The Mathematical Association of Victoria

Trial Examination 2019

MATHEMATICAL METHODS

Trial Written Examination 2 - SOLUTIONS

SECTION A: Multiple Choice

Question	Answer	Question	Answer
1	E	**11**	D
2	D	**12**	A
3	C	**13**	E
4	A	**14**	C
5	C	**15**	C
6	E	**16**	D
7	B	**17**	A
8	E	**18**	C
9	B	**19**	E
10	C	**20**	E

Question 1 **Answer E**

$y=-3\cos\left(2\pi x-\frac{\pi}{2}\right)$

period = $\frac{2\pi}{2\pi}=1$

Amplitude = 3

Question 2 **Answer D**

$f(x)=\sin(x)$ for $x\in[0,2\pi]$ and $g(x)=(x+1)^2$ for its maximal domain which is *R*.
$g(f(x))$ exists, with the domain of $g(f(x))$ equalling the domain of *f*, which is $[0,2\pi]$.

Question 3 **Answer C**

Solve $2\sin(2x)=1$ for *x*.

$\sin(2x)=\frac{1}{2}$

$2x=\frac{\pi}{6},\frac{5\pi}{6}...$

$x = \frac{\pi}{12}, \frac{5\pi}{12} \ldots$

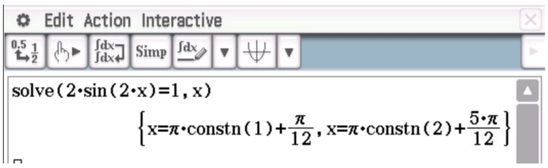
General solution

The period is π.

$x = \frac{\pi}{12} + n\pi, n \in Z$

or

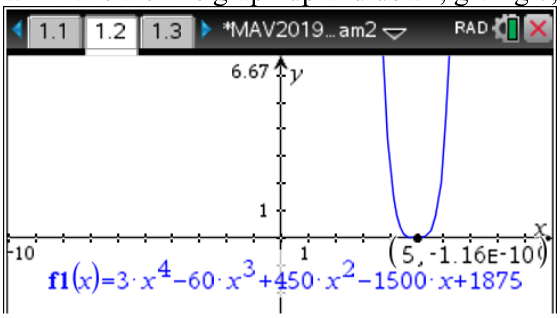
$x = \frac{5\pi}{12} + n\pi, n \in Z$

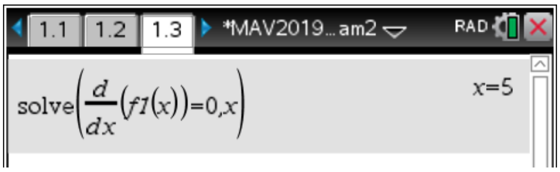


Question 4 **Answer A**

$f(x) = 3x^4 - 60x^3 + 450x^2 - 1500x + k + 1875$

The graph of $f_1(x) = 3x^4 - 60x^3 + 450x^2 - 1500x + 1875$ has only one turning point. The value of k will translate this graph up and down, giving 0, 1 or 2 x-intercepts.





Question 5 **Answer C**

$f : \left[\frac{3}{2}, \infty\right) \to R, f(x) = -\sqrt{2x-3} + 4$

Let $y = -\sqrt{2x-3} + 4$.

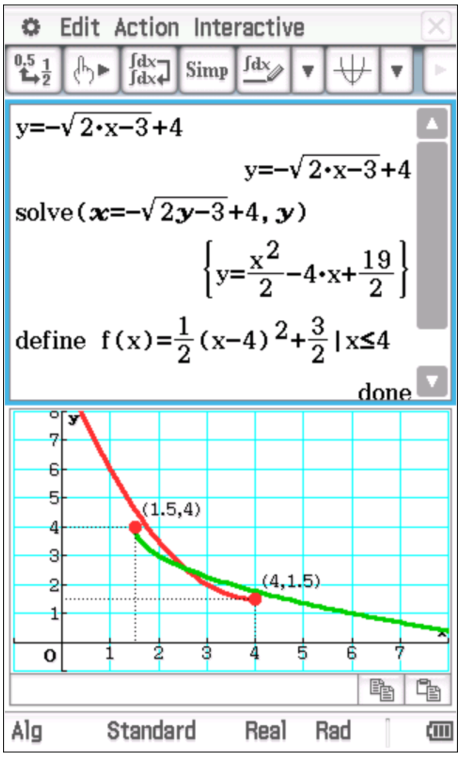
Inverse: swap x and y and solve for y.

$x = -\sqrt{2y-3} + 4$

$y=\frac{1}{2}(x-4)^2+\frac{3}{2}$

The range of f is $(-\infty,4]$. The domain of the inverse is $(-\infty,4]$.

$f^{-1}:(-\infty,4]\to R,\ f^{-1}(x)=\frac{1}{2}(x-4)^2+\frac{3}{2}$



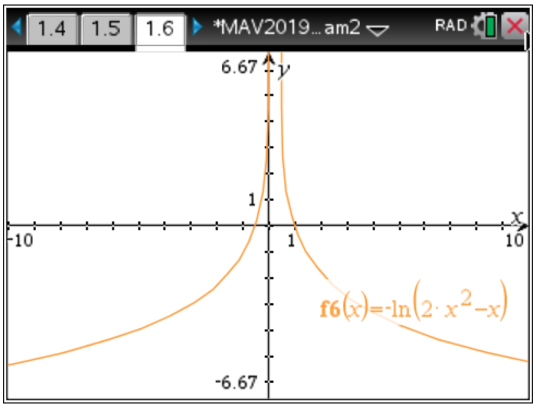
Question 6 **Answer E**

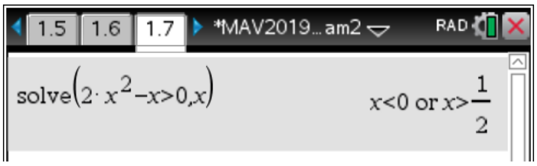
$y=e^x$ is $y_T=2e^{-2x}+3$

$y_1=2e^x$	a dilation from the x-axis by a factor of 2
$y_2=2e^{2x}$	followed by a dilation from the y-axis by a factor of 0.5
$y_3=2e^{-2x}$	then a reflection in the y-axis and
$y_T=2e^{-2x}+3$	a translation in the positive direction of the y-axis by 3 units.

