
Question 1 (3 marks)

a. $y = \log_e(4 - 2x^3)$

$$\begin{aligned}\frac{dy}{dx} &= \frac{-6x^2}{4 - 2x^3} \\ &= \frac{-6x^2}{2(2 - x^3)} \\ &= \frac{-3x^2}{2 - x^3}\end{aligned}$$

(1 mark)

b. $f(x) = \frac{e^x}{\tan(x)}$

$$f'(x) = \frac{\tan(x) \times e^x - \sec^2(x) \times e^x}{\tan^2(x)}$$

(1 mark)

$$\begin{aligned}f'\left(\frac{\pi}{4}\right) &= \frac{\tan\left(\frac{\pi}{4}\right)e^{\left(\frac{\pi}{4}\right)} - \sec^2\left(\frac{\pi}{4}\right)e^{\left(\frac{\pi}{4}\right)}}{\tan^2\left(\frac{\pi}{4}\right)} \\ &= e^{\left(\frac{\pi}{4}\right)} - 2 \times e^{\left(\frac{\pi}{4}\right)} \\ &= -e^{\left(\frac{\pi}{4}\right)}\end{aligned}$$

Now $\tan\left(\frac{\pi}{4}\right) = 1$, so $\tan^2\left(\frac{\pi}{4}\right) = 1$

$$\begin{aligned}\text{and } \sec^2\left(\frac{\pi}{4}\right) &= \frac{1}{\cos^2\left(\frac{\pi}{4}\right)} \\ &= \frac{1}{\left(\frac{1}{\sqrt{2}}\right)^2} \\ &= \frac{1}{1/2} \\ &= 2\end{aligned}$$

(1 mark)

Question 2 (3 marks)

$$g'(x) = \frac{1}{3x-1} - \frac{1}{4}, \quad x > \frac{1}{3}$$

$$\begin{aligned} g(x) &= \int \left(\frac{1}{3x-1} - \frac{1}{4} \right) dx \\ &= \frac{1}{3} \log_e(3x-1) - \frac{x}{4} + c \end{aligned} \quad \text{(1 mark)}$$

Since $g(1) = \frac{3}{4}$,

$$\begin{aligned} \frac{3}{4} &= \frac{1}{3} \log_e(2) - \frac{1}{4} + c \\ c &= 1 - \frac{1}{3} \log_e(2) \end{aligned} \quad \text{(1 mark)}$$

So $g(x) = \frac{1}{3} \log_e(3x-1) - \frac{x}{4} + 1 - \frac{1}{3} \log_e(2)$ (1 mark)

Question 3 (2 marks)

$$\left(\cos^2(\theta) - \frac{1}{2} \right) \left(\sin(\theta) - \frac{1}{2} \right) = 0, \quad 0 \leq \theta \leq \pi$$

$$\left(\cos(\theta) - \frac{1}{\sqrt{2}} \right) \left(\cos(\theta) + \frac{1}{\sqrt{2}} \right) \left(\sin(\theta) - \frac{1}{2} \right) = 0$$

$$\begin{aligned} \cos(\theta) &= \frac{1}{\sqrt{2}}, \quad \text{or} \quad \cos(\theta) = -\frac{1}{\sqrt{2}}, \quad \text{or} \quad \sin(\theta) = \frac{1}{2} \\ \theta &= \frac{\pi}{4}, \quad \theta = \frac{3\pi}{4}, \quad \theta = \frac{\pi}{6}, \frac{5\pi}{6} \end{aligned}$$



(1 mark) for solutions from first bracket
(1 mark) for solutions from second bracket

