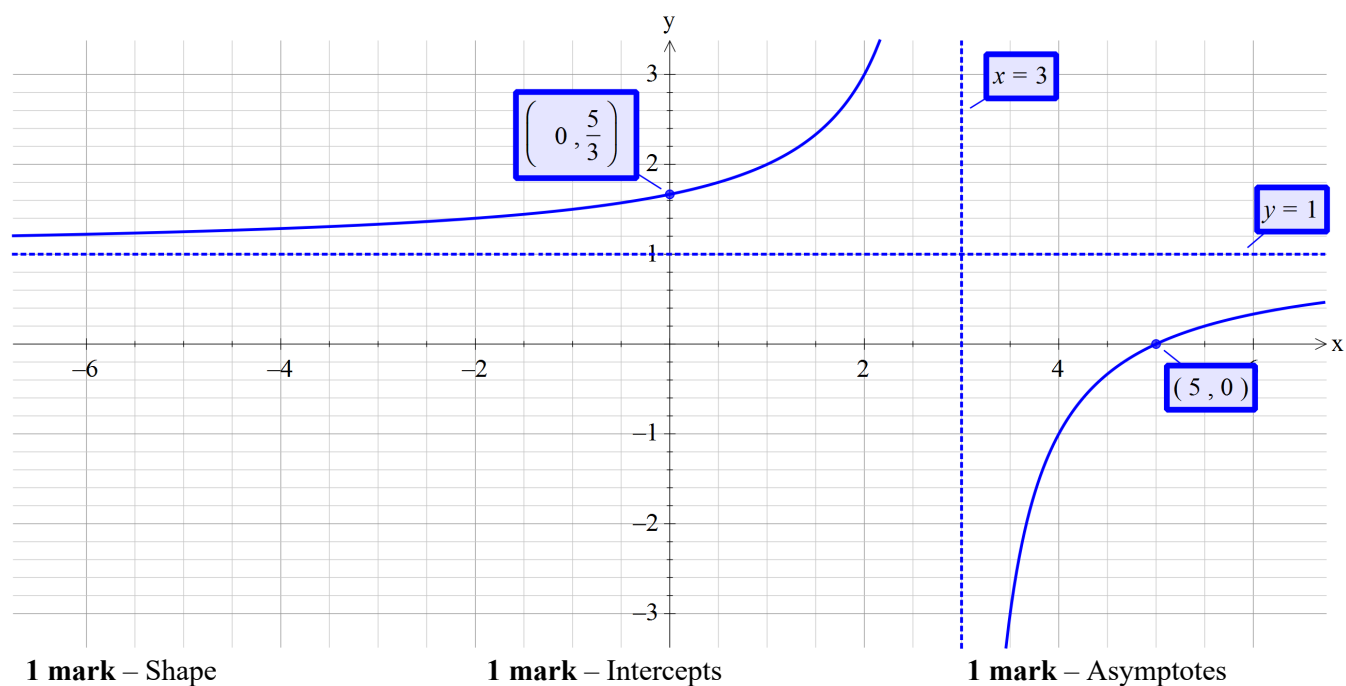
$$1-y = \frac{2}{x-3}$$

$$y = 1 - \frac{2}{x-3}$$

$$\therefore f^{-1} : R \setminus \{3\} \rightarrow R, f^{-1}(x) = 1 - \frac{2}{x-3}$$

A1

b.



c.

$$\int_5^7 1 - \frac{2}{x-3} dx$$

$$= [x - 2\log_e(x-3)]_5^7$$

M1

$$= [(7 - 2\log_e(4)) - (5 - 2\log_e(2))]$$

$$= 7 - 2\log_e(4) - 5 + 2\log_e(2)$$

$$= 2 - \log_e(16) + \log_e(4)$$

$$= 2 - \log_e(4)$$

A1

Question 6 (3 marks)

a.

$$\Pr(\text{all males}) = \frac{\binom{3}{3}\binom{4}{0}}{\binom{7}{3}} = \frac{1}{35}$$

$$\Pr(\text{all females}) = \frac{\binom{4}{3}\binom{3}{0}}{\binom{7}{3}} = \frac{4}{35}$$

M1

$$\Pr(\text{same sex}) = \frac{5}{35} = \frac{1}{7}$$

A1

b.

$$SD(X) = \sqrt{np(1-p)}$$

$$\therefore SD(X) = \sqrt{400 \times \frac{3}{5} \times \frac{2}{5}}$$

$$\therefore SD(X) = 4\sqrt{6}$$

A1

Question 7 (3 marks)

Equation 1

$$2y = (m + 1)x - k - 1$$

$$y = \frac{m+1}{2}x - \frac{k}{2} - \frac{1}{2}$$

Equation 2

$$4y = 2 - k + mx$$

$$y = \frac{2-k}{4} + \frac{m}{4}x \quad \text{M1}$$

Equate the gradients of the two straight lines.

$$\frac{m+1}{2} = \frac{2-k}{4} + \frac{m}{4}$$

$$4m + 4 = 2m$$

$$2m = -4$$

$$m = -2 \quad \text{A1}$$

Equate the y-intercept of the two straight lines.

