

Trial Examination 2018

VCE Mathematical Methods Units 3&4

Written Examination 2

Suggested Solutions

SECTION A – MULTIPLE-CHOICE QUESTIONS

1	☒ A	☐ B	☐ C	☐ D	☐ E
2	☐ A	☒ B	☐ C	☐ D	☐ E
3	☐ A	☐ B	☒ C	☐ D	☐ E
4	☐ A	☐ B	☐ C	☒ D	☐ E
5	☒ A	☐ B	☐ C	☐ D	☐ E
6	☒ A	☐ B	☐ C	☐ D	☐ E
7	☐ A	☒ B	☐ C	☐ D	☐ E
8	☐ A	☐ B	☒ C	☐ D	☐ E
9	☐ A	☐ B	☐ C	☐ D	☒ E
10	☐ A	☐ B	☒ C	☐ D	☐ E

11	☐ A	☐ B	☐ C	☐ D	☒ E
12	☒ A	☐ B	☐ C	☐ D	☐ E
13	☐ A	☒ B	☐ C	☐ D	☐ E
14	☐ A	☐ B	☐ C	☒ D	☐ E
15	☐ A	☐ B	☒ C	☐ D	☐ E
16	☐ A	☒ B	☐ C	☐ D	☐ E
17	☒ A	☐ B	☐ C	☐ D	☐ E
18	☐ A	☐ B	☐ C	☒ D	☐ E
19	☐ A	☐ B	☐ C	☐ D	☒ E
20	☐ A	☐ B	☐ C	☒ D	☐ E

Question 1 A

$$x = -5 \rightarrow f(-5) = 3$$

$$x = 4 \rightarrow f(4) = 0$$

$$\therefore x \in [0, 3)$$

Question 2 B

$$\begin{aligned}\text{average rate of change} &= \frac{f(2) - f(-1)}{2 - (-1)} \\ &= \frac{-1 - 2}{3} \\ &= -1\end{aligned}$$

Question 3 C

$$\begin{aligned}\text{distance} &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(2 - (-1))^2 + (-3 - 3)^2} \\ &= \sqrt{9 + 36} \\ &= \sqrt{45} \\ &= 3\sqrt{5}\end{aligned}$$

Question 4 D

Dilation factor of $\frac{1}{2}$ from x -axis:

$$\begin{aligned}y_1 &= \frac{1}{2}(2^{x+1} + 2) \\ &= 2^{-1} \times 2^{x+1} + 1 \\ &= 2^x + 1\end{aligned}$$

Reflection in the y -axis:

$$y_2 = 2^{-x} + 1$$

Question 5 A

$$\text{Let } g(x) = 2x^3 - 5x^2 + ax.$$

$$\text{If } (2x + a) \text{ is a factor, then } g\left(-\frac{a}{2}\right) = 0.$$

$$\Rightarrow 2\left(-\frac{a}{2}\right)^3 - 5\left(-\frac{a}{2}\right)^2 + a\left(-\frac{a}{2}\right) = 0$$

$$-\frac{a^3}{4} - \frac{7a^2}{4} = 0$$

$$a^2(a + 7) = 0$$

$$\therefore a = -7 \text{ as } a \neq 0$$

Question 6 **A**

$$\begin{aligned}E(X) &= a \times 0 + b \times 1 + 0.15 \times 2 + 0.04 \times 3 + 0.01 \times 4 \\&= 0.91\end{aligned}$$

$$b = 0.45$$

$$a + b + 0.15 + 0.04 + 0.01 = 1$$

$$a + 0.45 + 0.2 = 1$$

$$a = 0.35$$

Question 7 **B**

$$\begin{aligned}\int_1^4 2(1-f(x))dx &= \int_1^4 2dx - 2 \int_1^4 f(x)dx \\&= 6 - 2 \times 10 \\&= -14\end{aligned}$$