Question 4 (2 marks)

$$\text{sd}(\hat{P}) = \sqrt{\frac{p(1-p)}{n}}$$

We require $\sqrt{\frac{p(1-p)}{n}} \geq 0.04$

$$\sqrt{\frac{0.2 \times 0.8}{n}} \geq 0.04$$

(1 mark)

$$\frac{0.16}{n} \geq 0.0016$$

$$0.16 \geq 0.0016n \quad (n > 0)$$

$$n \leq \frac{0.16}{0.0016}$$

$$n \leq \frac{1600}{16}$$

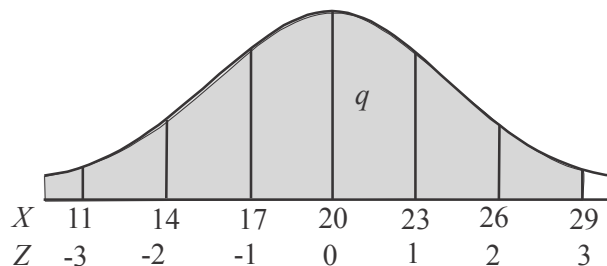
$$n \leq 100$$

So the largest sample size, i.e. the largest value of n is 100.

(1 mark)**Question 5** (3 marks)

- a. $\mu(X) = 20$, $\text{var}(X) = 9$ so $\text{sd}(X) = 3$

Draw a diagram. Note that the shaded area represents $\Pr(X < 29)$ which equals q .



$$\begin{aligned} \Pr(Z < -3) &= \Pr(Z > 3) \\ &= \Pr(X > 29) \\ &= 1 - q \end{aligned}$$

(1 mark)

- b. Method 1 – using the diagram

The sample space is reduced to $Z < 3$ i.e. $X < 29$ i.e. q . Also $\Pr(Z < 0) = 0.5$

So, $\Pr(Z > 0 | Z < 3)$

$$= \frac{q - 0.5}{q}$$

(1 mark) for numerator **(1 mark)** for denominator

Method 2 – using the formula

$$\Pr(Z > 0 | Z < 3)$$

$$= \frac{\Pr(Z > 0 \cap Z < 3)}{\Pr(Z < 3)}$$

$$= \frac{\Pr(0 < Z < 3)}{\Pr(Z < 3)}$$

(1 mark)

$$= \frac{q - 0.5}{q}$$

(1 mark)

Question 6 (6 marks)a. horizontal asymptote: $y = 1$ vertical asymptote: $x = 1$ x-intercepts occur when $y = 0$

$$0 = 1 + \frac{2}{x-1}$$

$$-1 = \frac{2}{x-1}$$

$$-1(x-1) = 2$$

$$-x + 1 = 2$$

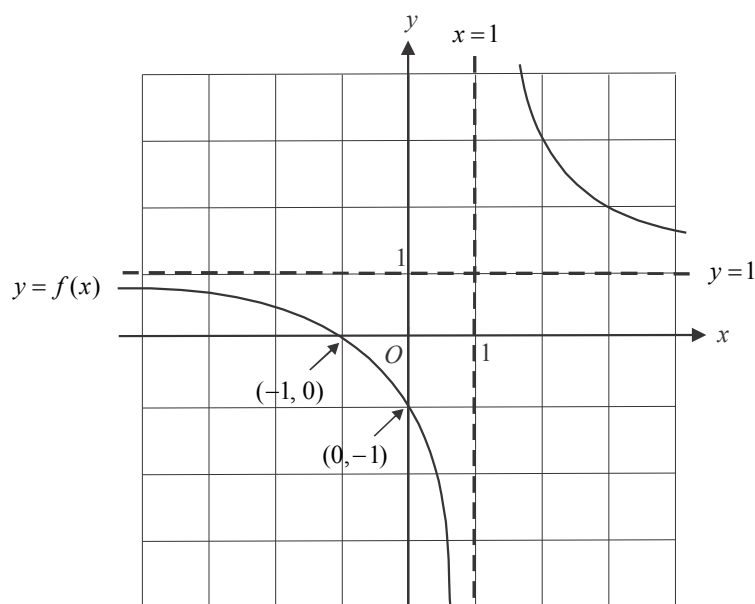
$$x = -1$$

y-intercepts occur when $x = 0$

$$y = 1 + \frac{2}{0-1}$$

$$y = 1 - 2$$

$$y = -1$$

**(1 mark)** – correct intercepts**(1 mark)** – correct asymptotes**(1 mark)** – correct shapes

b.
$$f(x) = 1 + \frac{2}{x-1}$$

Let
$$y = 1 + \frac{2}{x-1}$$

Swap x and y for inverse.

