

The Mathematical Association of Victoria

Trial Examination 2018

MATHEMATICAL METHODS

Trial Written Examination 2 - SOLUTIONS

SECTION A: Multiple Choice

Question	Answer	Question	Answer
1	D	11	B
2	A	12	E
3	E	13	B
4	C	14	D
5	C	15	B
6	E	16	D
7	C	17	C
8	E	18	A
9	D	19	A
10	A	20	B

Question 1

Answer D

$$f : \mathbb{R} \rightarrow \mathbb{R}, f(x) = -2 \sin(3x) + 1$$

Amplitude = 2

$$\text{period} = \frac{2\pi}{3}$$

Question 2

Answer A

$$f(x) = \frac{1}{(1-x)^2} + 2$$

- a reflection in the x -axis

$$y_1 = -\frac{1}{(1-x)^2} - 2$$

- a dilation of a factor of 2 from the y -axis.

$$g(x) = -\frac{1}{\left(1-\frac{x}{2}\right)^2} - 2 = -\frac{4}{(2-x)^2} - 2$$

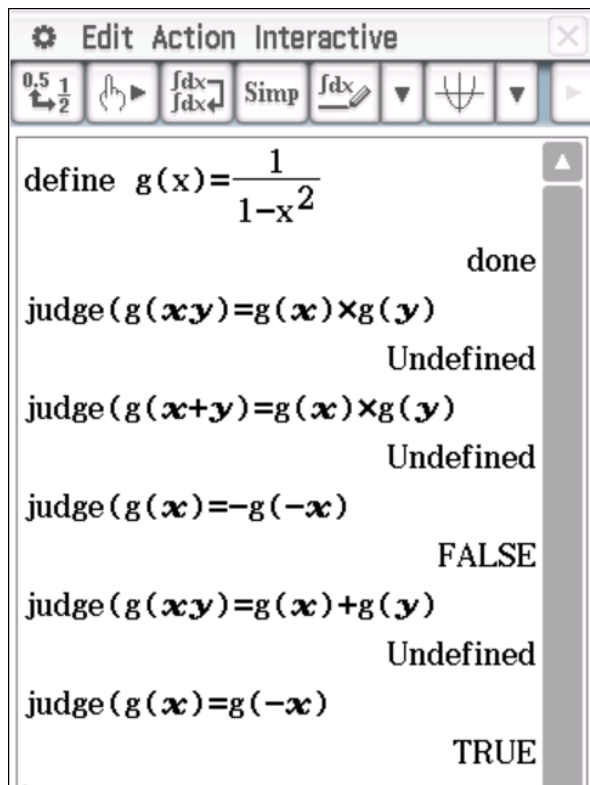
Question 3**Answer E**

$$g(x) = \frac{1}{1-x^2}$$

g is an even function, hence $g(x) = g(-x)$.

