



Victorian Certificate of Education – Free Trial Examinations

MATHEMATICAL METHODS

Free Trial Written Examination 1

SUGGESTED SOLUTIONS AND MARKING GUIDE BOOK

Abbreviations and Acronyms

WRT – with respect to; PDF – probability density function; n DP – correct to n decimal places

Marking instructions

- The relevant line(s) of working marked give an indication of what statement should be made in order to obtain that mark, but this is subject to the marker's discretion.
- Any mark related to the method can only be awarded if the student presents a convincing (and rigorous) argument.
- The final answer mark (if any) can only be awarded if the student provides the correct answer in either simplest form or the form required.
- Consequential marks can only be obtained for marks related to the method, not for any final answer.
- If elementary mathematical steps and/or logic are broken within a solution, it must be properly justified in order to obtain full marks.

Miscellaneous notes

Some questions may have multiple methods/solutions, including some that are beyond the scope of the course. The solutions provided are the ones that were intended by the examination authors.

Question 1a (2 marks)

Mark	Criteria
1	Applies quotient rule, or equivalent
2	Provides correct answer

$$\frac{dy}{dx} = \frac{\frac{d}{dx}[\sin(x)] \cdot 3x^2 - \frac{d}{dx}[3x^2] \sin(x)}{(3x^2)^2}$$

$$= \frac{3x^2 \cos(x) - 6x \sin(x)}{9x^4}$$

Mark 1

$$= \frac{x \cos(x) - 2 \sin(x)}{3x^3}$$

Mark 2**Question 1b** (2 marks)

Mark	Criteria
1	Differentiates f WRT x using the chain rule
2	Provides correct answer

$$f'(x) = 4 \frac{d}{dx}[\sqrt{x}] e^{\sqrt{x}}$$

$$= \frac{2}{\sqrt{x}} e^{\sqrt{x}}$$

Mark 1

$$\text{Therefore, } f'(4) = e^2.$$

Mark 2**Question 2a.i** (1 mark)

Mark	Criteria
1	Provides a correct method

$$1 + \frac{2}{x-2} = \frac{x-2+2}{x-2} = \frac{x}{x-2}, \text{ as required.}$$

Mark 1**Question 2a.ii** (1 mark)

Mark	Criteria
1	Provides correct answer

$$\int g(x) dx = \int \left(1 + \frac{2}{x-2}\right) dx$$

$$= x + 2 \log_e(x-2) \quad [\text{since } x-2 > 0]$$

Mark 1**Question 2b** (2 marks)

Mark	Criteria
1	Antidifferentiates f' WRT x
2	Provides correct answer

$$f(x) = \int (\pi \cos(\pi x) + 2x^{-1/2}) dx$$

$$= \sin(\pi x) + 4x^{1/2} + c \quad [c \in \mathbb{R}]$$

Mark 1

$$\text{Since } f\left(\frac{1}{4}\right) = \frac{1}{\sqrt{2}}, \text{ we have } \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} + 2 + c, \text{ and so } c = -2.$$

$$\text{Therefore, } f(x) = \sin(\pi x) + 4\sqrt{x} - 2.$$

Mark 2**Question 3a** (2 marks)

Mark	Criteria
1	Obtains correct reference angle, or equivalent
2	Provides correct answer

$$\sin(\pi x) = \frac{\sqrt{3}}{2}, \text{ where } -2\pi < \pi x < \pi$$

$$\Rightarrow \pi x = \frac{-5\pi}{3}, \frac{-4\pi}{3}, \frac{\pi}{3}, \frac{2\pi}{3}$$

Mark 1

$$\Rightarrow x = \frac{-5}{3}, \frac{-4}{3}, \frac{1}{3}, \frac{2}{3}$$

Mark 2**TURN OVER**

Question 3b (2 marks)

Mark	Criteria
1	Removes logarithms from equation
2	Provides correct answer with justification

$$\log_e \left[\frac{(a+3)^2}{b^2} \right] = 2$$

$$\Rightarrow \frac{(a+3)^2}{b^2} = e^2$$

Mark 1

$$\Rightarrow (a+3)^2 = e^2 b^2$$

$$\Rightarrow a+3 = -eb \quad [a+3 \neq +eb \text{ since we require } a+3 > 0 \text{ with } b < 0]$$

$$\text{Therefore, } a = -3 - eb.$$

Mark 2**Question 4a** (1 mark)

Mark	Criteria
1	Provides correct answer

$$\text{Let } L \sim N(100, 8^2)$$

$$\Pr(L > 108) = \Pr\left(Z > \frac{108-100}{8}\right)$$

$$= \Pr(Z > 1)$$

$$= 0.16 \quad (2DP)$$

Mark 1**Question 4b** (2 marks)