$$-\frac{k}{2} - \frac{1}{2} = \frac{2-k}{4}$$

$$4(-k - 1) = 2(2 - k)$$

$$-4k - 4 = 4 - 2k$$

$$-2k = 8$$

$$k = -4 \quad \text{A1}$$

OR

$$\begin{bmatrix} m+1 & -2 \\ -m & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} k+1 \\ 2-k \end{bmatrix}$$

Let $\det = 0$

$$4(m+1) - 2m = 0 \quad \text{M1}$$

$$4m + 4 - 2m = 0$$

$$2m = -4$$

$$\therefore m = -2 \quad \text{A1}$$

For equation 1,

$$y = \frac{(m+1)}{2}x - \frac{k+1}{2}$$

For equation 2,

$$y = \frac{m}{4}x + \frac{2-k}{4}$$

Equate y-int for both

$$-\frac{k+1}{2} = \frac{2-k}{4}$$

$$-4k - 4 = 4 - 2k$$

$$-8 = 2k$$

$$\therefore k = -4 \quad \text{A1}$$

Question 8 (5 marks)

a.

$$\Pr(B|A) = \frac{\Pr(A \cap B)}{\Pr(A)}$$

$$\Pr(A) = \frac{\Pr(A \cap B)}{\Pr(B|A)} \quad \mathbf{M1}$$

$$\Pr(A) = \frac{1}{3} \times \frac{q+1}{1} = \frac{q+1}{3} \quad \mathbf{A1}$$

b.

$$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}$$

$$\Pr(B) = \frac{\Pr(A \cap B)}{\Pr(A|B)}$$

$$\Pr(B) = \frac{1}{3} \times \frac{q}{2} = \frac{q}{6} \quad \mathbf{A1}$$

c.

$$\Pr(A \cap B) = \Pr(A) \times \Pr(B)$$

$$\frac{1}{3} = \frac{q+1}{3} \times \frac{q}{6} \quad \mathbf{M1}$$

$$\frac{1}{3} = \frac{q(q+1)}{18}$$

$$6 = q^2 + q$$

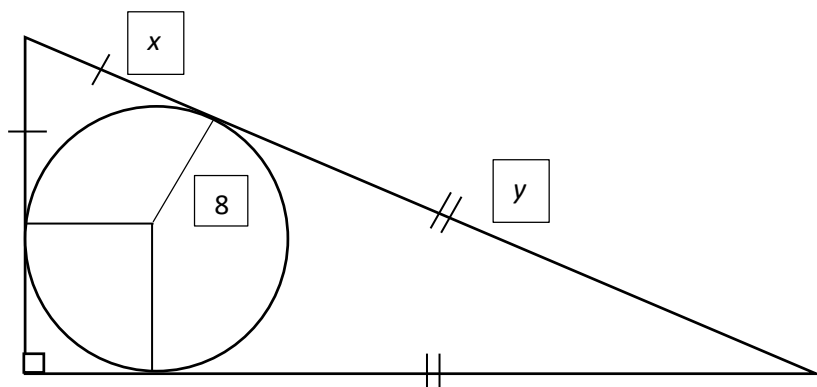
$$q^2 + q - 6 = 0$$

$$(q+3)(q-2) = 0$$

$$q = -3, 2 \quad \text{but } q \neq -3$$

$$\therefore q = 2 \quad \mathbf{A1}$$

Question 9 (5 marks)



a.

$$(x+y)^2 = (x+8)^2 + (y+8)^2$$

$$(x^2 + 2xy + y^2) = (x^2 + 16x + 64) + (y^2 + 16y + 64)$$

$$2xy - 16y = 16x + 128$$

$$y(2x - 16) = 16x + 128$$

M1

$$y = \frac{16x + 128}{2x - 16}$$

$$y = \frac{8x + 64}{x - 8}$$

Using division of polynomials

$$\begin{array}{r} 8 \\ x-8 \overline{) 8x+64} \end{array}$$

$$\underline{8x-64}$$

M1

$$128$$

$$\therefore y = \frac{128}{x-8} + 8$$

b.

$$A = \frac{1}{2} \left(\frac{8x+64}{x-8} + 8 \right) (x+8)$$

$$A = \frac{1}{2} \left(\frac{\cancel{8x+64} + \cancel{8x-64}}{x-8} \right) (x+8)$$

$$A = \left(\frac{8x}{x-8} \right) (x+8)$$

$$A = \frac{8x^2 + 64x}{x-8}$$

M1

For max area

$$\frac{dA}{dx} = \frac{(x-8)(16x+64) - (8x^2+64x)}{(x-8)^2} = 0$$

$$\cancel{16x^2+64x} - 128x - 512 - \cancel{8x^2-64x} = 0$$

$$8x^2 - 128x - 512 = 0$$

$$8(x^2 - 16x - 64) = 0$$

M1

Solve for x

$$x = \frac{16 \pm \sqrt{256 - 4(1)(64)}}{2(1)}$$

$$x = \frac{16 \pm \sqrt{512}}{2}$$

$$x = \frac{16 \pm 16\sqrt{2}}{2}$$

$$x = 8 \pm 8\sqrt{2} \quad \text{but } x > 0$$

$$\therefore x = 8 + 8\sqrt{2}$$

A1