Question 8 **C**

$$y = x^2 - ax$$

$$= \left(x - \frac{a}{4}\right)^2 - \frac{a^2}{4}$$

$$\Rightarrow -\frac{a^2}{4} = -4$$

$$a = \pm 4$$

$$\therefore a = 4 \text{ (only option)}$$

Question 9 **E**

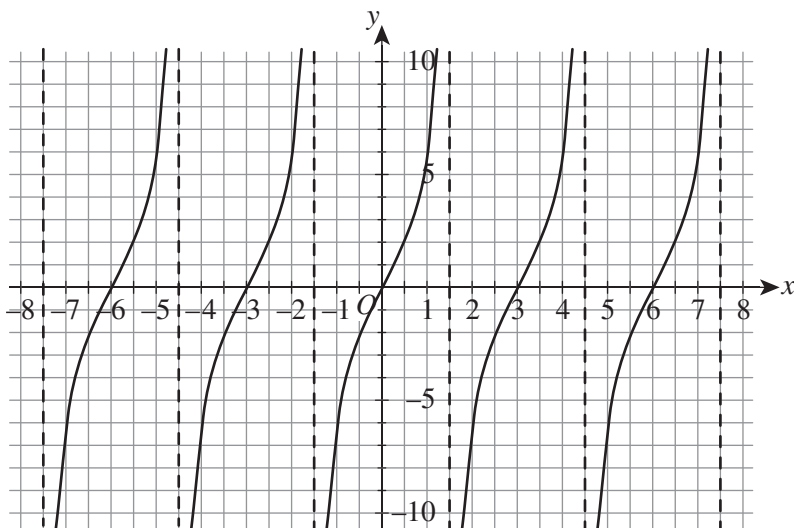
$$\begin{aligned}y_{\text{average}} &= \frac{1}{b-a} \int_a^b f(x)dx \\&= \frac{1}{2-0} \int_0^2 \log_e(2x+4)dx \\&= 4\log_e(2) - 1 \\&= \log_e(2^4) - 1 \\&= \log_e(16) - 1\end{aligned}$$

Question 10 C

	Pr(A)	Pr(A')	
Pr(B)	0.4	0.3	0.7
Pr(B')	0.1	0.2	0.3
	0.5	0.5	1

$$\begin{aligned}
 \Pr(A \cap B) &= \Pr(B) - \Pr(A' \cap B) \\
 &= 0.7 - 0.3 \\
 &= 0.4
 \end{aligned}$$

$$\begin{aligned}
 \Pr(A) &= \Pr(A \cap B) + \Pr(A \cap B') \\
 &= 0.4 + 0.1 \\
 &= 0.5
 \end{aligned}$$

Question 11 E

Vertical asymptotes are separated by new period and starting at $x = \frac{3}{2}$.

$$\begin{aligned}
 \text{period} &= \frac{\pi}{\frac{\pi}{3}} \\
 &= 3
 \end{aligned}$$

$$\therefore \text{option E expands to } R \setminus \left\{ \frac{3}{2} + 3n \right\}$$

Question 12 A

- reflection in the x -axis: $f_1(x) = -2\log_e(x) + 1$
- dilation factor of 4 from the y -axis: $f_2(x) = -2\log_e\left(\frac{x}{4}\right) - 1$
- translation of 2 units right: $g(x) = -2\log_e\left(\frac{x-2}{4}\right) - 1$

Question 13 B

$$2x - ay = a - 2$$

$$\Rightarrow y_1 = \frac{2}{a}x + \frac{2-a}{a}$$

$$ax - 8y = a$$

$$\Rightarrow y_2 = \frac{a}{8}x - \frac{a}{8}$$

For infinite solutions:

$$m_1 = m_2$$

$$\Rightarrow \frac{a}{8} = \frac{2}{a}$$

$$a = \pm 4$$

$$c_1 = c_2$$

$$\Rightarrow \frac{2-a}{a} = -\frac{a}{8}$$

$$a = 4$$

Question 14 D

$$\begin{aligned} \text{area} &= \int_0^b (f(x) - g(x))dx + \int_b^c (g(x) - f(x))dx - \int_c^d g(x)dx \\ &= \int_0^b (f(x) - g(x))dx + \int_c^b (f(x) - g(x))dx + \int_d^c g(x)dx \end{aligned}$$

Question 15 C

$$\Pr(X > a) = 0.3$$

$$\Rightarrow \Pr(X < a) = 0.7$$

$$X \sim N(20, 2^2)$$

**Question 16 B**

The derivative graph indicates three turning points at approximately -0.6 , 0 and 0.6 , so the solution could be either **B** or **C**.