Question 7 **Answer B**

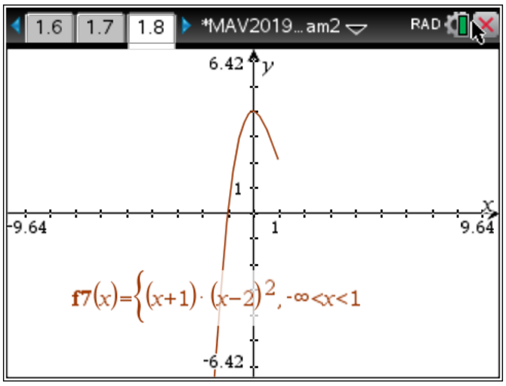
The graph of $y=-\log_e\left(2x^2-x\right)$ has two vertical asymptotes, $x=0$ and $x=\frac{1}{2}$.





Question 8 **Answer E**

$l:(-\infty,1)\rightarrow R,\ l(x)=(x+1)(x-2)^2$ is a many to 1 function. So the inverse is not a function.



Question 9 **Answer B**

$$T:R^2\rightarrow R^2, T\left(\begin{bmatrix}x\\y\end{bmatrix}\right)=\begin{bmatrix}-1 & 0\\0 & 0.5\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}+\begin{bmatrix}0\\2\end{bmatrix}$$

$x'=-x,\ x=-x'$

$y'=0.5y+2,\ y=2y'-4$

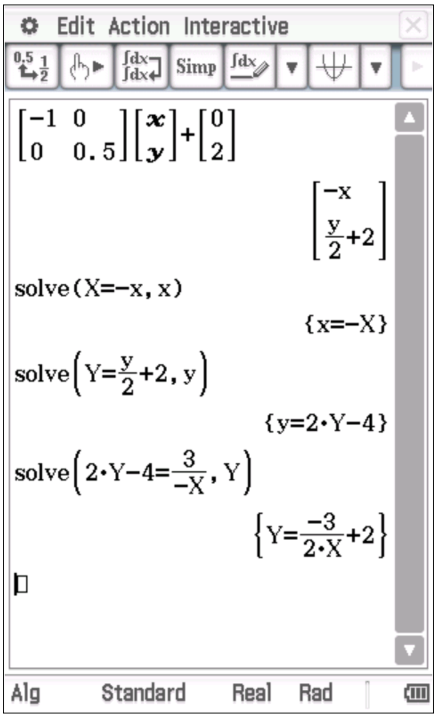
$$y=\frac{3}{x}$$

$$2y'-4=-\frac{3}{x'}$$

$$y'=-\frac{3}{2x'}+2$$

The equation of the image is

$$y = -\frac{3}{2x} + 2$$



OR

$y_1 = -\frac{3}{x}$ reflection in the y-axis

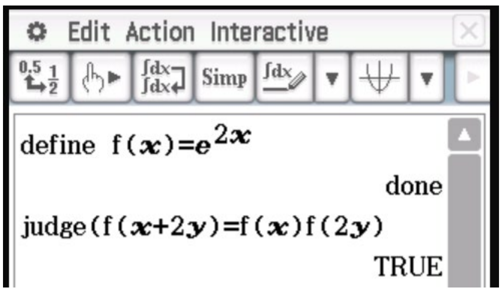
$y_2 = -\frac{3}{2x}$ dilation by a factor of 0.5 from the x-axis

$y_3 = -\frac{3}{2x} + 2$ translation of 2 units up

Question 10 **Answer C**

$$f(x+2y) = f(x)f(2y)$$
$$f(x) = e^{2x}$$

$$LHS = f(x+2y) = e^{2(x+2y)}$$
$$= e^{2x} \times e^{4y}$$
$$= f(x)f(2y) = RHS$$

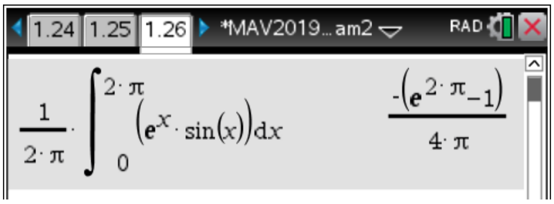


Question 11 **Answer D**

$f(x) = e^x \sin(x)$ over the interval $[0,\ 2\pi]$

Average value = $\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} \left(e^x \sin(x)dx\right)$

$= -\dfrac{e^{2\pi}-1}{4\pi}$

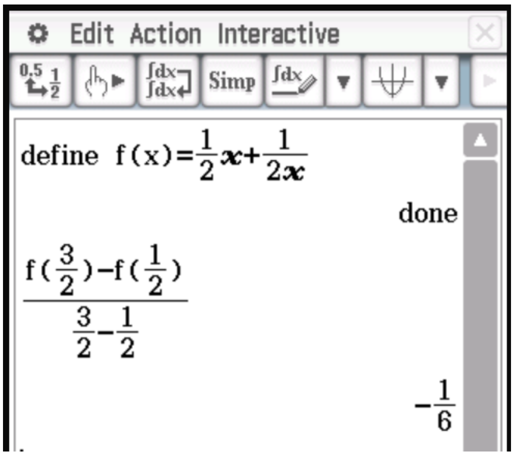


Question 12 **Answer A**

The average rate of change of $y = \dfrac{1}{2}x + \dfrac{1}{2x}$ from $x = \dfrac{1}{2}$ to $x = \dfrac{3}{2}$.

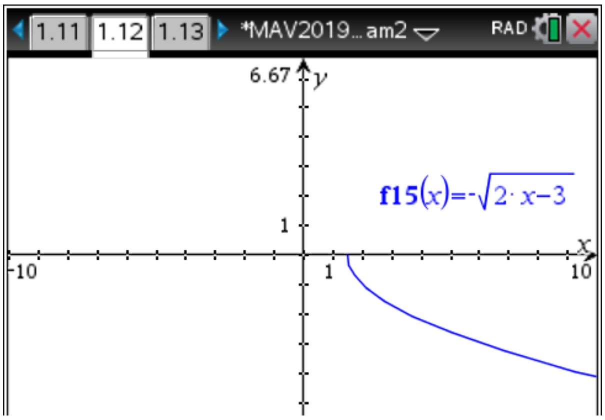
Let $f(x) = \dfrac{1}{2}x + \dfrac{1}{2x}$.

Average rate of change = $\dfrac{f\left(\frac{3}{2}\right) - f\left(\frac{1}{2}\right)}{\frac{3}{2} - \frac{1}{2}} = -\dfrac{1}{6}$



Question 13 **Answer E**

$f : \left(\dfrac{3}{2},\ \infty\right) \to R,\ f(x) = -\sqrt{2x-3}$ is a strictly decreasing function over the given domain.



