2019 Mathematical Methods Trial Exam 2 Solutions

SECTION A – Multiple-choice questions

1	2	3	4	5	6	7	8	9	10
C	C	D	A	B	B	C	E	E	A

11	12	13	14	15	16	17	18	19	20
C	E	E	A	A	A	C	B	D	D

Q1 $(b-x)^2 = a^c$, $b-x = \pm a^{\frac{c}{2}}$, $x = b \pm a^{\frac{c}{2}}$ **C**

Q2 $\cos\left(x+\frac{\pi}{4}\right)=0$, $x=-\frac{7\pi}{4},-\frac{3\pi}{4},\frac{\pi}{4},\frac{5\pi}{4}$

or $\tan\left(x-\frac{\pi}{4}\right)=0$, $x=-\frac{7\pi}{4},-\frac{3\pi}{4},\frac{\pi}{4},\frac{5\pi}{4},\frac{9\pi}{4}$ **C**

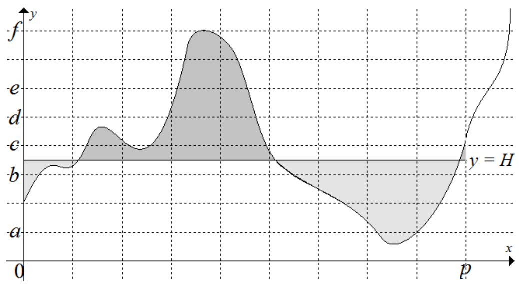
Q3 The domain of the inverse is the range of the function.
$D=\{-3,-2,-1,2\}$ **D**

Q4 $x=a\left(1\pm\frac{1}{y-a}\right)$, $x=a\pm\frac{a}{y-a}$, $\pm(x-a)=\frac{a}{y-a}$

$y-a=\pm\frac{a}{x-a}$, $y=a\left(1\pm\frac{a}{x-a}\right)$ **A**

Q5 **B**

Q6 **B**

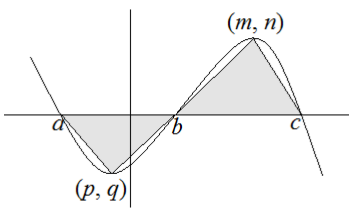


Q7 Average rate $=\frac{c-\frac{a+b}{2}}{p-0}=\frac{2c-a-b}{2p}$ **C**

Q8 $f(a-x)=-f(x-a)$, $f(-x)=-f(x)$
$\therefore$ $f(x)$ is an odd function. **E**

Q9 **E**

Q10 Area $\approx\frac{1}{2}(-q)(b-a)+\frac{1}{2}n(c-b)=\frac{1}{2}(nc-(n+q)b+qa)$ **A**



Q11 Binomial:
$n=18$, $p=\frac{1}{3}$, $\Pr(X=5)+\Pr(X=6)\approx 0.3774$ **C**

Q12 **E**

Q13 $\frac{p(1-p)}{256}=0.025^2$, $p^2-p+0.16=0$, $p=0.20$ **E**

Q14 $\hat{P}$: mean ≈ 0.25, standard dev $\approx\sqrt{\frac{0.25\times0.75}{300}}=0.025$
Proportion of random samples $\approx \Pr(\hat{P}>0.2)\approx 0.9772$ **A**

Q15 $y=f(x)\rightarrow y+b=f(x)\rightarrow -y+b=f(x)\rightarrow -\frac{y}{a}+b=f(x)$
$\therefore$ $y=ab-af(x)$ **A**

Q16 $\sum \Pr=1$, $a^2+0.2a+0.85=1$, $a^2+0.2a-0.15=0$
$\therefore$ $a=0.3$, $\overline{X}=0.50\times1+0.3^2\times2+0.35\times3+0.2\times0.3\times4=1.97$
$\mathrm{Var}(X)=0.50\times1^2+0.3^2\times2^2+0.35\times3^2+0.2\times0.3\times4^2-1.97^2$
$=1.0891$
$\therefore$ $\mathrm{sd}(X)=\sqrt{1.0891}\approx1.0436$ **A**

Q17 $\int_0^a\left(\frac{1}{1+a-x}\right)dx=1$, $[-\log_e(1+a-x)]_0^a=1$, $a=e-1$ **C**