For $x > 0.6$, $f'(x) > 0$, option **B** is thus the correct solution.

Question 17 A

p = probability that traveller does not have ticket

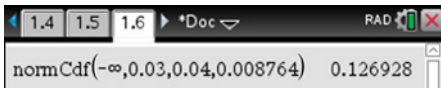
$$= 0.04$$

$$n = 500$$

$$\begin{aligned} \text{sd}(\hat{p}) &= \sqrt{\frac{p(1-p)}{n}} \\ &= \sqrt{\frac{0.04(1-0.04)}{500}} \\ &= 0.008764 \end{aligned}$$

$$X \sim N(0.04, 0.008764^2)$$

$$\begin{aligned} \Pr\left(\hat{p} \leq \frac{3}{100}\right) &= \Pr(X < 0.03) \\ &= 0.1269 \end{aligned}$$

**Question 18 D**

$g(f(x))$ is defined if $\text{range}_f \subseteq \text{domain}_g$.

$$\text{domain}_g = \left(-\infty, \frac{1}{2}\right)$$

We need to restrict range of f to $\left(-\infty, \frac{1}{2}\right)$.

$$f(x) < \frac{1}{2}$$

$$x^2 < \frac{1}{2}$$

$$x \in \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$$

Question 19 E

require new period $> 2\pi$

$$\Rightarrow -1 < k < 1$$

This is relevant for both options **D** and **E**.

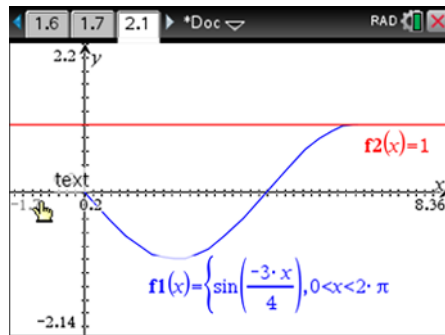
$$k = \frac{1}{4}$$

$$\Rightarrow \sin\left(\frac{1}{4}x\right) = 1 \text{ has a solution for } x = 2\pi$$

$\therefore$ option **E** is the correct solution.

Note that $k \neq -\frac{3}{4}$.

The graph for $k = -\frac{3}{4}$ is shown below.

**Question 20 D**

$$f(u) - f(-u) = e^{2u} - e^{-2u}$$

$$\frac{(f(u))^2 - 1}{f(u)} = \frac{(e^{2u})^2 - 1}{e^{2u}}$$

$$= \frac{e^{4u} - 1}{e^{2u}}$$

$$= e^{2u} - \frac{1}{e^{2u}}$$

$$= e^{2u} - e^{-2u}$$

SECTION B**Question 1** (14 marks)**a. i.** Let $f(x) = h$.

$$x(2-x) = h$$

$$x^2 - 2x + h = 0$$

M1

$$x = \frac{2 \pm \sqrt{4-4h}}{2}$$

$$= 1 \pm \sqrt{1-h}$$

$$x_A < x_B$$

$$\therefore x_A = 1 - \sqrt{1-h}$$

M1

$$\text{point A: } (1 - \sqrt{1-h}, h)$$

$$\text{ii. } x_B = 1 + \sqrt{1-h}$$

$$x_B - x_A = 1 + \sqrt{1-h} - (1 - \sqrt{1-h})$$

M1

$$\therefore \overline{AB} = 2\sqrt{1-h}$$

b. area = lw

$$= 2\sqrt{1-h} \times h$$

$$= 2h\sqrt{1-h}$$

A1

$$\text{c. } A(h) = 2h\sqrt{1-h}$$

$$A'(h) = 2\sqrt{1-h} - \frac{h}{\sqrt{1-h}}$$

A1

$$\text{Let } A'(h) = 0.$$

$$2\sqrt{1-h} - \frac{h}{\sqrt{1-h}} = 0$$

$$h = \frac{2}{3}$$

A1

$$A\left(\frac{2}{3}\right) = \frac{4\sqrt{3}}{9}$$

$$\therefore \text{maximum area} = \frac{4\sqrt{3}}{9}$$

A1

d. Let $g(x) = h$.