OR

$$g(-x) = \frac{1}{1-(-x)^2} = \frac{1}{1-x^2} = g(x)$$

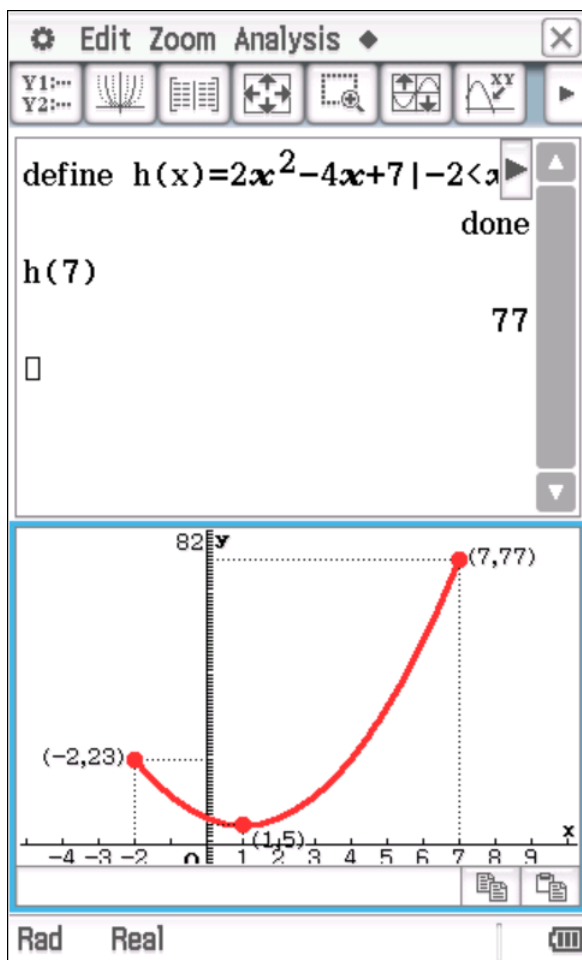
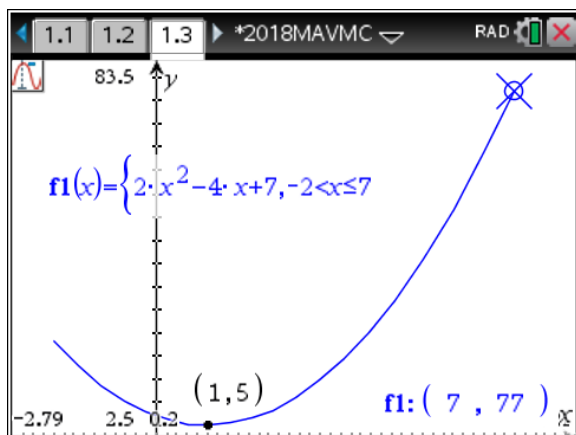
**Question 4****Answer C**

$$h: (-2, 7] \rightarrow \mathbb{R}, h(x) = 2x^2 - 4x + 7$$

$$h(x) = 2(x-1)^2 + 5$$

The minimum value of h is at the turning point and the maximum occurs when $x = 7$.

The range is $[5, 77]$.



Question 5**Answer C**

$$g(x) = \frac{1}{1+3x} \text{ and } f(x) = (x)^{\frac{2}{3}}$$

For $g(f(x))$, test that the range $f \subseteq \text{domain } g$

$$\text{Giving } [0, \infty) \subseteq \mathbb{R} \setminus \left\{ -\frac{1}{3} \right\}$$

The domain of $g(f(x))$ is the same as the domain of f which is $\mathbb{R}$.

$$\text{The rule is } g(f(x)) = \frac{1}{1+3x^{\frac{2}{3}}}.$$

$$g(f(x)): \mathbb{R} \rightarrow \mathbb{R}, \text{ where } g(f(x)) = \frac{1}{1+3x^{\frac{2}{3}}}$$

Question 6**Answer E**

$$2\sin(2x) + 1 = 0, x \in [0, \pi]$$

$$\sin(2x) = -\frac{1}{2}$$

$$2x = \pi + \frac{\pi}{6}, 2\pi - \frac{\pi}{6}$$

$$2x = \frac{7\pi}{6}, \frac{11\pi}{6}$$

$$\text{Giving } x = \frac{7\pi}{12}, \frac{11\pi}{12}$$

$$\text{Product} = 77 \left(\frac{\pi}{12} \right)^2$$

The screenshot shows a TI-84 Plus calculator interface. The top status bar indicates '1.1', '1.2', and 'RAD'. The main display shows the equation 'solve(2 sin(2x) + 1 = 0, x) | 0 ≤ x ≤ π'. Below this, the solutions are shown as 'x = 7π/12 or x = 11π/12'. At the bottom, the product calculation is shown as (7π/12) * (11π/12) = (77π²)/144.