Q18 $D=\sqrt{x^2+y^2}=\sqrt{x^2+\frac{(x-2)^6}{9}}$, min $D\approx0.99$ **B**

Q19 $\frac{dy}{dx}=9x^2+2ax+b^2$
No stationary points, $\Delta=4a^2-36b^2<0$, $-6b<2a<6b$ **D**

Q20 $y=\frac{a}{x-a}+b$, $\therefore$ $(x-a)(y-b)=a$
If $a=b$, the inverse is the function itself, $\therefore$ infinitely many solutions.
For intersections, solve $(x-a)(y-b)=a$ and $y=x$
$\therefore$ $x^2-(a+b)x+(ab-a)=0$, it is a quadratic equation, it cannot have three solutions (intersections)
$\Delta=(a+b)^2-4(ab-a)=(a-b)^2+4a$
No intersections if $\Delta<0$, e.g. $b=0$ and $a=-1$
One intersection if $\Delta=0$, e.g. $b=0$ and $a=-4$
Two intersections if $\Delta>0$, e.g. $b=0$ and $a=1$ **D**

SECTION B

Q1a $y=a(x-2)^2$ and $(3,1)$

.: $a=1$ and $y=(x-2)^2=x^2-4x+4$

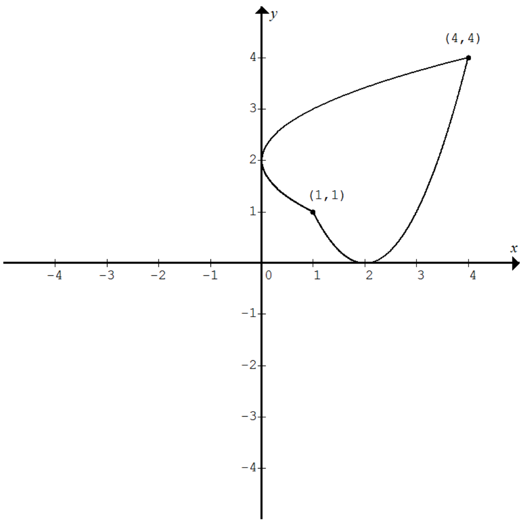
Q1b Inverse $x=(y-2)^2$, $y=2\pm\sqrt{x}$

.: $f(x)=2+\sqrt{x}$ and $g(x)=2-\sqrt{x}$

Q1c $y=x$ and $y=x^2-4x+4$

.: $x=x^2-4x+4$, $x^2-5x+4=0$, $x=1,4$, .: $(1,1)$, $(4,4)$

Q1di



Q1dii Area $=2\times\int_1^4\left(x-(x-2)^2\right)dx=2\times\left[\frac{x^2}{2}-\frac{(x-2)^3}{3}\right]_1^4=9$

Q1e Same as $x=d$ dividing the bounded region.

$\int_d^4\left(2+\sqrt{x}-(x-2)^2\right)dx=\frac{9}{2}$, $d\approx 2.08$

Q1f Maximum length of line segment $y=-x+c$ occurs when its endpoint is the point where the gradient of the parabola is 1.

Let $\frac{dy}{dx}=2(x-2)=1$, .: $x=2.5$ and $y=0.25$, $(2.5,0.25)$.

The other endpoint is $(0.25,2.5)$.

Max length $=\sqrt{(2.5-0.25)^2+(0.25-2.5)^2}=\frac{9\sqrt{2}}{4}$

$y=-x+c$ and $(2.5,0.25)$, .: $c=\frac{11}{4}$

Q2a $\left\{x: x=\frac{n}{\cos 20°}, n=1,2,3,...,22\right\}$

Q2b $a^2=(3+x\cos 20°)^2+(x\sin 20°+1.5-3)^2$

$=9+6x\cos 20°+x^2\cos^2 20°+x^2\sin^2 20°-3x\sin 20°+2.25$

$=(\sin^2 20°+\cos^2 20°)x^2+(6\cos 20°-3\sin 20°)x+11.25$

$=x^2+(6\cos 20°-3\sin 20°)x+11.25$

Q2c $b^2=(3+x\cos 20°)^2+(8+3-1.5-x\sin 20°)^2$

$=9+6x\cos 20°+x^2\cos^2 20°+90.25-19x\sin 20°+x^2\sin^2 20°$

$=(\sin^2 20°+\cos^2 20°)x^2+(6\cos 20°-19\sin 20°)x+99.25$

$=x^2+(6\cos 20°-19\sin 20°)x+99.25$

Q2di $\cos\theta$

$$=\frac{x^2+(6\cos 20°-3\sin 20°)x+11.25+x^2+(6\cos 20°-19\sin 20°)x+99.25-8^2}{2\sqrt{x^2+(6\cos 20°-3\sin 20°)x+11.25}\sqrt{x^2+(6\cos 20°-19\sin 20°)x+99.25}}$$