$$x = 1 + \frac{2}{y-1}$$

$$x - 1 = \frac{2}{y-1}$$

$$y - 1 = \frac{2}{x-1}$$

$$y = 1 + \frac{2}{x-1}$$

$$f^{-1}(x) = 1 + \frac{2}{x-1}$$

(1 mark)

$$d_f = \mathbb{R} \setminus \{1\} \quad \text{and} \quad r_f = \mathbb{R} \setminus \{1\}$$

So $d_{f^{-1}} = \mathbb{R} \setminus \{1\}$

(1 mark)c. Note that f and f^{-1} are self-inverses so $f(x) = f^{-1}(x)$ for $x \in \mathbb{R} \setminus \{1\}$.**(1 mark)**

Question 7 (5 marks)

- a. Set up a table showing the discrete distribution.

x	0	1	2	3
$\Pr(X=x)$	k	k	$4k$	$9k$

(1 mark)

$$k + k + 4k + 9k = 1$$

$$15k = 1$$

$$k = \frac{1}{15}$$

(1 mark)

- b.
- $E(X) = \frac{12}{5}$
- (given)

$$E(X^2) = 0^2 \times \frac{1}{15} + 1^2 \times \frac{1}{15} + 2^2 \times \frac{4}{15} + 3^2 \times \frac{9}{15}$$

$$= \frac{1}{15} + \frac{16}{15} + \frac{81}{15}$$

$$= \frac{98}{15}$$

(1 mark)

$$\text{var}(X) = E(X^2) - \{E(X)\}^2$$

$$= \frac{98}{15} - \left(\frac{12}{5}\right)^2$$

$$= \frac{98}{15} - \frac{144}{25}$$

$$= \frac{980 - 864}{150}$$

$$= \frac{116}{150}$$

$$= \frac{58}{75}$$

(1 mark)

- c.
- Method 1
- using the table

x	0	1	2	3
$\Pr(X=x)$	$\frac{1}{15}$	$\frac{1}{15}$	$\frac{4}{15}$	$\frac{9}{15}$

$$\Pr(X \leq 2 | X > 0)$$

$$= \left(\frac{1}{15} + \frac{4}{15}\right) \div \left(\frac{1}{15} + \frac{4}{15} + \frac{9}{15}\right)$$

$$= \frac{5}{15} \div \frac{14}{15}$$

$$= \frac{5}{15} \times \frac{15}{14}$$

$$= \frac{5}{14}$$

(1 mark)

- Method 2
- using the formula

$$\Pr(X \leq 2 | X > 0)$$

$$= \frac{\Pr(X \leq 2 \cap X > 0)}{\Pr(X > 0)} \quad (\text{formula sheet})$$

$$= \frac{\Pr(X=1) + \Pr(X=2)}{\Pr(X > 0)}$$

$$= \left(\frac{1}{15} + \frac{4}{15}\right) \div \left(\frac{1}{15} + \frac{4}{15} + \frac{9}{15}\right)$$

$$= \frac{5}{15} \times \frac{15}{14}$$

$$= \frac{5}{14}$$

(1 mark)

Question 8 (5 marks)

- a. We have $P(x, y)$ i.e. $P(x, \sqrt{x})$ and $Q(1, 0)$.

Let D = the distance from P to Q .

$$D = \sqrt{(x-1)^2 + (\sqrt{x}-0)^2} \quad (\text{distance formula})$$

$$= \sqrt{x^2 - 2x + 1 + x}$$

$$= \sqrt{x^2 - x + 1}$$

$$= (x^2 - x + 1)^{\frac{1}{2}}$$

(1 mark)

$$\frac{dD}{dx} = \frac{1}{2} (x^2 - x + 1)^{-\frac{1}{2}} \times (2x - 1) \quad (\text{chain rule})$$

$$= \frac{2x - 1}{2\sqrt{x^2 - x + 1}}$$

$$\frac{dD}{dx} = 0 \text{ for min/max.}$$

$$\frac{2x - 1}{2\sqrt{x^2 - x + 1}} = 0$$

So $2x - 1 = 0$ (Note, if the denominator equals zero then $\frac{dD}{dx}$ is undefined.)