Question 14 **Answer C**

$a = 5t + 3$

$v = \int (5t + 3)dt$

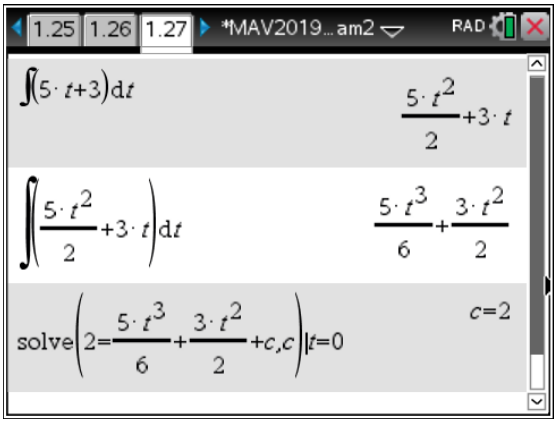
$= \frac{5t^2}{2} + 3t + c, c = 0$

$= \frac{5t^2}{2} + 3t$

$x = \int \left(\frac{5t^2}{2} + 3t \right) dt$

$= \frac{5t^3}{6} + \frac{3t^2}{2} + d$, when $t = 0, x = 2$

$x = \frac{5t^3}{6} + \frac{3t^2}{2} + 2$



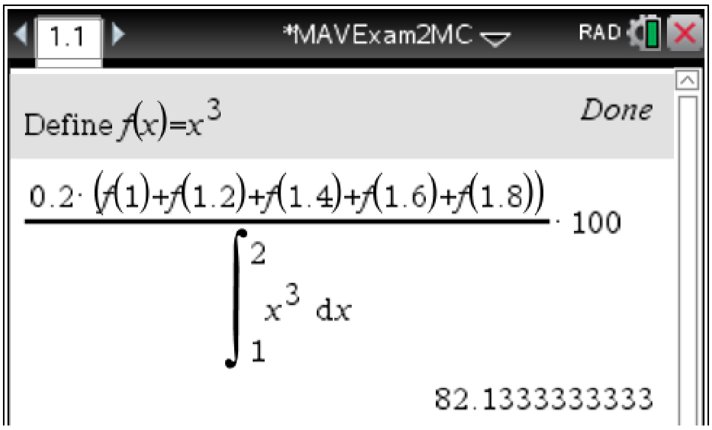
Question 15 **Answer C**

Let $f(x) = x^3$

Area of the rectangles = $0.2\left(f(1) + f(1.2) + f(1.4) + f(1.6) + f(1.8)\right) = 3.08$

Actual area = $\int_1^2 f(x)dx = \frac{15}{4}$

$\frac{3.08}{3.75} \times 100\% \approx 82.13\%$



Question 16 **Answer D**

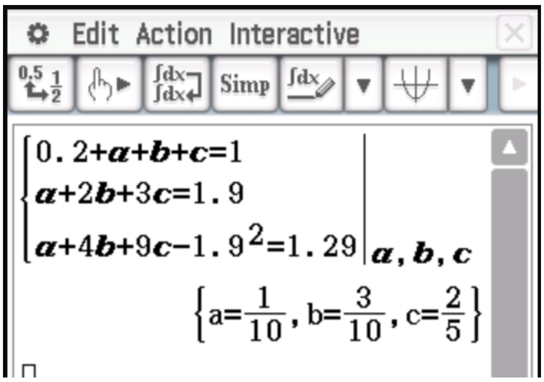
x	0	1	2	3
$\Pr(X = x)$	0.2	a	b	c

$0.2 + a + b + c = 1 \ldots(1)$

$a + 2b + 3c = 1.9 \ldots(2)$

$a + 4b + 9c - 1.9^2 = 1.29 \ldots(3)$

$a = 0.1,\ b = 0.3,\ c = 0.4$

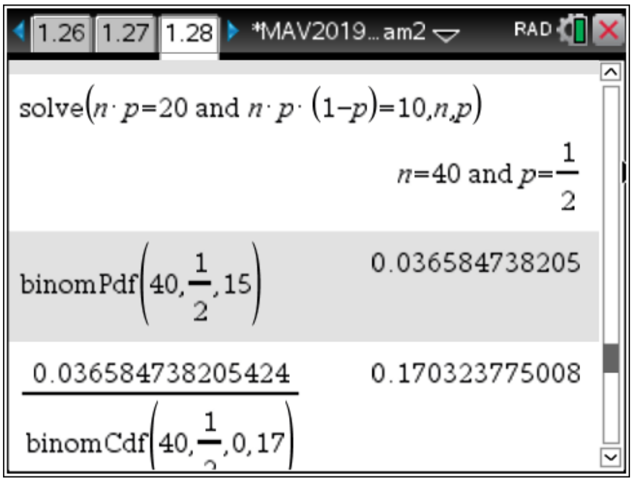


Question 17 **Answer A**

$np = 20, np(1-p) = 10$

$n = 40, p = \frac{1}{2}$

$\Pr(X = 15 \mid X < 18) = \dfrac{\Pr(X = 15)}{\Pr(0 \le X \le 17)} = 0.170$ correct to three decimal places



Question 18 **Answer C**

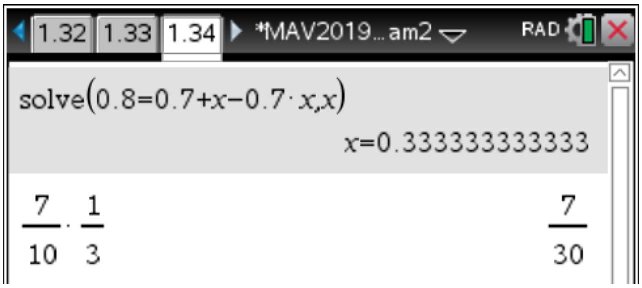
$\Pr(A) = 0.7$ and $\Pr(A' \cap B') = 0.2$

For independent events $\Pr(A \cap B) = \Pr(A) \times \Pr(B)$

$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A) \times \Pr(B)$

Solve $0.8 = 0.7 + \Pr(B) - 0.7 \times \Pr(B)$ for $\Pr(B)$

$\Pr(A \cap B) = \frac{7}{10} \times \frac{1}{3} = \frac{7}{30}$



Question 19 **Answer E**

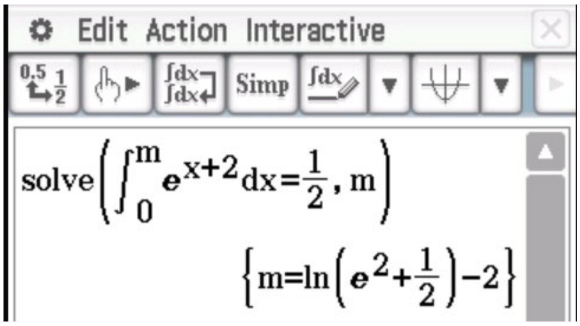
$$f(x) = \frac{1}{4\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-3}{4}\right)^2}, \quad x \in R$$