$$=\frac{x^2+(6\cos 20°-11\sin 20°)x+23.25}{\sqrt{(x^2+(6\cos 20°-3\sin 20°)x+11.25)(x^2+(6\cos 20°-19\sin 20°)x+99.25)}}$$

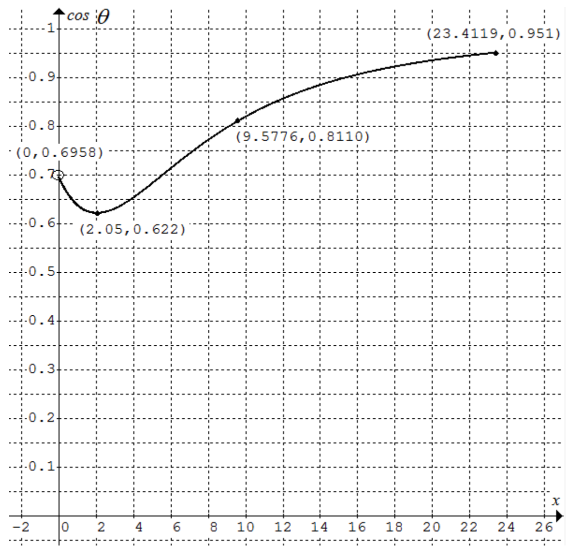
Q2dii $b=6\cos 20°-19\sin 20°+6\cos 20°-3\sin 20°\approx 3.75$

Q2e $n=9$ for the tenth row, $x=\frac{9}{\cos 20°}\approx 9.5776$

.: $\cos\theta\approx\frac{x^2+1.88x+23.25}{\sqrt{x^4+bx^3+106.53x^2+448.07x+1116.56}}\approx 0.8110$

$\theta\approx 36°$

Q2f



Q2g θ is the greatest when $\cos\theta$ is the smallest.

$\cos\theta\approx 0.622$, greatest $\theta\approx 52°$ when $x\approx 2$

Q2h $n\approx x\cos 20°\approx 2\cos 20°\approx 2$, .: the third row.

Q3a $f(x)=e^x-mx=0$ has exactly one solution if the graphs of $y=e^x$ and $y=mx$ intersect at one point only, the point where the two curves have the same gradient, .: $f'(x)=e^x-m=0$, $m=e^x$

.: $e^x-e^x x=0$, $e^x(1-x)=0$, .: $x=1$ and $m=e$.

Q3b $m>e$

Q3c $\int_{x_1}^{x_2}(-f(x))dx=\int_{x_1}^{x_2}(mx-e^x)dx$

Q3d $\left[\frac{mx^2}{2}-e^x\right]_{x_1}^{x_2}=\frac{mx_2^{\,2}}{2}-\frac{mx_1^{\,2}}{2}-e^{x_2}+e^{x_1}$

Q3e $\frac{mx_2^{\,2}}{2}-\frac{mx_1^{\,2}}{2}-e^{x_2}+e^{x_1}>0$, $e^{x_1}-mx_1=0$ and $e^{x_2}-mx_2=0$

$\frac{m(x_2^{\,2}-x_1^{\,2})}{2}-e^{x_2}+e^{x_1}>0$, $\frac{m(x_2-x_1)(x_2+x_1)}{2}-e^{x_2}+e^{x_1}>0$

$\frac{(e^{x_2}-e^{x_1})(x_2+x_1)}{2}-(e^{x_2}-e^{x_1})>0$, $(e^{x_2}-e^{x_1})\left(\frac{(x_2+x_1)}{2}-1\right)>0$

Since $e^{x_2}-e^{x_1}>0$, .: $\frac{(x_2+x_1)}{2}-1>0$, .: $x_1+x_2>2$

Q3f $g(x)=\log_e x-nx^2=0$ has exactly one solution if the graphs of $y=\log_e x$ and $y=nx^2$ intersect at one point only, where $x=a$, the point where the two curves have the same gradient

.: $f'(a)=\frac{1}{a}-2na=0$ and $g(a)=\log_e a-na^2=0$

.: $na^2=\frac{1}{2}$, $\log_e a=\frac{1}{2}$.: $a=\sqrt{e}$ and $n=\frac{1}{2e}$.