$$x = \frac{1}{2}$$

From the graph we see that we must have a minimum rather than a maximum at

$$x = \frac{1}{2}.$$

(1 mark)

Substitute $x = \frac{1}{2}$ into

$$D = \sqrt{x^2 - x + 1}$$

$$= \sqrt{\frac{1}{4} - \frac{1}{2} + 1}$$

$$= \sqrt{\frac{1}{4} - \frac{2}{4} + \frac{4}{4}}$$

$$= \sqrt{\frac{3}{4}}$$

$$= \frac{\sqrt{3}}{2}$$

Minimum distance is $\frac{\sqrt{3}}{2}$ units.

(1 mark)

- b. Let an image point be (x', y') .

$$\begin{aligned}\begin{bmatrix} x' \\ y' \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} x \\ -2y \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} x \\ -2y + 2 \end{bmatrix}\end{aligned}$$

$$\text{So } x' = x \quad \text{and} \quad y' = -2y + 2$$

$$x = x' \quad 2y = 2 - y'$$

$$y = \frac{2 - y'}{2}$$

$$\text{Since } f(x) = \sqrt{x}$$

$$\text{let } y = \sqrt{x}$$

The image equation is

$$\frac{2 - y'}{2} = \sqrt{x'}$$

$$2 - y' = 2\sqrt{x'}$$

$$-y' = -2 + 2\sqrt{x'}$$

$$y' = 2 - 2\sqrt{x'}$$

(1 mark)

$$\text{So } h(x) = 2 - 2\sqrt{x}$$

Q is the point $(1, 0)$.

$$h(1) = 2 - 2\sqrt{1}$$

$$= 0$$

So Q lies on h .

(1 mark)

Question 9 (5 marks)

a. $f(x) = 2x \cos(x), \quad x \geq 0$

$$\frac{d}{dx} (2x \sin(x)) = f(x) + 2 \sin(x)$$

$$\begin{aligned} LS &= \frac{d}{dx} (2x \sin(x)) \\ &= 2 \sin(x) + 2x \cos(x) \quad (\text{product rule}) \\ &= 2 \sin(x) + f(x) \\ &= RS \end{aligned}$$

as required.

(1 mark)

b. $\int_0^{n\pi} f(x) dx$

$$\begin{aligned} &= \int_0^{n\pi} \frac{d}{dx} (2x \sin(x)) dx - \int_0^{n\pi} 2 \sin(x) dx \quad (\text{using part a.}) \\ &= [2x \sin(x)]_0^{n\pi} + [2 \cos(x)]_0^{n\pi} \\ &= (2n\pi \sin(n\pi) - 0) + (2 \cos(n\pi) - 2 \cos(0)) \\ &= 2n\pi \times 0 + (2 \times 1 - 2 \times 1) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

(1 mark)

(1 mark)

Note that since n is a positive, even integer,
 $\sin(n\pi) = 0$ and $\cos(n\pi) = 1$. Also, $\sin(0) = 0$ and $\cos(0) = 1$.

c. Method 1 – using part b.

From part b., $\int_0^{n\pi} f(x) dx = 0$ where $n = 2, 4, 6, \dots$

So, $\int_0^{2\pi} f(x) dx = 0$

From the graph we have $\int_0^a f(x) dx + \int_a^{2\pi} f(x) dx = 0$ (1 mark)

$$\begin{aligned} &-(3\pi + 2) + \int_a^{2\pi} f(x) dx = 0 \\ &\int_a^{2\pi} f(x) dx = 3\pi + 2 \end{aligned}$$

Required area is $3\pi + 2$ square units.