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, \quad x \in R$$

mean is 3, variance is 16

Question 20 **Answer E**

Solve $\int_0^m \left(e^{x+2}\right) dx = \frac{1}{2}$ for m



SECTION B

Question 1

a. Total Surface Area = $4 \times \frac{1}{2} xs + x^2$ with TSA given as 1100 cm^2

$$4 \times \frac{1}{2} xs + x^2 = 1100$$

$$2xs + x^2 = 1100$$

Giving $s = \frac{1100 - x^2}{2x}$ as required **1M Show that**

b. Right angled triangle formed by vertical height, h, half the side length, x and slant height, s.

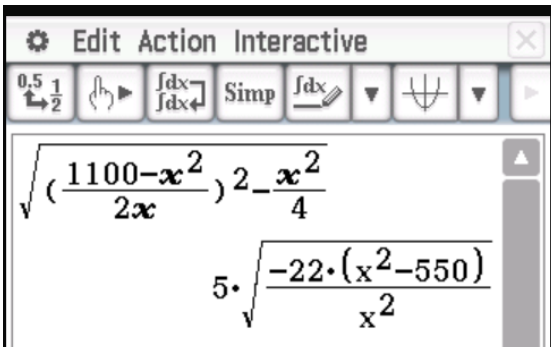
$$\left(\frac{x}{2}\right)^2 + h^2 = s^2$$

Giving $h^2 = s^2 - \frac{x^2}{4}$

From $s = \frac{1100 - x^2}{2x}$ we have $h^2 = \left(\frac{1100 - x^2}{2x}\right)^2 - \frac{x^2}{4}$

Positive vertical height, $h = \sqrt{\left(\frac{1100 - x^2}{2x}\right)^2 - \frac{x^2}{4}}$

Simplified $h = 5\sqrt{\frac{-22\left(x^2 - 550\right)}{x^2}} = \frac{5\sqrt{-22(x^2 - 550)}}{x}$ (accept different forms) **1A**



c. Volume of square based pyramid = $\frac{1}{3} x^2 h$

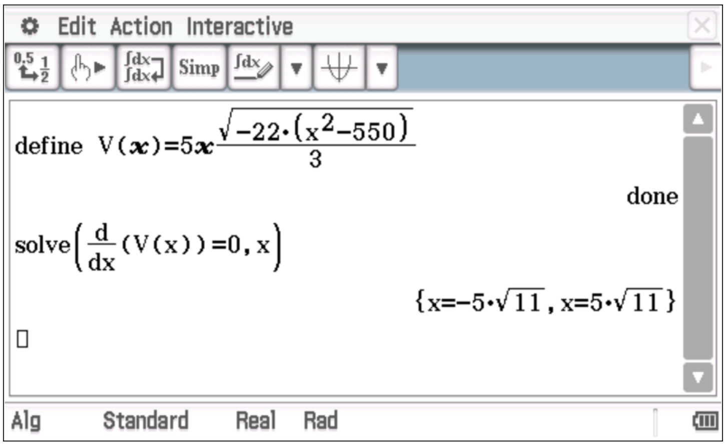
giving $V = \frac{1}{3} x^2 \times 5\sqrt{\frac{-22\left(x^2 - 550\right)}{x^2}}$

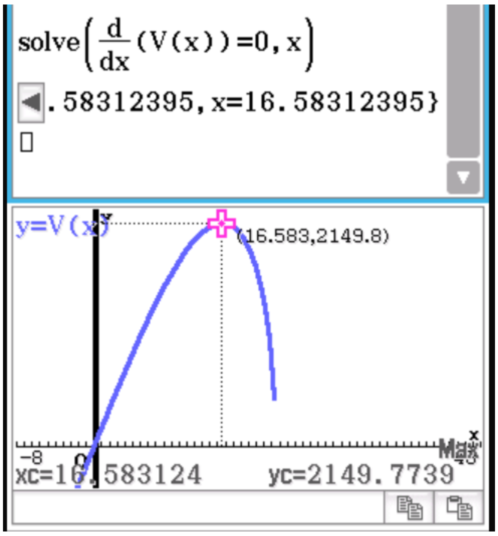
for $x > 0$, $V = \frac{5x\sqrt{-22(x^2 - 550)}}{3}$ **1M Show that**

d. For maximum volume, $\frac{dV}{dx} = 0$

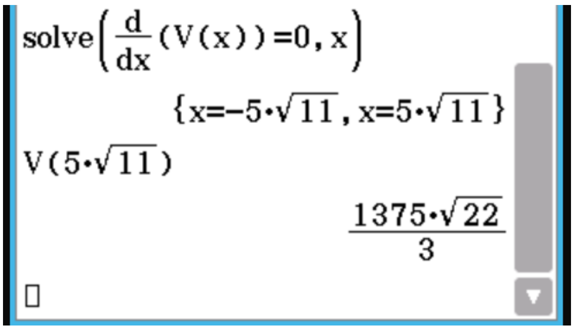
gives $x = \pm 5\sqrt{11}$

from graph of V against x, and $x > 0$, maximum volume occurs at $x = 5\sqrt{11}$ **1A**





e. Maximum volume, $V = \frac{1375\sqrt{22}}{3}$ **1A**



f. Given $\dfrac{dV}{dt} = 0.2t$

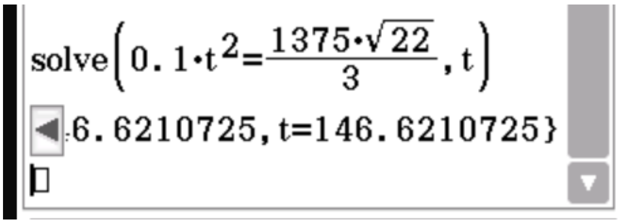
$V = \int (0.2t)\,dt = 0.1t^2 + c$

$t = 0,\ V = 0,\ c = 0$

$V = 0.1t^2$ **1M**

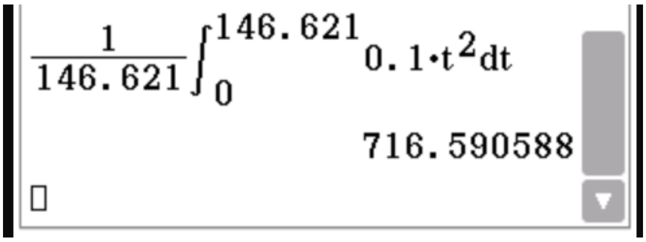
Equating $0.1t^2 = \dfrac{1375\sqrt{22}}{3}$ for full volume