Mark	Criteria
1	Applies conditional probability definition and utilises symmetry, either algebraically or graphically
2	Provides correct answer

$$\Pr(L > 92 \mid L < 100) = \frac{\Pr(92 < L < 100)}{\Pr(L < 100)}$$

$$= \frac{\Pr(-1 < Z < 0)}{\Pr(Z < 0)}$$

$$= \frac{0.5 - 0.16}{0.5}$$

Mark 1

$$= 0.68 \quad (2DP)$$

Mark 2**Question 4c** (2 marks)

Mark	Criteria
1	Writes down an inequation involving n , or equivalent
2	Provides correct answer

$$\Pr(L < 100) = 0.5, \text{ so we have}$$

$$\sqrt{\frac{1/2 \times 1/2}{n}} \leq \frac{1}{48}$$

Mark 1

$$\Rightarrow \frac{1}{2\sqrt{n}} \leq \frac{1}{48}$$

$$\Rightarrow \sqrt{n} \geq 24$$

$$\text{Therefore, the smallest value of } n \text{ is } 576.$$

Mark 2**TURN OVER**

Question 5a (2 marks)

Mark	Criteria
1	Finds zeros of f'
2	Provides correct answer

$$\text{Let } f'(x) = \frac{1}{4}(3x^2 - 3) = 0.$$

$$\Rightarrow x^2 - 1 = 0$$

$$\Rightarrow x = -1, 1$$

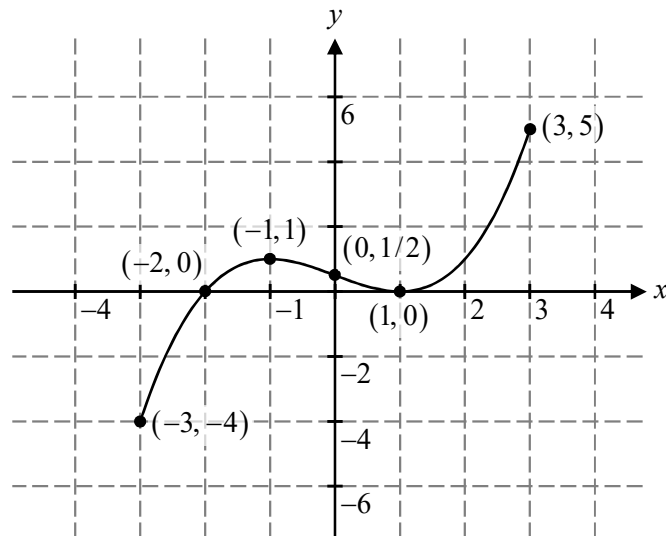
Mark 1

$$f(-1) = 1 \text{ and } f(1) = 0.$$

The stationary points of f are $(-1, 1)$ and $(1, 0)$.

Mark 2**Question 5b** (2 marks)

Mark	Criteria
1	Labels all points correctly
2	Sketches correct graph shape

**Question 5c** (1 mark)

Mark	Criteria
1	Provides correct answer

From the graph, $\bar{f} = \frac{1}{2}$.

Mark 1**Question 6a** (3 marks)

Mark	Criteria
1	Finds expression for n in terms of m
2	Expands resulting quadratic for $\text{Var}(X)$
3	Provides correct answer

Since $0.1 + m + n = 1$, we have $n = 0.9 - m$.

Mark 1

$$\text{Var}(X) = E(X^2) - [E(X)]^2$$

$$= m + 4n - (m + 2n)^2$$

$$= -3m + 3.6 - (1.8 - m)^2$$

$$= -3m + 3.6 - (3.24 - 3.6m + m^2)$$

Mark 2

$$= -m^2 + 0.6m + 0.36$$

Hence, $a = -1$, $b = 0.6$ and $c = 0.36$.

