



THE SCHOOL FOR EXCELLENCE (TSFX)
UNITS 3 & 4 MATHEMATICAL METHODS 2018
WRITTEN EXAMINATION 1 – SOLUTIONS

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Marking Legend:

- $\left(A \frac{1}{2} \times 4 \downarrow \right)$ means four answer half-marks rounded **down** to the next integer.
Rounding occurs at the end of a part of a question.
- **M1** = 1 Method mark.
- **A1** = 1 Answer mark (it **must** be this or its equivalent).
- **H1** = 1 consequential mark (**His/Her** mark...correct answer from incorrect statement or slip, arithmetic slip preventing an **A** mark).

QUESTION 1

a.
$$f'(x) = -3e^{-3x} \cos(2x) - 2e^{-3x} \sin(2x)$$
$$= -e^{-3x} (3\cos 2x + 2\sin 2x)$$

M1 Product Rule + A1

b.
$$f'\left(\frac{\pi}{3}\right) = -e^{-\pi} \left(3\cos\left(\frac{2\pi}{3}\right) + 2\sin\left(\frac{2\pi}{3}\right) \right)$$

$$= -e^{-\pi} \left(3\left(-\frac{1}{2}\right) + 2 \times \frac{\sqrt{3}}{2} \right)$$

A1 for 1 correct trig ratios

$$= -e^{-\pi} \left(-\frac{3}{2} + \sqrt{3} \right)$$

$$= e^{-\pi} \left(\frac{3}{2} - \sqrt{3} \right)$$

Hence $a = \frac{3}{2}$ and $b = -1$

A1 (both)

QUESTION 2

a. $\int \frac{1}{(3x+2)^4} - e^{1-2x} dx = \int (3x+2)^{-4} - e^{1-2x} dx$

Now $\int (ax+b)^n dx = \frac{1}{a(n+1)}(ax+b)^{n+1} + c, \quad n \neq -1$

$\therefore \frac{1}{3(-4+1)}(3x+2)^{-3} - \frac{e^{1-2x}}{-2} + c$

Answer: $-\frac{1}{9}(3x+2)^{-3} + \frac{e^{1-2x}}{2} + c$

A2 (must have +c)

b. $\int_0^{\frac{\pi}{6}} \left(\sin\left(\frac{3\pi}{2} + x\right) + \cos\left(\frac{\pi}{2} - x\right) \right) dx = \int_0^{\frac{\pi}{6}} (-\cos(x) + \sin(x)) dx$

$= [-\sin x - \cos x]_0^{\frac{\pi}{6}}$ **M1**

$= \left(-\sin\left(\frac{\pi}{6}\right) - \cos\left(\frac{\pi}{6}\right) \right) - (-\sin(0) - \cos(0))$

$= \left(-\frac{1}{2} - \frac{\sqrt{3}}{2} \right) - (0 - 1)$

$= \frac{1}{2} - \frac{\sqrt{3}}{2}$ **A1**

Alternatively:

$$\begin{aligned} \int_0^{\frac{\pi}{6}} \left(\sin\left(\frac{3\pi}{2} + x\right) + \cos\left(\frac{\pi}{2} - x\right) \right) dx &= \left[-\cos\left(\frac{3\pi}{2} + x\right) - \sin\left(\frac{\pi}{2} - x\right) \right]_0^{\frac{\pi}{6}} \\ &= \left(-\cos\left(-\frac{\pi}{3}\right) - \sin\left(\frac{\pi}{3}\right) \right) - \left(-\cos\left(\frac{3\pi}{2}\right) - \sin\left(\frac{\pi}{2}\right) \right) \\ &= -\frac{1}{2} - \frac{\sqrt{3}}{2} - (0 - 1) \\ &= \frac{1 - \sqrt{3}}{2} \end{aligned}$$

