

MATHEMATICAL METHODS

Units 3 & 4 – Written examination 2



2019 Trial Examination

SOLUTIONS

SECTION A: Multiple-choice questions (1 mark each)

Question 1

Answer: C

Explanation:

$$\text{Period: } \frac{2\pi}{\frac{1}{4}} = 2\pi \times 4 = 8\pi$$

$$\text{Range: } [c - a, c + a] = [-1 - 2, -1 + 2] = [-3, 1]$$

Question 2

Answer: B

Explanation:

negative quartic graph

x-intercepts at $x = a$, $x = b$ and $x = c$ which gives brackets of $(x - a)(x - b)(x - c)$

repeated factor at $x = c$, means $(x - c)^2$

Question 3

Answer: C

Explanation:

Solving by hand gives C, or use solve function on CAS and equate each option to the answer until the CAS offers a true response

The screenshot shows a CAS window with the input $\text{solve}(y=\ln(1-3\cdot x)+1,x)$. The output is $x = \frac{-e^{-1} \cdot (e^y - e)}{3}$. Below this, the equation $\frac{1 - e^{y-1}}{3} = \frac{-e^{-1} \cdot (e^y - e)}{3}$ is displayed, and the CAS returns 'true'.

Question 4

Answer: E

Explanation:

For inverse swap x and y and solve for y on CAS, take the equation of $y^{-1} = 3 - \frac{1}{\sqrt{x}}$ and domain of $(0, \infty)$ as $\text{dom } f^{-1} = \text{ran } f$

Question 5

Answer: C

Explanation:

	$f(x)$	$g(x)$
Domain	$[0, 10]$	R
Range	$[0, 10]$	$(0, \infty)$

Firstly for $g(f(x))$ to exist $\text{range } f \subseteq \text{domain } g$

The $\text{range } g(f(x)) = \text{range } f$

Question 6

Answer: D

Explanation:

$$\begin{bmatrix} m & 5 \\ 1 & m+4 \end{bmatrix}$$

$$m^2 + 4m - 5 = 0$$

$$m = -5, m = 1$$

For infinitely many solutions same gradient and same y-intercept

Sub in $m = -5, m = 1$ into both equations to find the answer

Question 7

Answer: B

Explanation:

$\text{solve}(-\sqrt{3} \cdot \sin(2 \cdot x) = \cos(2 \cdot x), x) | 0 \leq x \leq 2 \cdot \pi$
 $x = \frac{5 \cdot \pi}{12} \text{ or } x = \frac{11 \cdot \pi}{12} \text{ or } x = \frac{17 \cdot \pi}{12} \text{ or } x = \frac{23 \cdot \pi}{12}$
 $\frac{5 \cdot \pi}{12} + \frac{11 \cdot \pi}{12} + \frac{17 \cdot \pi}{12} + \frac{23 \cdot \pi}{12} = \frac{14 \cdot \pi}{3}$

Question 8

Answer: D

Explanation:

Independent events $\Pr(A \cap B) = \Pr(A) \times \Pr(B) = 0.3 \times 0.45 = 0.135$

$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B) = 0.3 + 0.45 - 0.135 = 0.615$

Question 9

Answer: E

Explanation:

$f(x) \rightarrow -f(x)$ vertical reflection

$-f(x) \rightarrow -f(-x)$ horizontal reflection

$-f(x) \rightarrow -f(-x) + 1$ vertical translation

Question 10

Answer: C

Explanation:

Perpendicular gradient = $-\frac{1}{6}$

CAS interface showing the following steps:

- Input: $\frac{d}{dx}((2 \cdot x + 1)^3)$
- Result: $6 \cdot (2 \cdot x + 1)^2$
- Input: $6 \cdot (2 \cdot -1 + 1)^2$
- Result: 6
- Input: $\text{solve}\left(y - -1 = \frac{-1}{6} \cdot (x - -1), y\right)$
- Result: $y = \frac{-x}{6} - \frac{7}{6}$