Q3g $0<n<\frac{1}{2e}$

Q3h $g(b)=\log_e b-nb^2=0$, $\log_e b=nb^2$

Since $n<0$ and $b^2>0$, .: $nb^2<0$

.: $\log_e b<0$ and $0<b<1$

Q4a $\tan\theta_1=\frac{30}{25}$, $\tan\theta_2=\frac{30}{20}$, $\tan(\theta_1+\theta_2)=\frac{\frac{30}{25}+\frac{30}{20}}{1-\frac{30}{25}\times\frac{30}{20}}=\frac{-27}{8}$

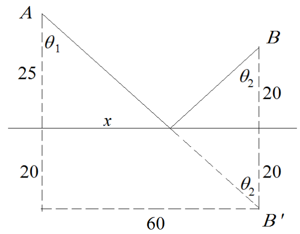
.: $\theta_1+\theta_2=\tan^{-1}\left(\frac{-27}{8}\right)$

Q4b $\tan\theta_1=\frac{x}{25}$, $\tan\theta_2=\frac{60-x}{20}$, $\tan(\theta_1+\theta_2)=\frac{\frac{x}{25}+\frac{60-x}{20}}{1-\frac{x}{25}\times\frac{60-x}{20}}$

When $\theta_1+\theta_2=90°$, $1-\frac{x}{25}\times\frac{60-x}{20}=0$, $500-x(60-x)=0$

.: $x=10, 50$

Q4c When $\theta_1=\theta_2$, AB' is a straight line and it is the shortest.



Q4d $\theta_1=\tan^{-1}\left(\frac{60}{25+20}\right)=\tan^{-1}\left(\frac{4}{3}\right)$

Q4e Minimum total length $=\sqrt{(25+20)^2+60^2}=75$ m

Q5a $k\left(\int_0^1 e^{-2}x\,dx+\int_1^9 e^{-\frac{(x-5)^2}{8}}dx-\int_9^{10}e^{-2}(x-10)dx\right)=1$

$k(0.06766764+4.7851521+0.06766764)=1$, $k\approx 0.203232$

Q5b $\Pr(X<6\mid X>1)=\frac{\Pr(1<X<6)}{\Pr(X>1)}$

$\approx\frac{3.3524265k}{(4.7851521+0.06766764)k}\approx 0.6908$

Q5c $p=\Pr(X>6)=\Pr(6<X\le 9)+\Pr(9<X\le 10)$

$\approx k(1.4327256+0.06766764)\approx 0.3049$

Q5d Mean$(\hat{P})\approx p\approx 0.3049$, sd$(\hat{P})\approx\sqrt{\frac{0.3049(1-0.3049)}{100}}\approx 0.0460$

Q5e $\Pr(\hat{P}<0.4)\approx 0.9807$ (Normal: $\mu=0.3049, \sigma=0.0460$)

Q5f $\Pr(X>c)=\Pr(c<X\le 9)+\Pr(9<X\le 10)=p=0.3$

$\Pr(c<X\le 9)+0.06766764k\approx 0.3$, $\Pr(c<X\le 9)\approx 0.28625$

$\int_c^9 0.203232e^{-\frac{(x-5)^2}{8}}dx\approx 0.28625$, $c\approx 6.03$

Q5g $\sqrt{\frac{0.3(1-0.3)}{n}}\le 0.01$, $n\ge 2100$

Q5h For $Y<8$, $p\approx\hat{p}=\frac{400-100}{400}=0.75$

Q5i $\sqrt{\frac{0.75(1-0.75)}{400}}\approx 0.02$

Q5j For 70%, $z=\left|\text{invNorm}\left(\frac{1-0.70}{2}\right)\right|=|\text{invNorm}(0.15)|\approx 1.04$

70% interval $\approx(0.75-1.04\times 0.02, 0.75+1.04\times 0.02)\approx(0.73, 0.77)$

Please inform mathline@itute.com re conceptual and/or mathematical errors