Question 7**Answer C**

$$(k+1)x^2 - 2kx - (k-1) = 0$$

$$\Delta = (-2k)^2 - 4(k+1) \times (-(k-1))$$

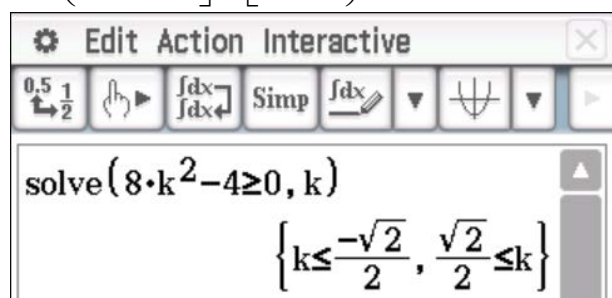
$$\Delta = 4k^2 + 4(k^2 - 1)$$

$$\Delta = 8k^2 - 4$$

Real roots when $\Delta \geq 0$

$$\text{Giving } 8k^2 - 4 \geq 0$$

$$k \in \left(-\infty, -\frac{\sqrt{2}}{2} \right] \cup \left[\frac{\sqrt{2}}{2}, \infty \right)$$

**Question 8****Answer E**

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

Gives the equations

$$-x + 1 = x', \quad 4y - 2 = y'$$

Rearrange to get

$$-x + 1 = x', \quad x = 1 - x'$$

$$4y - 2 = y', \quad y = \frac{y' + 2}{4}$$

In the equation $y = 2\sqrt{x}$

$$\frac{y' + 2}{4} = 2\sqrt{1 - x'}$$

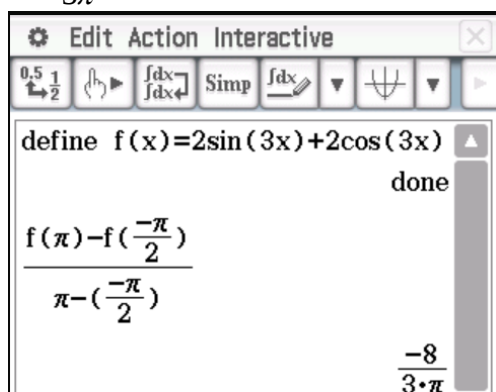
$$\text{Giving } y' = 8\sqrt{1 - x'} - 2$$

Question 9**Answer D**

$$f(x) = 2\sin(3x) + 2\cos(3x)$$

Average rate of change for $x \in \left[-\frac{\pi}{2}, \pi\right]$

$$\begin{aligned} &= \frac{f(\pi) - f\left(-\frac{\pi}{2}\right)}{\pi - \left(-\frac{\pi}{2}\right)} \\ &= -\frac{8}{3\pi} \end{aligned}$$