Question 11

Answer: E

Explanation:

Use CAS to solve

CAS interface showing the following steps:

- Input: $\text{Define } f(x) = \frac{2}{3 \cdot x}$
- Result: Done
- Input: $f(x+y) = f(x) \cdot f(y)$
- Result: $\frac{2}{3 \cdot (x+y)} = \frac{4}{9 \cdot x \cdot y}$
- Input: $\text{Define } f(x) = 3^{2 \cdot x}$
- Result: Done
- Input: $f(x+y) = f(x) \cdot f(y)$
- Result: true

Question 12

Answer: A

Explanation:

$$X \sim \text{Bi}\left(10, \frac{1}{6}\right)$$

$$\Pr(X \leq 4) =$$

**Question 13**

Answer: C

Explanation:

$$x' = x - 3 \rightarrow x = x' + 3$$

$$y' = -2y + 4 \rightarrow y = 2 - \frac{y'}{2}$$

$$y = \frac{1}{x^2}$$

$$2 - \frac{y'}{2} = \frac{1}{(x'+3)^2}$$

$$-\frac{y'}{2} = \frac{1}{(x'+3)^2} - 2$$

$$\frac{y'}{2} = \frac{-1}{(x'+3)^2} + 2$$

$$y' = \frac{-2}{(x'+3)^2} + 4$$

Question 14

Answer: A

Explanation:

$$p + 2p + p + \frac{p}{2} + \frac{3p}{2} = 1$$

$$6p = 1$$

$$p = \frac{1}{6}$$

$$E(X) = 1 \times \frac{1}{6} + 2 \times 2 \left(\frac{1}{6}\right) + 3 \times \frac{1}{6} + 4 \times \frac{1}{6} + 5 \times \frac{3 \times \frac{1}{6}}{2}$$

$$E(X) = \frac{1}{6} + \frac{2}{3} + \frac{1}{2} + \frac{1}{3} + \frac{5}{4}$$

$$E(X) = \frac{35}{12}$$

Question 15

Answer: E

Explanation:

$p = 1$ two intersections

$p = 4$ two intersections

Between $p = 1$ and $p = 4$ three intersections therefore $1 < p < 4$

Question 16

Answer: D

Explanation:

integration by recognition

$$\int \frac{2x}{x^2+2} dx = \log_e(x^2 + 2) + c$$

$$2 \int \frac{x}{x^2+2} dx = \log_e(x^2 + 2) + c$$

$$\int \frac{x}{x^2+2} dx = \frac{1}{2} \log_e(x^2 + 2) + c$$

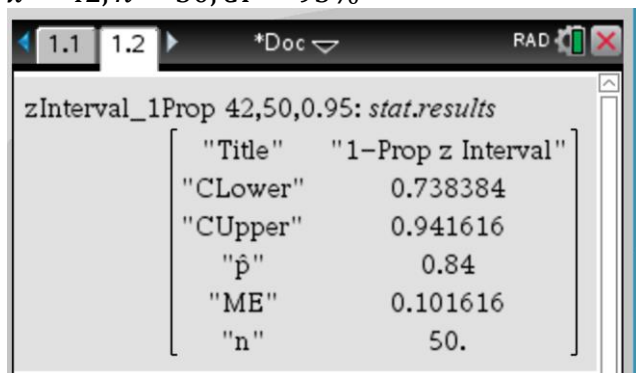
$$\int \frac{3x}{x^2+2} dx = \frac{3}{2} \log_e(x^2 + 2) + c$$

Question 17

Answer: A

Explanation:

$x = 42, n = 50, CI = 95\%$



zInterval_1Prop 42,50,0.95: stat.results	
"Title"	"1-Prop z Interval"
"CLower"	0.738384
"CUpper"	0.941616
"p"	0.84
"ME"	0.101616
"n"	50.