Gives $t = 146.62$ minutes correct to two decimal places **1A**

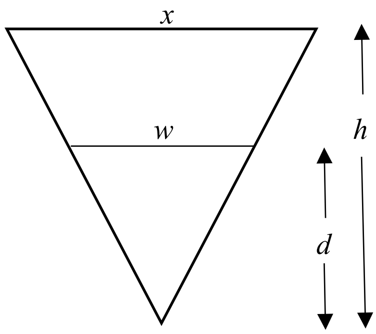


g. Average volume $= \dfrac{1}{146.621 - 0}\displaystyle\int_0^{146.621} \left(0.1t^2\right) dt$ **1M**

Average volume = 716.6 cm^3 correct to one decimal place **1A**



h. Using similar triangles $\frac{w}{d}=\frac{x}{h}$ **1M**



With side length $x=5\sqrt{11}$, we have $h=5\sqrt{\frac{-22\left(\left(5\sqrt{11}\right)^2-550\right)}{\left(5\sqrt{11}\right)^2}}=5\sqrt{22}$ **1A**

$\frac{w}{d}=\frac{5\sqrt{11}}{5\sqrt{22}},\ w=\frac{d}{\sqrt{2}}$ **1M**

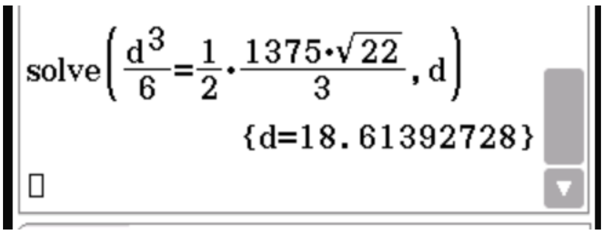
Volume of water $=\frac{1}{3}w^2d$

Giving $V_w=\frac{1}{3}w^2d=\frac{1}{3}\left(\frac{d}{\sqrt{2}}\right)^2d$

Giving $V_w=\frac{d^3}{6}$ with domain $0\le d\le 5\sqrt{22}$ **1A**

i. Let $\frac{d^3}{6}=\frac{1}{2}\times\frac{1375\sqrt{22}}{3}$

giving d = 18.6 cm, correct to one decimal place **1A**

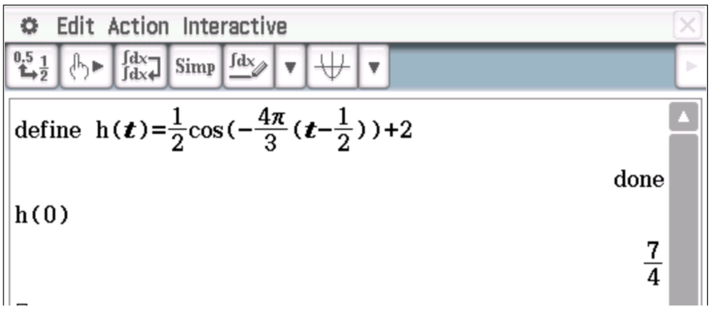


Question 2

$h:[0,\ 60] \to R, h(t) = \frac{1}{2}\cos\left(-\frac{4\pi}{3}\left(t-\frac{1}{2}\right)\right)+2$

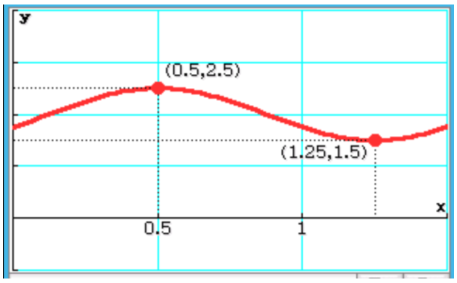
a. $h(0) = \frac{7}{4}$

Initial height 1.75 metres **1A**



b. period $= \dfrac{2\pi}{\frac{4\pi}{3}} = \frac{3}{2}$ **1A**

range $= \left[-\frac{1}{2}+2, \frac{1}{2}+2\right] = [1.5, 2.5]$ **1A**



c. 1A 3 correct, 2A all correct

- dilate from t-axis by factor of $\frac{1}{2}$
- dilate from h-axis by factor of $\frac{3}{4\pi}$
- reflect in the h-axis
- translate in positive direction of t-axis by $\frac{1}{2}$ units
- translate in positive direction of h-axis by 2 units

d. $T\left(\begin{bmatrix} t \\ h \end{bmatrix}\right) = \begin{bmatrix} b & 0 \\ 0 & c \end{bmatrix}\begin{bmatrix} t \\ h \end{bmatrix}$.

Dilated by a factor of $\frac{2}{3}$ from the t-axis gives $b = 1,\ c = \frac{2}{3}$. **1A**

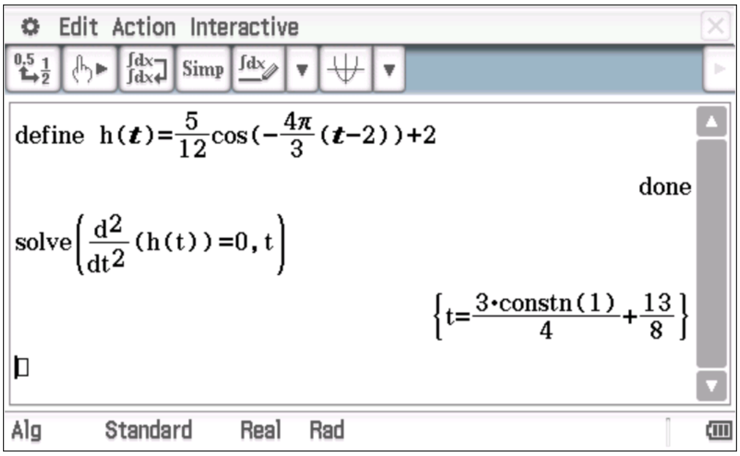
e. $h_1(t) = \frac{2}{3}\left(\frac{1}{2}\cos\left(-\frac{4\pi}{3}\left(t-\frac{1}{2}\right)\right)+2\right)$