**Question 10****Answer A**

$$f(x) = \sqrt{m - x^2}$$

$$f'(x) = \frac{1}{2\sqrt{m - x^2}} \times -2x$$

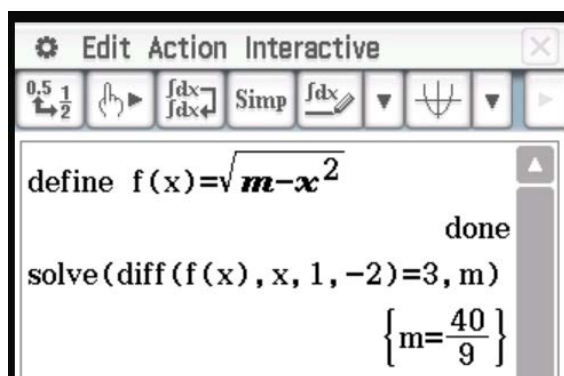
$$= \frac{-x}{\sqrt{m - x^2}}$$

Tangent has gradient of 3 when $x = -2$

Solve

$$\frac{2}{\sqrt{m - 4}} = 3$$

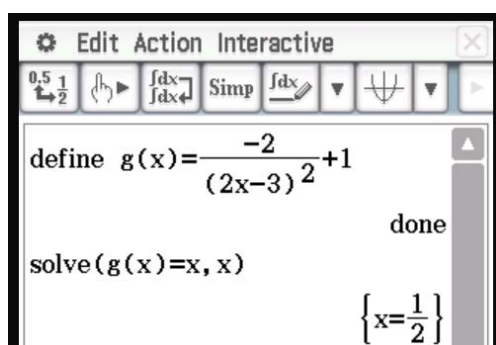
$$m = \frac{40}{9}$$

**Question 11****Answer B**

$$g(x) = \frac{-2}{(2x-3)^2} + 1$$

$$\text{Solve } \frac{-2}{(2x-3)^2} + 1 = x$$

$$\text{Intersection } g^{-1} \text{ and } g \text{ is } \left(\frac{1}{2}, \frac{1}{2} \right)$$

**Question 12****Answer E**

$$h(x) = \log_e(kx^2 + 2)$$

The derivative of $g(h(x))$

$$= \text{derivative of } g(h(x)) \times \text{derivative of } h(x)$$

$$= \frac{2kx}{kx^2 + 2} g'(h(x))$$

Question 13**Answer B**

The area enclosed by the graphs of f and g

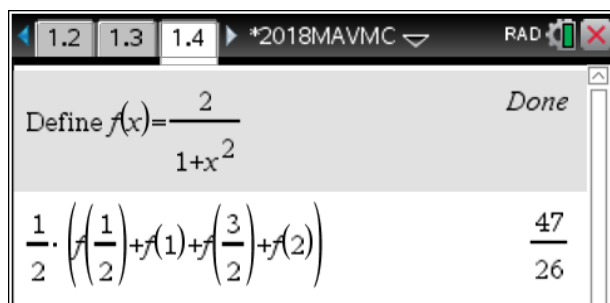
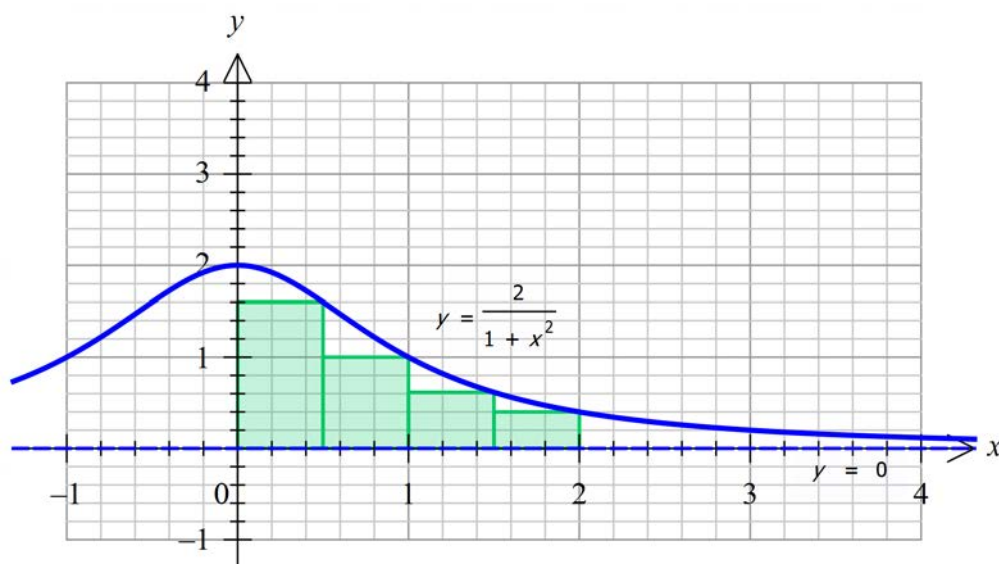
$$= \int_2^{11} (g(x) - f(x)) dx$$