Question 18

Answer: B

Explanation:

$$y_{ave} = \frac{1}{b-a} \int_a^b f(x) dx$$

$$0.4 = \frac{1}{a-(-a)} \int_{-a}^a \sin^2(x) dx \text{ solve on CAS for } a$$

Question 19

Answer: C

Explanation:

400 pens initially, 10 run out in first 6 weeks

$$\Pr(X < 6) = \frac{10}{400} = 0.025$$

$$\Pr(Z < z) = 0.025 \quad \text{invNorm}(0.025, 0, 1) = -1.9599639 \dots$$

$$z = \frac{x - \mu}{\sigma} \quad -1.9599 = \frac{6 - 14}{\sigma}$$

$$\sigma = 4.08 \approx 4 \text{ weeks}$$

Question 20

Answer: E

Explanation:

Each area between the two graphs is the same

$g(x)$ starts above $f(x)$ hence E is correct.

SECTION B: Extended response questions**Question 1 (9 marks)**

a. $B(23) = 2A$

$2A = Ae^{23k} \quad (1M)$

$2 = e^{23k}$

$\log_e(2) = 23k \quad (1M)$

$k = \frac{1}{23} \log_e(2)$

$k = 0.030137 \dots \quad (1M)$

$k = 0.03 \text{ as required}$

3 marks

b. $15000 = Ae^{0.03 \times 60} \quad (1M)$

$15000 = Ae^{1.8}$

$A = 2479.48$

$A = 2479 \text{ bacteria} \quad (1A)$

2 marks

c. $B(90) = 36887 \text{ bacteria}$

1 mark

d. 3900 bacteria

1 mark

e. $P(t) = B(t)$

$3900e^{0.02t} = 2479e^{0.03t} \quad (1M)$

$t = 45.3121$

$t = 45 \text{ mins and } 19 \text{ seconds} \quad (1A)$

2 marks

Total 9 marks

Question 2 (15 marks)

a. $f(x) = 0$

$(-1, 0), (0, 0) \text{ and } \left(\frac{3}{2}, 0\right) \quad (1A)$

1 mark

b. $f'(x) = 0 \quad (1M)$

$8x^3 - 3x^2 - 6x = 0$

$x = \frac{-\sqrt{201}+3}{16}, x = 0, x = \frac{\sqrt{201}+3}{16}$

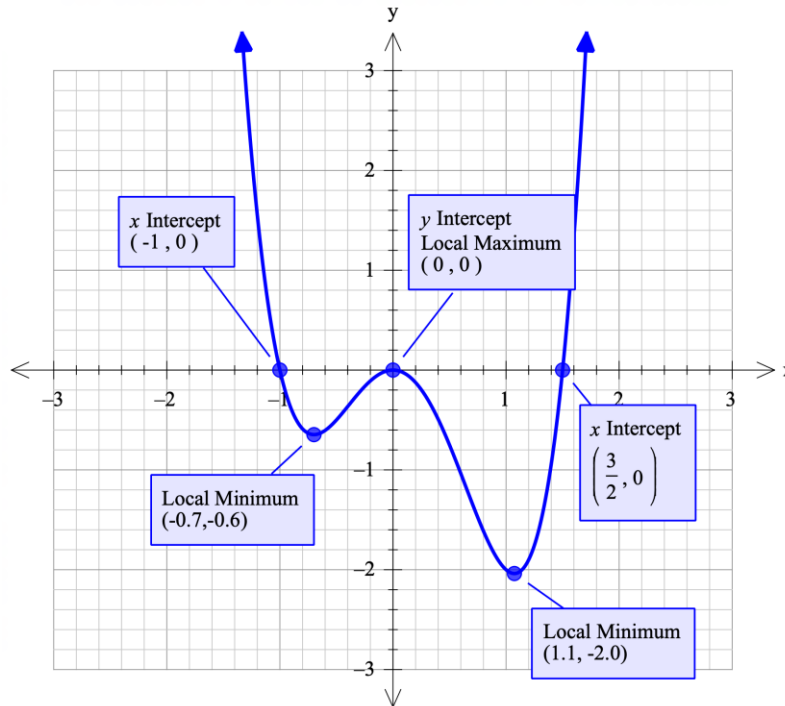
$(-0.7, -0.6), (0, 0) \text{ and } (1.1, -2.0)$