(1 mark)

Method 2 – “otherwise”

Find a by solving $f(x) = 0$, for $x \geq 0$

$$2x \cos(x) = 0$$

$$x = 0 \text{ or } \cos(x) = 0$$

$$x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2} \dots$$

From the graph, we see that $a = \frac{3\pi}{2}$.

$$\text{shaded area} = \int_{\frac{3\pi}{2}}^{2\pi} f(x) dx \quad (1 \text{ mark})$$

$$= \left[2x \sin(x) \right]_{\frac{3\pi}{2}}^{2\pi} + \left[2 \cos(x) \right]_{\frac{3\pi}{2}}^{2\pi} \quad (\text{using part b.'s solution})$$

$$= \left(4\pi \sin(2\pi) - 3\pi \sin\left(\frac{3\pi}{2}\right) \right) + \left(2 \cos(2\pi) - 2 \cos\left(\frac{3\pi}{2}\right) \right)$$

$$= 0 - 3\pi \times -1 + 2 \times 1 - 2 \times 0$$

$$= 3\pi + 2$$

Required area is $3\pi + 2$ square units.

(1 mark)

Question 10 (6 marks)

a. $f(x) - g(x) = 1$

$$2 \log_e(x) - \log_e\left(x - \frac{e}{4}\right) = 1$$

$$\log_e(x^2) - \log_e\left(x - \frac{e}{4}\right) = 1$$

$$\log_e\left(\frac{x^2}{x - \frac{e}{4}}\right) = 1$$

$$e^1 = \frac{x^2}{x - \frac{e}{4}}$$

(1 mark)

$$e\left(x - \frac{e}{4}\right) = x^2$$

$$ex - \frac{e^2}{4} = x^2$$

$$x^2 - ex + \frac{e^2}{4} = 0$$

$$\left(x - \frac{e}{2}\right)\left(x - \frac{e}{2}\right) = 0$$

$$x = \frac{e}{2}$$

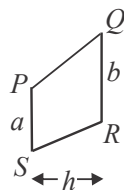
as required

(1 mark)

b. $A = \int_{\frac{e}{2}}^{\frac{e(e+1)}{4}} (f(x) - g(x)) dx$

(1 mark)

c.



area of a trapezium

$$= \frac{1}{2}(a+b) \times h \quad (\text{where } a \text{ and } b \text{ are the parallel sides ie } PS \text{ and } QR)$$

From part a., when $x = \frac{e}{2}$, $f(x) - g(x) = 1$, so $d(PS) = 1$.

The y-coordinate of Q is $f\left(\frac{e(e+1)}{4}\right) = 2\log_e\left(\frac{e(e+1)}{4}\right)$.

The y-coordinate of R is $g\left(\frac{e(e+1)}{4}\right) = \log_e\left(\frac{e(e+1)}{4} - \frac{e}{4}\right)$

$$= \log_e\left(\frac{e^2 + e - e}{4}\right)$$

$$= \log_e\left(\frac{e^2}{4}\right)$$

(1 mark)

$$d(QR) = 2\log_e\left(\frac{e(e+1)}{4}\right) - \log_e\left(\frac{e^2}{4}\right)$$

$$= \log_e\left(\frac{e^2(e+1)^2}{16}\right) - \log_e\left(\frac{e^2}{4}\right)$$

$$= \log_e\left(\frac{e^2(e+1)^2}{16} \div \frac{e^2}{4}\right)$$

$$= \log_e\left(\frac{e^2(e+1)^2}{16} \times \frac{4}{e^2}\right)$$

$$= \log_e\left(\frac{(e+1)^2}{4}\right)$$

$$\text{area of trapezium} = \frac{1}{2}(a+b) \times h$$

$$= \frac{1}{2}(d(PS) + d(QR)) \times \left(\frac{e(e+1)}{4} - \frac{e}{2}\right)$$

$$= \frac{1}{2}\left(1 + \log_e\left(\frac{(e+1)^2}{4}\right)\right) \times \left(\frac{e(e+1)}{4} - \frac{e}{2}\right)$$

$$= \frac{1}{2}\left(\frac{e^2 + e - 2e}{4}\right) \times \left(1 + \log_e\left(\frac{(e+1)^2}{4}\right)\right)$$

$$= \left(\frac{e^2 - e}{8}\right) \times \left(1 + \log_e\left(\frac{(e+1)^2}{4}\right)\right)$$

(1 mark)

$$\text{So } c = \frac{e^2 - e}{8} \text{ and } d = \frac{(e+1)^2}{4}.$$

(1 mark)