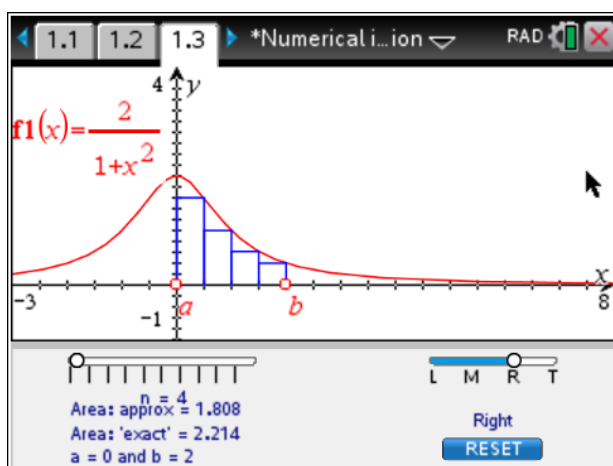
$$= \int_{\frac{11}{3}}^2 (f(x) - g(x)) dx$$

Question 14**Answer D**

Let $y = f(x) = \frac{2}{1+x^2}$

$$A = \frac{1}{2} \left(f\left(\frac{1}{2}\right) + f(1) + f\left(\frac{3}{2}\right) + f(2) \right) = \frac{47}{26}$$



**Question 15****Answer B**

Area of a circle is $A = \pi r^2$

In the first quadrant there is $\frac{1}{4}$ of a circle and $r = 2$.

So the total area is $\frac{1}{4} \times \pi \times 4$.

Rearrange $x^2 + (y-2)^2 = 4$ to make x the subject

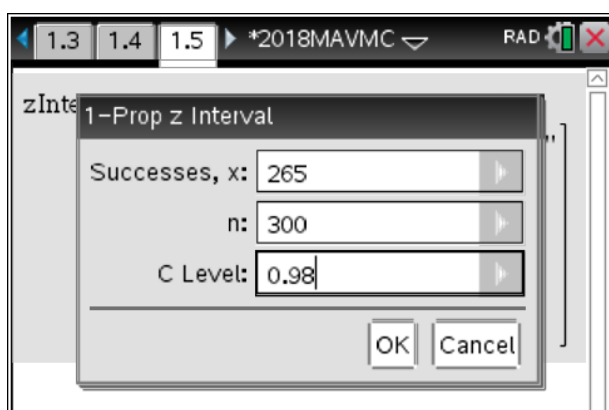
$x = \sqrt{4 - (y-2)^2}$, $x > 0$ in the first quadrant.

$$\frac{1}{3} \times \frac{1}{4} \times \pi \times 4 = \int_0^d \sqrt{4 - (y-2)^2} dy$$

Question 16**Answer D**

$n = 300$, the number of successes is 265

(0.840, 0.926)



zInterval_1Prop 265,300,0.98: stat.results	
"Title"	"1-Prop z Interval"
"CLower"	0.840216
"CUpper"	0.92645
"p"	0.883333
"ME"	0.043117
"n"	300.

Question 17**Answer C**

$$\Pr(A) = 0.6 \text{ and } \Pr(A \cap B) = 0.1$$

For independent events $\Pr(A) \times \Pr(B) = \Pr(A \cap B)$

$$0.6 \times \Pr(B) = 0.1$$

$$\Pr(B) = \frac{1}{6}$$

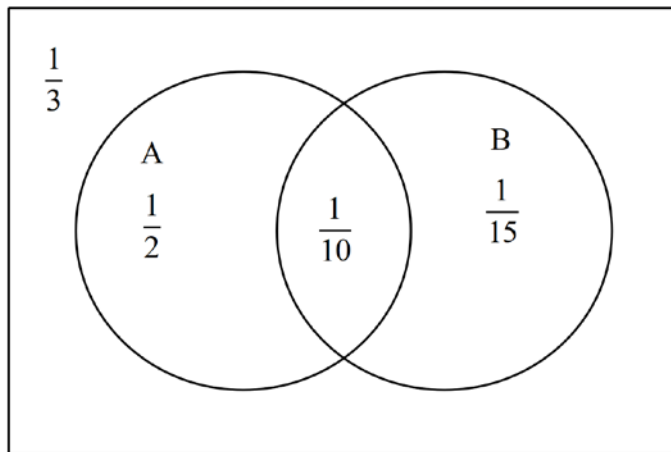
$$\Pr(A \cap B') = \frac{3}{5} - \frac{1}{10} = \frac{1}{2}$$