QUESTION 3

a. $g(x) = x^2 - 1$

Let $y = x^2 - 1$. Interchange x and y :

$$x = y^2 - 1$$

$$y^2 = x + 1$$

$$y = \pm\sqrt{x+1}$$

$$\text{As } d_g = R^+ = r_{g^{-1}} \quad \therefore g^{-1}(x) = \sqrt{x+1}$$

$$g^{-1} : (-1, \infty) \rightarrow R, \quad g^{-1}(x) = \sqrt{x+1}$$

A1 for domain, A1 for rule

b. For $g^{-1}(f(x))$ to exist, $r_f \subseteq d_{g^{-1}}$ **M1**

$R \not\subset (-1, \infty)$ therefore, $g^{-1}(f(x))$ does not exist. **M1**

c. For $g^{-1}(f(x))$ to exist, $r_f = d_{g^{-1}}$

$$r_f = (-1, \infty)$$

M1

Find x when $f(x) = -1$

$$\log_e x = -1$$

$$x = e^{-1} = \frac{1}{e}$$

$$\therefore k = \frac{1}{e}$$

A1

d. $g^{-1}(f(x)) = \sqrt{\log_e x + 1}, \quad x \in \left(\frac{1}{e}, \infty\right)$

A1 for domain, A1 for rule

QUESTION 4

a. $M_A = 500e^{-0.05t}$
 $M_B = 100e^{-kt}$

Let $M_A = M_B$

$$100e^{-kt} = 500e^{-0.05t}$$

M1

$$\frac{e^{-kt}}{e^{-0.05t}} = \frac{500}{100}$$

$$e^{-kt+0.05t} = 5$$

$$\log_e e^{-kt+0.05t} = \log_e 5$$

$$-kt + 0.05t = \log_e 5$$

$$t(0.05 - k) = \log_e 5$$

$$t = \frac{\log_e 5}{0.05 - k}$$

M1

b. $M(t) = M_0 e^{-kt}$

When half the mass disappears, $\frac{M}{M_0} = \frac{1}{2}$ and $t = 1590$.

Therefore, $\frac{M}{M_0} = e^{-kt}$

$$\frac{1}{2} = e^{-1590k}$$

M1

$$\log_e \left(\frac{1}{2} \right) = \log_e e^{-1590k}$$

$$\log_e \left(\frac{1}{2} \right) = -1590k$$

$$-\log_e (2) = -1590k$$

$$k = \frac{\log_e 2}{1590}$$

A1

QUESTION 5

$$3f'(x) + f(x) = \pi$$

$$3\cos\left(\frac{x}{3}\right) + 3\sin\left(\frac{x}{3}\right) + \pi = \pi$$

$$3\sin\left(\frac{x}{3}\right) = -3\cos\left(\frac{x}{3}\right)$$

H1

Dividing both sides by $3\cos\left(\frac{x}{3}\right)$ gives $\tan\left(\frac{x}{3}\right) = -1$

$$\text{If } 0 \leq x \leq 9\pi \text{ then } 0 \leq \frac{x}{3} \leq 3\pi$$

Tan is negative in the 2nd and 4th quadrants: $\frac{x}{3} = \frac{3\pi}{4}, \frac{7\pi}{4}, 2\pi + \frac{3\pi}{4}$

$$\text{Answer: } x = \frac{9\pi}{4}, \frac{21\pi}{4}, \frac{33\pi}{4}$$

A1(1 correct) A2(all correct) (any extra solutions- minus 1)

QUESTION 6

a. $f(x) = 4x - x^2$

Reflect in the X axis: $f(x) = -4x - x^2$

$$\begin{aligned} &= -(x^2 + 4x) \\ &= -(x^2 + 4x + 4 - 4) \\ &= -((x+2)^2 - 4) \\ &= -(x+2)^2 + 4 \end{aligned}$$

$$\begin{aligned} g(x) &= 9 - 6x - 3x^2 \\ &= -3(x^2 + 2x - 3) \\ &= -3(x^2 + 2x + 1 - 1 - 3) \\ &= -3((x+1)^2 - 4) \\ &= -3(x+1)^2 + 12 \end{aligned}$$

Reflection in the y-axis.