x	-1	-0.7	-0.2	0	1	1.1	2
sign	-	0	+	0	-	0	+
slope	\	-	/	-	\	-	/

Local Minimum at $(-0.7, -0.6)$ (1A)
 Local Maximum at $(0, 0)$ and (1A)
 Local Minimum at $(1.1, -2.0)$ (1A)

4 marks

c.



1M – correct shape
 1M – correct intercepts
 1M – correct turning points
 3 marks

d. $c > 2$ (1A)

1 mark

e. $f'(x) = 8x^3 - 3x^2 - 6x$
 $f'(1) = -1$ (1M)
 $f(1) = -2$
 $y - -2 = -1(x - 1)$
 $y = -x - 1$ (1A)

2 marks

f. $-x - 1 = 2x^4 - x^3 - 3x^2$ (1M)
 $x = -1, x = -\frac{1}{2}, x = 1$
 $(-1, 0), (-\frac{1}{2}, -\frac{1}{2}), (1, -2)$ (1A)

2 marks

$$g. \int_{-1}^{-\frac{1}{2}} ((-x-1) - f(x)) dx + \int_{-\frac{1}{2}}^1 (f(x) - (-x-1)) dx \quad (1M)$$

$$\frac{41}{320} + \frac{297}{320} = \frac{169}{160} \text{ units}^2 \quad (1A)$$

2 marks
Total 15 marks

Question 3 (12 marks)**a.**

i. Period $\frac{2\pi}{\frac{2}{3}} = 2\pi \times \frac{2}{3} = \frac{4\pi}{3} \quad (1A)$

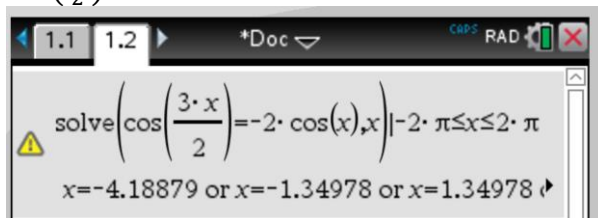
Amplitude 1 (1A)

ii. Period $2\pi \quad (1A)$

Amplitude 2 (1A)

2 + 2 = 4 marks

b. $\cos\left(\frac{3x}{2}\right) = -2 \cos(x) \quad (1M)$



$$x = -4.18879, x = -1.34978, x = 1.34978, x = 4.18879$$

$$(-4.189, 1.000), (-1.350, -0.438), (1.350, -0.438), (4.189, 1.000) \quad (1/2A \text{ each coordinate})$$

3 marks

c. $[-4.189, 4.189]$

1 marks

d.

i. $[-1, 1] \quad (1A)$

ii. $[-2, 2] \quad (1A)$

1 + 1 = 2 marks

e. $2 \int_{-4.189}^{-1.350} (g(x) - f(x)) dx + \int_{-1.350}^{1.350} (f(x) - g(x)) dx \quad (1M)$

$$2 \times 4.28257 + 5.10104 = 13.6662$$

$$14m^2 \quad (1A)$$

2 marks
Total 12 marks

Question 4 (15 marks)