$$\text{then } \Pr(A' \cap B') = \frac{5}{6} - \frac{1}{2} = \frac{1}{3}$$

	A	A'	
B	$\frac{1}{10}$	$\frac{1}{15}$	$\frac{1}{6}$
B'	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{5}{6}$
	$\frac{3}{5}$	$\frac{2}{5}$	1

Or using a Venn diagram

$$\Pr(A \cap B) = \frac{1}{10}, \Pr(B \cap A') = \frac{1}{15}, \Pr(A \cap B') = \frac{1}{2}, \Pr(A' \cap B') = \frac{1}{3}$$

**Question 18****Answer A**

$$X \sim \text{Bi}(n, 0.2)$$

$$\Pr(X > 1) > 0.8$$

$$1 - (\Pr(X = 0) + \Pr(X = 1)) > 0.8$$

$$\Pr(X = 0) + \Pr(X = 1) < 0.2$$

$$0.8^n + \binom{n}{1} 0.2 \times 0.8^{n-1} < 0.2$$

$$n = 14$$

Question 19**Answer A**

$$X \sim N(\mu, \sigma^2)$$

0.6% of giraffes have a gestation period greater than 155 days

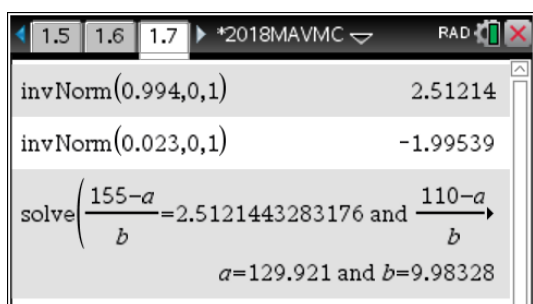
$$\Pr(X > 155) = 0.006$$

and 2.3% of giraffes have a gestation period less than 110 days

$$\Pr(X < 110) = 0.023$$

Solve $\frac{155-\mu}{\sigma} = 2.512\dots$ and $\frac{110-\mu}{\sigma} = -1.995\dots$ for μ and σ

$$\mu = 130 \text{ and } \sigma = 10$$

**Question 20****Answer B**

$$f(x) = \begin{cases} \frac{3}{34}x & 0 \leq x \leq 2 \\ \frac{3}{68}x^2 & 2 < x \leq 4 \\ 0 & \text{elsewhere} \end{cases}$$

$$\text{sd}(X) = \sqrt{E(X^2) - (E(X))^2}$$

$$\text{sd}(X) = \sqrt{\left(\int_0^4 x^2 \times f(x) - \left(\int_0^4 x \times f(x) \right)^2 \right)} = \frac{\sqrt{5765}}{85}$$

TI-84 Plus calculator screen showing the definition of a piecewise function $f(x)$ and the calculation of a definite integral.

Define $f(x) = \begin{cases} \frac{3}{34} \cdot x, & 0 \leq x \leq 2 \\ \frac{3}{68} \cdot x^2, & 2 < x \leq 4 \end{cases}$

Calculation: $\int_0^4 (x^2 \cdot f(x)) dx - \left(\int_0^4 (x \cdot f(x)) dx \right)^2 = 0.893266$

Result: $\frac{\sqrt{5765}}{85} = 0.893266$

TI-84 Plus calculator screen showing the calculation of a definite integral.