Dilation by a factor of 3 from the x-axis.

Translation 1 unit to the right.

Order must have translation last.

(none – given)

A1

A1

A1

b.
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$
 A2

c.
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -x \\ 3y \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1-x \\ 3y \end{bmatrix}$$

$$\begin{aligned} x' &= 1-x & y' &= 3y \\ \therefore x &= 1-x' & \therefore y &= \frac{y'}{3} \end{aligned}$$

$$y = e^{x+1}$$

$$\frac{y'}{3} = e^{(1-x')+1}$$

$$y' = 3e^{(2-x')}$$

$$y = 3e^{(2-x)} \quad \text{A1}$$

QUESTION 7

a. Values of $\hat{p} = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$ **A1**

b. If the probability of the proportion of red marbles is greater than 0.5 then there must be either 3 red marbles or 4 red marbles in the sample of 4. **M1**

$$\Pr(3 \text{ red}) = \Pr(\text{RRRB or RRBR or RBRR or BRRR}) = 4 \times \left(\frac{4}{12} \times \frac{3}{11} \times \frac{2}{10} \times \frac{8}{9} \right) = \frac{64}{990}$$

$$\Pr(4 \text{ red}) = \Pr(\text{RRRR}) = \frac{4}{12} \times \frac{3}{11} \times \frac{2}{10} \times \frac{1}{9} = \frac{2}{990} \quad \text{M1 For 1 correct probability}$$

So this makes a probability of $\frac{66}{990} = \frac{1}{15}$.

Answer: $\frac{1}{15}$ **A1**

QUESTION 8

a. (a, b) lies on the curve $f(x) = \log_e \left(\frac{x}{3} \right)$

$$\therefore b = \log_e \left(\frac{a}{3} \right)$$

$$e^b = \frac{a}{3} \text{ and therefore, } a = 3e^b \quad \text{A1}$$

b. If $f(x) = \log_e \left(\frac{x}{3} \right)$ then $f'(x) = \frac{1}{x}$ A1

As (a, b) lies on the tangent:

$$y - b = \frac{1}{a}(x - a) \quad \text{A1}$$

$$y = \frac{x}{a} + b - 1$$

$$ay - ab + a = x$$

As tangent passes through the origin:

$$-ab + a = 0$$

$$b = \frac{a}{a} = 1$$

Therefore, $ay - ab + a = x$

$$ay - a + a = x$$

$$x = ay$$

M1

QUESTION 9

$$Area = \int_{-6}^{-3} (x+5-(-1)) dx + \int_{-3}^0 (2-(-1)) dx + \int_0^4 (2-\sqrt{x}) dx + \int_0^1 (-\sqrt{x}-(-1)) dx$$

$$Area = \int_{-6}^{-3} (x+6) dx + \int_{-3}^0 (3) dx + \int_0^4 (2-\sqrt{x}) dx + \int_0^1 (1-\sqrt{x}) dx$$

OR

$$Area = \text{Trapezium} + \int_0^4 (2-\sqrt{x}) dx + \int_0^1 (1-\sqrt{x}) dx$$

$$Area = \frac{1}{2}(3+6)3 + \int_0^4 (2-\sqrt{x}) dx + \int_0^1 (1-\sqrt{x}) dx$$

$$= \frac{27}{2} + \left[2x - \frac{2x^{3/2}}{3} \right]_0^4 + \left[x - \frac{2x^{3/2}}{3} \right]_0^1$$

$$= \frac{27}{2} + \left(2(4) - \frac{2(4)^{3/2}}{3} \right) + \left(1 - \frac{2(1)^{3/2}}{3} \right)$$

$$= \frac{27}{2} + 8 - \frac{16}{3} + \frac{1}{3}$$

$$= \frac{81+48-32+2}{6}$$

$$= \frac{99}{6}$$

$$= \frac{33}{2} \text{ units}^2$$