$h_1(t) = \frac{1}{3}\cos\left(-\frac{4\pi}{3}\left(t-\frac{1}{2}\right)\right)+\frac{4}{3}$ **1A**

f. $T\left(\begin{bmatrix} t \\ h \end{bmatrix}\right) = \begin{bmatrix} -\frac{3}{4\pi} & 0 \\ 0 & \frac{1}{3} \end{bmatrix}\begin{bmatrix} t \\ h \end{bmatrix} + \begin{bmatrix} \frac{1}{2} \\ \frac{4}{3} \end{bmatrix}$ **1A**

g. Solve $\frac{d}{dt}\left(h_2{}'(t)\right) = 0$ or observe from the graph of the derivative

Solve $\frac{d}{dt}\left(h_2{}'(t)\right) = 0$ and substitute a suitable constant **1M**



First at a maximum at $t = \frac{1}{8}$ minutes **1A**

OR

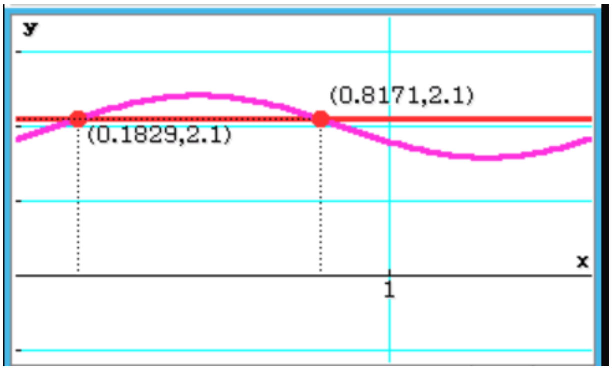
Sketch derivative graph **1M**

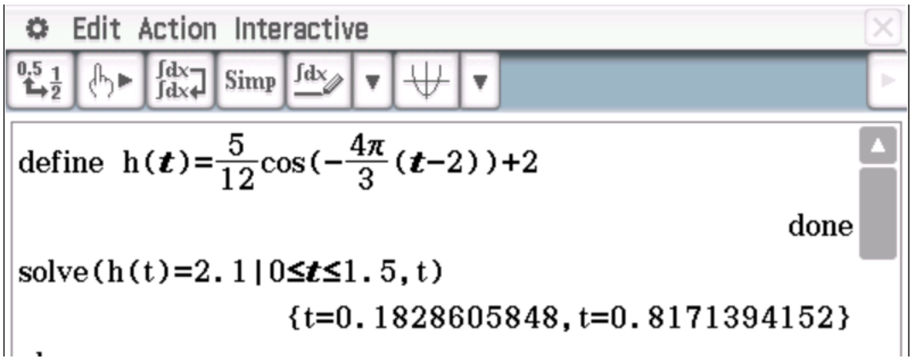
First at a maximum at $t = \frac{1}{8}$ minutes **1A**

h. one wave cycle is 1.5 minutes

Solve $h_2(t) > 2.1,\ t \in [0,1.5]$ **1M**

A graph helps to visualise the situation





northern path under water for $\frac{0.8171 - 0.1829}{1.5} = 0.4228$

proportion of one wave cycle = 0.42 correct to two decimal places **1A**

Question 3

a. No, only the first cage was selected. **1A**

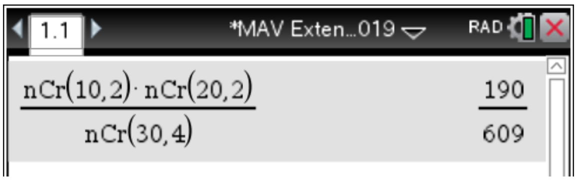
b. Give every mouse a number from 1 to 90 and generate 12 numbers using a suitable random number generator. **1A**

OR

Randomly select a tub and then randomly select 4 mice from each tub (stratified sampling). **1A**

c. $\dfrac{\binom{10}{2}\binom{20}{2}}{\binom{30}{4}}$ **1M**

$=\dfrac{190}{609}$ **1A**



OR

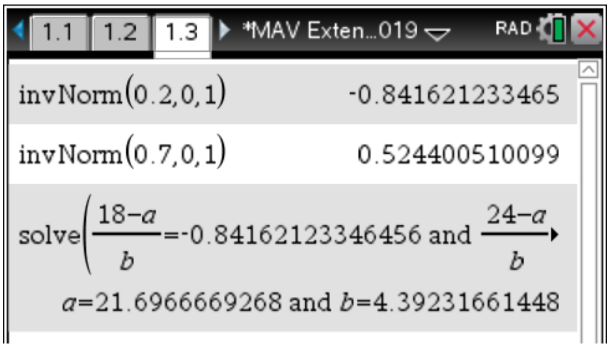
$6\times\dfrac{10}{30}\times\dfrac{9}{29}\times\dfrac{20}{28}\times\dfrac{19}{27}$ **1M**

$=\dfrac{190}{609}$ **1A**

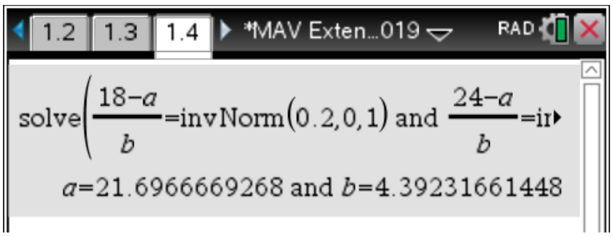
d. Solve $\dfrac{18-\mu}{\sigma}=-0.841...$ and $\dfrac{24-\mu}{\sigma}=0.524...$ **1M**

$\mu=21.7$ correct to one decimal place **1A**

$\sigma=4.4$ correct to one decimal place **1A**



OR



e. $\mathrm{E}\left(\hat{P}\right) = p = 0.8$ **1A**

Standard deviation of $\hat{P} = \sqrt{\dfrac{p(1-p)}{n}}$ where $p = 0.8,\ n = 100$