$$x(k-x) = h$$

$$x = \frac{k \pm \sqrt{k^2 - 4h}}{2}$$

M1

$$\overline{QR} = \frac{k + \sqrt{k^2 - 4h}}{2} - \left(\frac{k - \sqrt{k^2 - 4h}}{2} \right)$$

$$= \sqrt{k^2 - 4h}$$

A1

e. i. $A(h) = h\sqrt{k^2 - 4h}$

A1

$$A'(h) = \sqrt{k^2 - 4h} - \frac{2h}{\sqrt{k^2 - 4h}}$$

A1

Let $A'(h) = 0$.

$$\sqrt{k^2 - 4h} - \frac{2h}{\sqrt{k^2 - 4h}} = 0$$

$$k^2 - 4h = 2h$$

$$h = \frac{k^2}{6}$$

M1

$$A\left(\frac{k^2}{6}\right) = \frac{k^2}{6} \sqrt{k^2 - \frac{4k^2}{6}}$$

$$= \frac{k^2}{6} \sqrt{\frac{k^2}{3}}$$

$$= \frac{\sqrt{3}|k^3|}{18}$$

$$k > 0, \therefore \text{maximum area} = \frac{\sqrt{3}k^3}{18}$$

A1

ii. $A = h^2$ and $h = \frac{k^2}{6}$ for maximum area.

$$\Rightarrow A = \frac{k^4}{36}$$

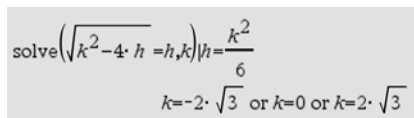
From **part e. i.**, $A = \frac{\sqrt{3}k^3}{18}$.

$$\Rightarrow \frac{k^4}{36} = \frac{\sqrt{3}k^3}{18}$$

$$\therefore k = 2\sqrt{3}$$

A1

Graphically:



As $k > 0$, $k = 2\sqrt{3}$.

A1

Question 2 (13 marks)

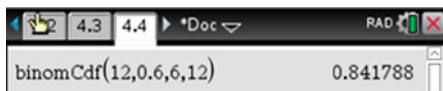
a. $X \sim \text{Bi}(n, p)$

$$X \sim \text{Bi}(12, 0.6)$$

A1

$$\Pr(X \geq 6) = 0.8418$$

A1



b. $\Pr(X = 8 | X \geq 6) = \frac{\Pr(X = 8 \cap X \geq 6)}{\Pr(X \geq 6)}$

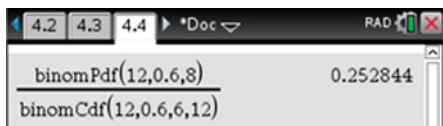
$$= \frac{\Pr(X = 8)}{\Pr(X \geq 6)}$$

M1

$$= \frac{0.212841...}{0.841788...}$$

$$= 0.2528$$

A1



c. $Y \sim \text{Bi}(n, p)$

$Y \sim \text{Bi}(4, p)$

$\Pr(Y = 2) = {}^4C_2 p^2 (1-p)^2$

$= 0.0486$

M1

$6p^2(1-p)^2 = 0.0486$

$p = 0.1$ as $0 < p < 0.4$

A1

$R \sim \text{Bi}(12, 0.1)$

$E(R) = np$

$= 12 \times 0.1$

$= 1.2$

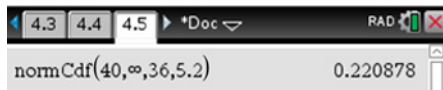
A1

d. $F \sim N(36, (5.2)^2)$

A1

$\Pr(F > 40) = 0.2209$

A1



e. $L \sim \text{Bi}(7, 0.2209)$

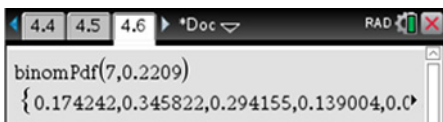
Number of days (n)	$\Pr(L = n)$
0	0.1742...
1	0.3458...
2	0.2941...
3	
4	
5	
6	
7	