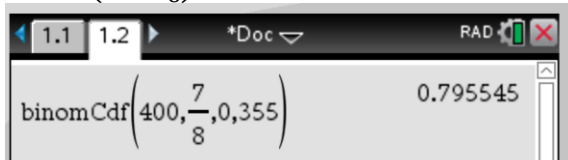
a. $n = 1000, p = \frac{7}{8}$

$$E(X) = np = 1000 \times \frac{7}{8} \quad (1M)$$

$$E(X) = 875 \quad (1A)$$

2 marks

b. $X \sim Bi\left(400, \frac{7}{8}\right)$ (1M)



$$\Pr(X \leq 355) = 0.796 \quad (1A)$$

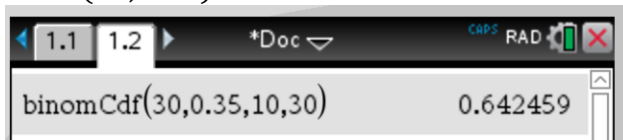
2 marks

c. $\Pr(X < 350 | X > 100) = \frac{\Pr(X < 350 \cap X > 100)}{\Pr(X > 100)}$ (1M)

$$\frac{Ps(101 \leq X \leq 349)}{\Pr(X \geq 101)} = 0.462 \quad (1A)$$

2 marks

d. $Y \sim Bi(30, 0.35)$ (1M)

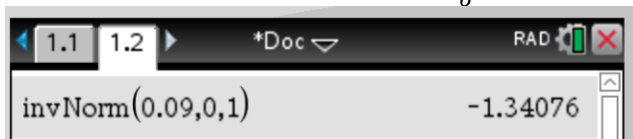


$$\Pr(Y \geq 10) = 0.642 \quad (1A)$$

2 marks

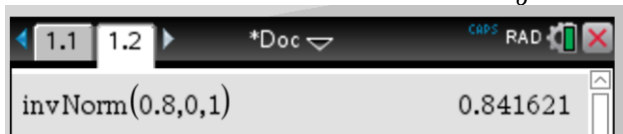
e. Using $z = \frac{x - \mu}{\sigma}$ and simultaneous equations

$$\Pr(Z < a) = 0.09, \text{ where } a = \frac{16.57 - \mu}{\sigma}$$



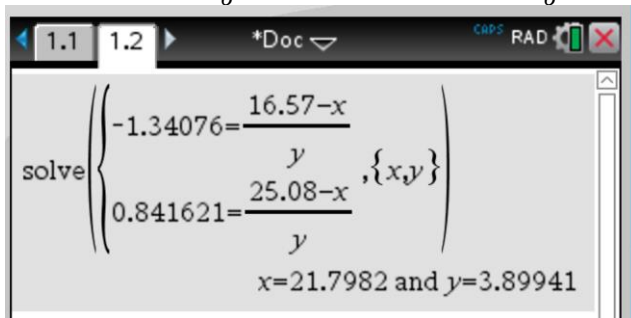
$$a = -1.34076 \dots \quad (1M)$$

$$\Pr(b < 25.08) = 0.80, \text{ where } b = \frac{25.08 - \mu}{\sigma}$$



$$b = 0.841621 \dots \quad (1M)$$

$$-1.34076 = \frac{16.57 - \mu}{\sigma} \text{ and } 0.841621 = \frac{25.08 - \mu}{\sigma}$$



TI-84 Plus calculator screen showing the solve function for the system of equations:

$$\text{solve} \left(\begin{cases} -1.34076 = \frac{16.57 - x}{y} \\ 0.841621 = \frac{25.08 - x}{y} \end{cases}, \{x, y\} \right)$$

The solution found is $x = 21.7982$ and $y = 3.89941$.

$$\mu = 21.80^\circ\text{C} \text{ and } \sigma = 3.90^\circ\text{C}$$

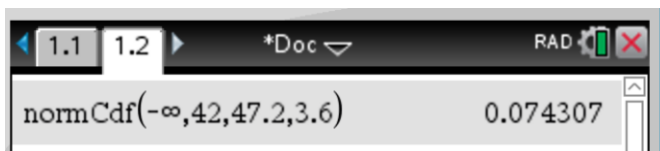
(1A)

3 marks

f. $X \sim N(47.2, 3.6^2)$

$$\Pr(X < 42) = 0.074307$$

(1M)