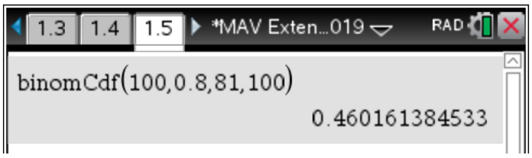
$= \sqrt{\dfrac{0.8 \times 0.2}{100}} = 0.04$ **1A**

f. Let $X \sim \mathrm{Bi}(100,\ 0.8)$

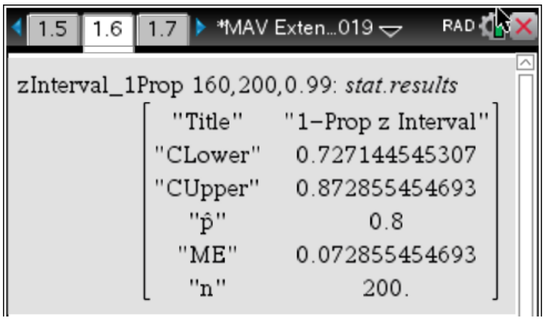
$\Pr\left(\hat{P} > p\right) = \Pr\left(\hat{P} > 0.8\right)$

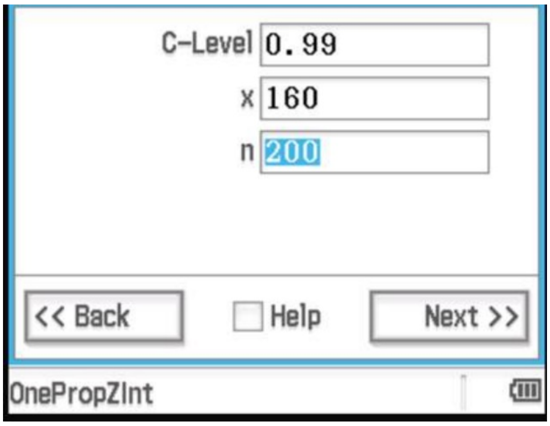
$= \Pr(X > 80)$ **1M**

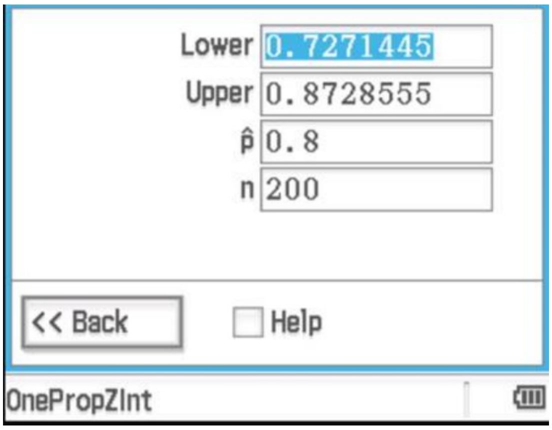
= 0.460 correct to three decimal places **1A**



g. (0.727, 0.873) **1A**



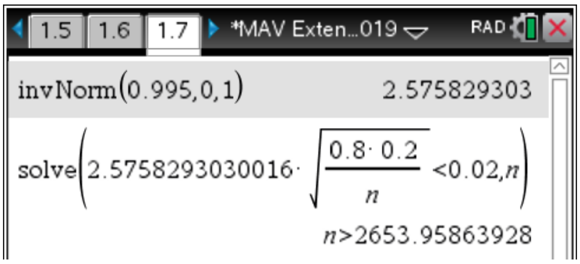




h. $0.99 \times 300 = 297$ **1A**

i. Solve $2.575...\sqrt{\dfrac{0.8 \times 0.2}{n}} < 0.02$ **1M**

$n = 2654$ **1A**



Note: $Z \sim \mathrm{N}(0,1)$, $\Pr(Z < z_{0.99}) = 0.995$, $z_{0.99} = 2.575...$

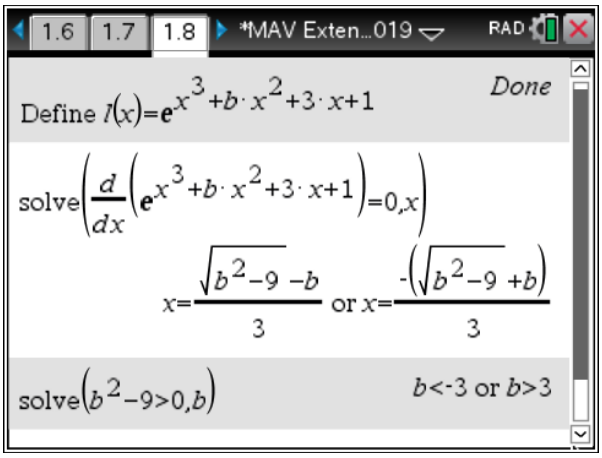
Question 4

a. Solve $\frac{d}{dx}\left(l_b(x)\right)=0$ for x.

$x=\frac{-b\pm\sqrt{b^2-9}}{3}$ **1M**

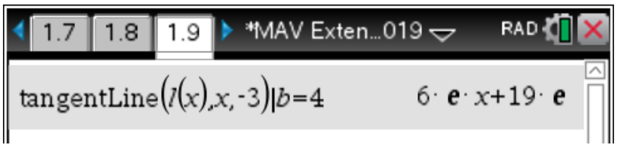
For two solutions $b^2-9>0$.

$b<-3$ or $b>3$ **1A**

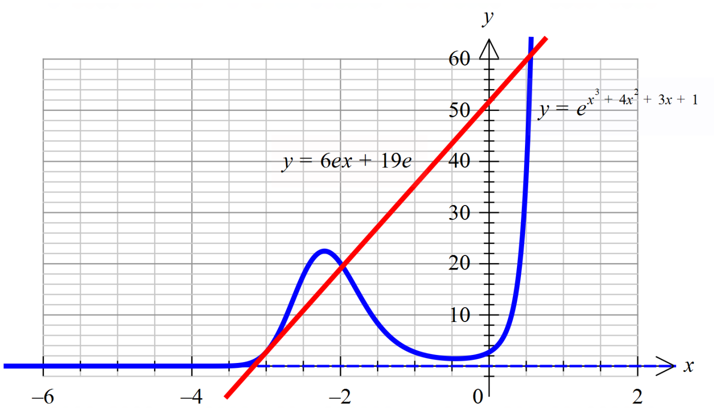


b. $-3\le b\le 3$ **1A**

c. $y=6ex+19e$ **1A**

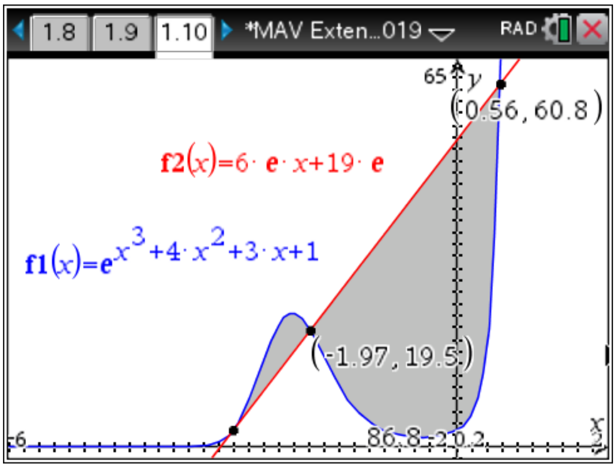


The tangent line drawn correctly on the graph. **1A**



d. $\int_{-3}^{-1.97}\left(l_4(x)-y_t\right)dt+\int_{-1.97}^{0.56}\left(y_t-l_4(x)\right)dt$ **1M**