$\Pr(L = 0) + \Pr(L = 1) = 0.5160...$ (> fiftieth percentile)

M1

$\therefore$ median = 1 day

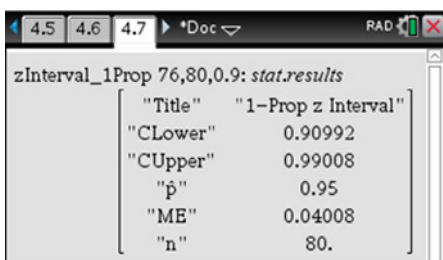
A1



f. $n = 80, p = \frac{76}{80}$

90% confidence interval: $[0.9099, 0.9901]$

A1



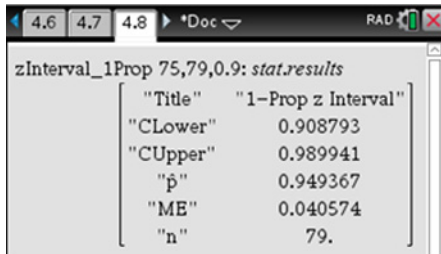
- g. The new sample proportion must be either $\frac{75}{79}$ or $\frac{76}{79} \rightarrow$ test each value.

$$n = 79, \hat{p} = \frac{75}{79}$$

90% confidence interval: [0.9088, 0.9899]

$$\therefore \hat{p} = \frac{75}{79}$$

A1



zInterval_1Prop 75,79,0.9: stat.results	
"Title"	"1-Prop z Interval"
"CLower"	0.908793
"CUpper"	0.989941
"p"	0.949367
"ME"	0.040574
"n"	79.

Question 3 (8 marks)

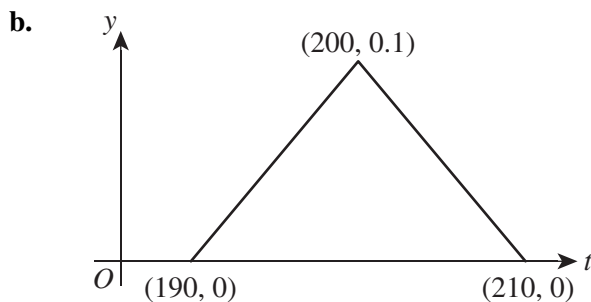
a.
$$\int_{190}^{200} m(t-190)dt + \int_{200}^{210} -m(t-210)dt = 1$$

M1

$$100m = 1$$

$$m = 0.01$$

A1



correct shape A1
correct coordinates A1

c.
$$\begin{aligned} \Pr(192 < F < 208) &= 1 - 2\Pr(190 < F < 192) \\ &= 1 - 2 \int_{190}^{192} 0.01(t-190)dt \\ &= 1 - 2 \times 0.02 \\ &= 0.9600 \end{aligned}$$

M1

A1

$$\begin{aligned}
 \text{d. } \Pr(192 < F < 208 | F > 205) &= \frac{\Pr(192 < F < 208 \cap F > 205)}{\Pr(F > 205)} \\
 &= \frac{\Pr(205 < F < 208)}{\Pr(F > 205)} \\
 &= \frac{\int_{205}^{208} -0.01(t - 210)dt}{\int_{205}^{210} -0.01(t - 210)dt} \\
 &= \frac{0.105}{0.125} \\
 &= 0.84
 \end{aligned}$$

M1
A1

Question 4 (14 marks)

$$\begin{aligned}
 \text{a. } g(x) &= \sqrt{x} \times (\sqrt{x} - 4) \\
 &= x - 4\sqrt{x}
 \end{aligned}$$