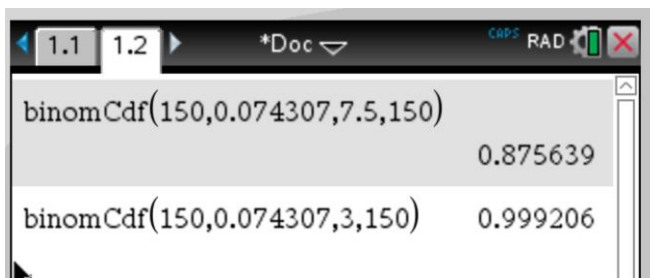
TI-84 Plus calculator screen showing the normCdf function:

$$\text{normCdf}(-\infty, 42, 47.2, 3.6) = 0.074307$$

$$\Pr(\hat{p} \geq 0.05 | \hat{p} \geq 0.02) = \Pr(X \geq 150 \times 0.05 | X \geq 150 \times 0.02)$$

$$\Pr(X \geq 7.5 | X \geq 3) = \frac{\Pr(X \geq 7.5 \cap X \geq 3)}{\Pr(X \geq 3)} \quad (1M)$$

$$\Pr(X \geq 7.5 | X \geq 3) = \frac{\Pr(X \geq 7.5)}{\Pr(X \geq 3)}$$



TI-84 Plus calculator screen showing the binomCdf function calculations:

$$\text{binomCdf}(150, 0.074307, 7.5, 150) = 0.875639$$

$$\text{binomCdf}(150, 0.074307, 3, 150) = 0.999206$$

$$\Pr(X \geq 7.5 | X \geq 3) = 0.8763$$

(1A)

3 marks

g. $\hat{p} = \frac{1}{15}$

$$\left(\frac{1}{15} - 1.96\sqrt{\frac{\frac{1}{15} \times \frac{14}{15}}{150}}, \frac{1}{15} + 1.96\sqrt{\frac{\frac{1}{15} \times \frac{14}{15}}{150}} \right)$$

OR use CAS

zInterval_1Prop 10,150,0.95: stat.results	
"Title"	"1-Prop z Interval"
"CLower"	0.026748
"CUpper"	0.106585
"p"	0.066667
"ME"	0.039919
"n"	150.

(0.0267, 0.1066)

1 mark

Total 15 marks

Question 5 (9 marks)

a. $f'(x) = 0$

$$0 = -2x(2x^2 - n^2)$$

$$\left(\frac{n\sqrt{2}}{2}, \frac{n^4}{4} \right), \left(-\frac{n\sqrt{2}}{2}, \frac{n^4}{4} \right), (0,0) \quad (1A)$$

1 mark

b. $\int_{-n}^n f(x) dx = \frac{4}{15}$

$$\frac{4n^5}{15} = \frac{4}{15}$$

$$n = 1 \quad (1A)$$

$$x = 1, x = 0, x = -1 \quad (1A)$$

2 marks

c. $0 = -mx^2 + m$

$$-m = -mx^2$$

$$1 = x^2$$

$$x = \pm 1 \text{ as required}$$

1 mark

d. $Area = 2 \int_{-1}^{-\sqrt{m}} (f(x) - g(x)) dx + \int_{-\sqrt{m}}^{\sqrt{m}} (g(x) - f(x)) dx \quad (1M)$

$$= \frac{32m^2 - 20m + 4}{15} \quad (1A)$$

2 marks

e.

i. $f(b) = -b^2(b - n)(b + n)$
 $f'(b) = -2b(2b^2 - n^2) \quad (1A)$

ii. $y - f(b) = f'(b)(x - b) \quad (1M)$
 $y - (-b^2(b - n)(b + n)) = (-2b(2b^2 - n^2))(x - b)$
 $y = -2b(2(2b^2 - n^2)x + 3b^4 - b^2n^2) \quad (1A)$

1 + 2 = 3 marks

Total 9 marks