$=86.8$ **1A**



e. $g : R \to R,\ g(x) = e^x$ and $h : [-1, \infty) \to R,\ h(x) = x^2 + 2x + 1$

$f(x) = g\left(h(x)\right)$ exists because the range of h, $[0,\ \infty)$ is a subset of the domain of g, R. **1A**

f. $f(x) = g\left(h(x)\right) = e^{x^2+2x+1}$

Let $y = e^{x^2+2x+1}$

Inverse swap x and y

$x = e^{y^2+2y+1}$

$y^2 + 2y + 1 = \log_e(x)$ **1M**

$(y+1)^2 = \log_e(x)$

$y = \sqrt{\log_e(x)} - 1$ as the range is $[-1,\ \infty)$

$f^{-1}(x) = \sqrt{\log_e(x)} - 1$ **1M Show that**

g. Let $y = f_a(x) = e^{x^2+2x+1} + a$

Inverse swap x and y

$(y+1)^2 = \log_e(x-a)$

$f_a^{-1}(x) = \sqrt{\log_e(x-a)} - 1$ **1A**

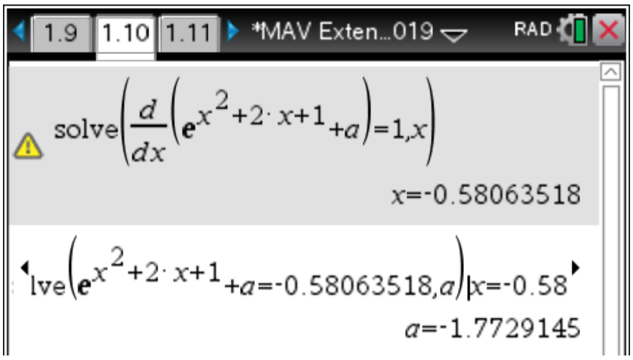
h. The graphs will be tangential when they touch the line $y = x$. The gradient will be equal to 1.

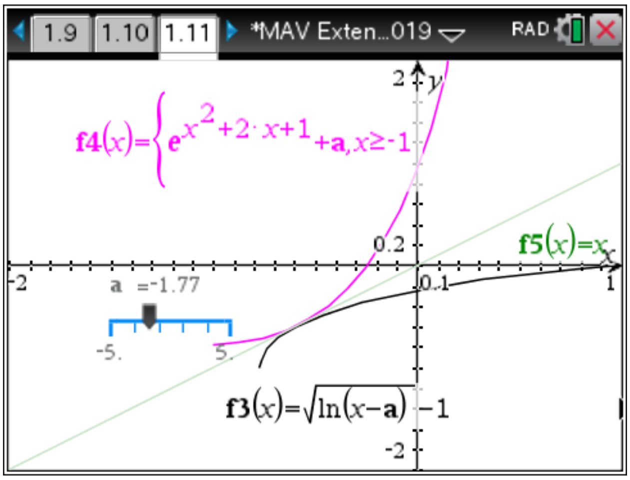
Solve $\dfrac{d}{dx}\left(e^{x^2+2x+1} + a\right) = 1$ for x.

$x = -0.58$ correct to two decimal places **1A**

Solve $e^{(-0.58...)^2+2\times-0.58..+1} + a = -0.58...$ for a. **1M**

$a = -1.77$ correct to two decimal places **1A**



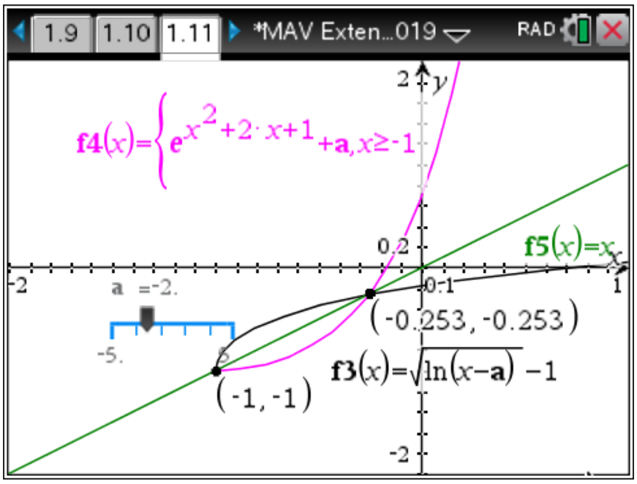


i. Translate the graph of f, 2 units down to get the maximum bounded area.

$a = -2$ **1A**

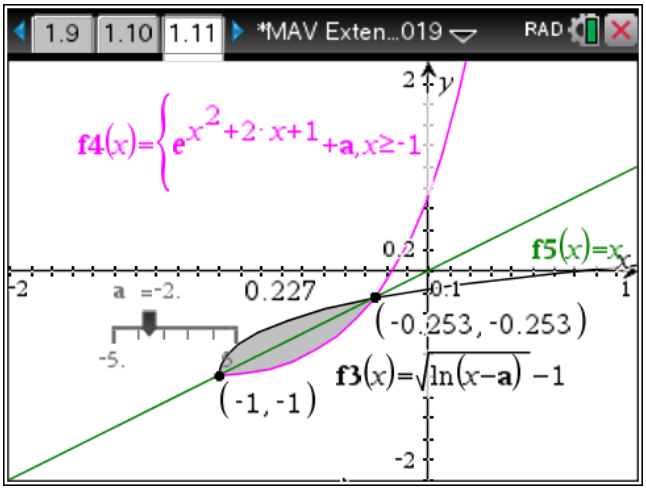
$x = -1$ **1A**

$x = -0.253$ correct to three decimal places **1A**



j. area $= 2\int_{-1}^{-0.253...}\left(x - f_{-2}(x)\right)dx$ **1M**

$= 0.23$ correct to two decimal places **1A**



END OF SOLUTIONS