M1

$$\text{b. } g'(x) = 1 - \frac{2}{\sqrt{x}}$$

A1

$$\text{Let } g'(x) = 0.$$

$$1 - \frac{2}{\sqrt{x}} = 0$$

$$\Rightarrow x = 4$$

$$g(4) = -4$$

$$\therefore \text{SP} = (4, -4)$$

A1

A1

$$\text{c. domain: } [0, \infty), \text{ range: } [-4, \infty)$$

A1

$$\text{d. } s = 4$$

A1

$$\text{e. Let } y = x - 4\sqrt{x}.$$

$$y = (\sqrt{x} - 2)^2 - 4$$

For inverse, swap x and y .

$$x = (\sqrt{y} - 2)^2 - 4$$

$$(\sqrt{y} - 2)^2 = x + 4$$

$$\sqrt{y} = \sqrt{x + 4} + 2 \text{ (as } y \geq 4)$$

$$y = (\sqrt{x + 4} + 2)^2$$

$$\therefore h^{-1}(x) = (\sqrt{x + 4} + 2)^2$$

M1

A1

$$\text{f. domain: } [-4, \infty), \text{ range: } [4, \infty)$$

A1

- g. i.** domain: $[4, \infty)$ A1
- ii.** $d(x) = (\sqrt{x+4} + 2)^2 - (x - 4\sqrt{x})$
 $= (\sqrt{x+4} + 2)^2 - x + 4\sqrt{x}$ A1
- iii.** $d(x) = (\sqrt{x+4} + 2)^2 - x + 4\sqrt{x}$
 $= 4\sqrt{x+4} + 4\sqrt{x} + 8$ M1
- $4\sqrt{x+4} > 0$ and $4\sqrt{x} > 0$ for $x \in [4, \infty)$.
 $\Rightarrow d(x) > 0$ for $x \in [4, \infty)$.
 $\therefore$ vertical distance > 0 and graphs do not intersect A1
- h.** The minimum vertical distance between $h(x)$ and $h^{-1}(x)$ occurs at the end point of $h(x)$, where $x = 4$.
 $d(4) = 16 + 8\sqrt{2}$
 $q(x)$ is a transformation of the graph of $y = h(x)$ by $-c$ units upwards.
 $\therefore c = -16 - 8\sqrt{2}$ A1

Question 5 (11 marks)

- a. i.** $f'(x) = (x - k + 1)e^x$ A1
- ii.** Let $f'(x) = 0$.
 $\Rightarrow x = k - 1$ A1
 $f(k - 1) = -e^{k-1}$
 $\therefore$ stationary point: $(k - 1, -e^{k-1})$ A1
- b.** Two solutions occur between the stationary point and x -axis, which is an asymptote for $f(x)$.
 $n \in (-e^{k-1}, 0)$ A1
- c. i.** $\frac{d}{dx}[xe^x] = x \times e^x + 1 \times e^x$ *use product rule* M1
 $= (x + 1)e^x$ as required

ii. $\int (x+1)e^x dx = xe^x + c$

$$\int xe^x dx + \int e^x dx = xe^x + c$$

$$\int xe^x dx = xe^x - e^x + c$$

M1

$$\int f(x) dx = \int (x-k)e^x dx$$

$$= \int (xe^x - ke^x) dx$$

$$= xe^x - e^x - ke^x + c$$

$$= (x-k-1)e^x + c$$

A1

d. $\text{area} = -\int_0^k (x-k)e^x dx$

M1

$$= -\left[(x-k-1)e^x\right]_0^k$$

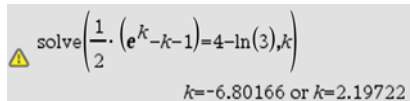
$$= e^k - k - 1$$

A1

- e. Using transformations: $f(x) \rightarrow$ dilation factor of 2 from x -axis and $\frac{1}{4}$ from y -axis $\rightarrow g(x)$
 Therefore the area is $2 \times \frac{1}{4} = \frac{1}{2}$ of area found in **part d**.

$$\Rightarrow \frac{1}{2}(e^k - k - 1) = 4 - \log_e(3)$$

M1



As $k \geq 1$, $k = 2.1972$ (correct to four decimal places).

A1