Mark 3**Question 6b** (1 mark)

Mark	Criteria
1	Provides a correct method

Since the $\text{Var}(X)$ is given by a 'negative' quadratic, provided m is suitable, maximum variance occurs when

$$m = \frac{-0.6}{2 \times (-1)} = 0.3, \text{ as required.}$$

Mark 1**TURN OVER**

Question 6c (2 marks)

Mark	Criteria
1	Writes down the possible event combinations, and their associated probabilities, or equivalent
2	Provides correct answer

$$\Pr(E) = \Pr(1, 2) + \Pr(2, 1) + \Pr(2, 2)$$

$$= 0.3 \times 0.6 + 0.6 \times 0.3 + 0.6 \times 0.6$$

$$= 0.18 + 0.18 + 0.36$$

$$= 0.72$$

Mark 1**Mark 2****Question 7a.i** (1 mark)

Mark	Criteria
1	Provides a correct method

$$h(x) = \sqrt{x^2 - 2x + 5}$$

$$= \sqrt{(x-1)^2 - 1 + 5}$$

$$= \sqrt{(x-1)^2 + 4}, \text{ as required.}$$

Mark 1**Question 7a.ii** (2 marks)

Mark	Criteria
1	Provides correct domain
2	Provides correct range

$$\text{domain}(g) = \text{domain}(h) = (-\infty, 1]$$

Mark 1

We have $\text{range}(g) = [-1, \infty) \xrightarrow{f} [2, \infty)$ since f is strictly increasing.

$$\text{Hence, } \text{range}(h) = [2, \infty).$$

Mark 2**Question 7b** (2 marks)

Mark	Criteria
1	Provides correct rule of inverse function of h
2	Provides correct domain and range

$$\text{Let } y = h^{-1}(x).$$

$$x^2 = (y-1)^2 + 4 \Rightarrow y-1 = \pm\sqrt{x^2-4}$$

However, since we require $h^{-1}(x) \leq 1$, we have $h^{-1}(x) = 1 - \sqrt{x^2-4}$. **Mark 1**

$$\text{domain}(h^{-1}) = [2, \infty) \text{ and } \text{range}(h^{-1}) = (-\infty, 1].$$

Mark 2**Question 8a.i** (2 marks)

Mark	Criteria
1	Provides one correct transformation
2	Provides correct answer

• Dilation by factor $\frac{1}{a}$ from the x -axis

Mark 1

• Dilation by factor $\frac{1}{a}$ from the y -axis

Mark 2

Note: in any order

TURN OVER

Question 8a.ii (1 mark)

Mark	Criteria
1	Provides a correct method

Method 1:

The first positive x -axis intercept of the graph of $y = x \cos(x)$ is $\left(\frac{\pi}{2}, 0\right)$,

and so applying the transformations from **part a.i**, we have

$$(b, 0) = \left(\frac{\pi}{2} \times \frac{1}{a}, 0 \times \frac{1}{a}\right) = \left(\frac{\pi}{2a}, 0\right). \quad \text{Mark 1}$$

Thus, $b = \frac{\pi}{2a}$, as required.

Method 2:

Let $x \cos(ax) = 0$, where $x > 0$.

$$\cos(ax) = 0$$

$$\Rightarrow ab = \frac{\pi}{2} \quad [\text{for first positive } x\text{-axis intercept}] \quad \text{Mark 1}$$

Thus, $b = \frac{\pi}{2a}$, as required.

Question 8b (4 marks)

Mark	Criteria
1	Differentiates $x \sin(ax)$ WRT x using the product rule.
2	Forms an equation involving a definite integral and finds an antiderivative of $x \cos(ax)$ WRT x
3	Substitutes $a = \pi/(2b)$
4	Provides correct answer

$$\begin{aligned} \frac{d}{dx}[x \sin(ax)] &= \frac{d}{dx}[x] \sin(ax) + x \frac{d}{dx}[\sin(ax)] \\ &= \sin(ax) + ax \cos(ax) \end{aligned}$$

Mark 1

Since f is a PDF, we have

$$\begin{aligned} 1 &= \int_0^b x \cos(ax) dx \\ &= \frac{1}{a} [x \sin(ax)]_0^b - \frac{1}{a} \int_0^b \sin(ax) dx \quad [\text{using above result}] \\ &= \frac{1}{a} [x \sin(ax)]_0^b + \frac{1}{a} \left[\frac{1}{a} \cos(ax) \right]_0^b \end{aligned}$$

Mark 2

Substituting $a = \frac{\pi}{2b}$ gives

$$\left[\frac{2bx}{\pi} \sin\left(\frac{\pi x}{2b}\right) + \frac{4b^2}{\pi^2} \cos\left(\frac{\pi x}{2b}\right) \right]_0^b = 1$$

Mark 3

$$\Rightarrow \frac{2b^2}{\pi} + 0 - 0 - \frac{4b^2}{\pi^2} = 1$$

$$\Rightarrow b^2(2\pi - 4) = \pi^2$$

$$\Rightarrow b = \frac{\pi}{\sqrt{2\pi - 4}} \quad [b > 0]$$

Mark 4**END OF SUGGESTED SOLUTIONS AND MARKING GUIDE BOOK**