Calculation: $\int_0^2 \left(\frac{3}{34} \cdot x^3 \right) dx + \int_2^4 \left(\frac{3}{68} \cdot x^4 \right) dx - \left(\int_0^2 \left(\frac{3}{34} \cdot x^2 \right) dx \right)^2$

Result: $\frac{\sqrt{5765}}{85}$

SECTION B**Question 1**

$$f: \mathbb{R} \rightarrow \mathbb{R}, f(x) = 18x^2 - 36x^4$$

a. $(0, 0)$ **1A**

b. Solve $f(x) = 0$

$(0, 0), \left(\frac{\sqrt{2}}{2}, 0\right), \left(-\frac{\sqrt{2}}{2}, 0\right)$ **1A** any two correct **2A** all correct

c. $f'(x) = 36x - 144x^3$ **1A**

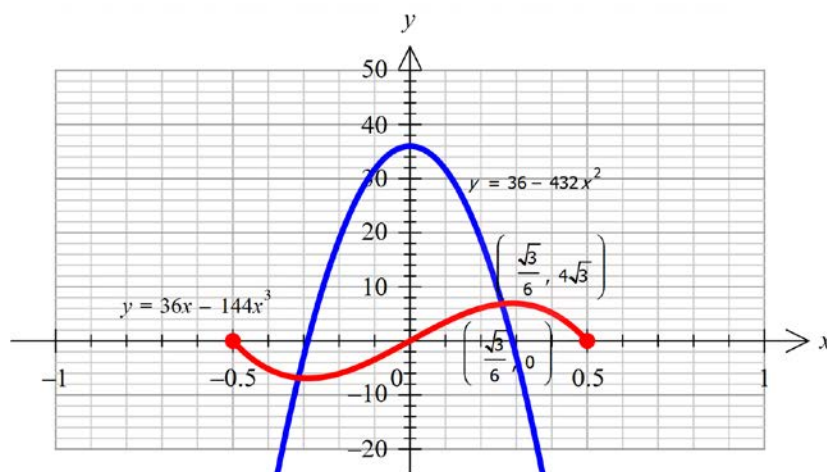
Solve $f'(x) = 0$

Turning points $(0, 0), \left(\frac{1}{2}, \frac{9}{4}\right), \left(-\frac{1}{2}, \frac{9}{4}\right)$ **1A**

d. Sketch a graph of the derivative function with equation $f'(x) = 36x - 144x^3$. The maximum occurs at the turning point.

Solving $f''(x) = 36 - 432x^2 = 0$ for points of inflexion gives $x = \pm \frac{\sqrt{3}}{6}$.

From the graph given, gradient at its maximum at $x = \frac{\sqrt{3}}{6}$. **1A**



e. Line joining $A(-k, g(-k))$ and $B(k, g(k))$ where

$$g(k) = 18k^2 - 36k^4, \quad g(-k) = 18k^2 - 36k^4$$

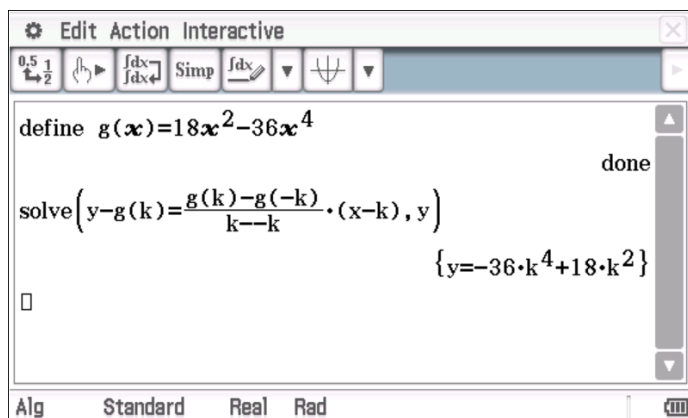
$$\text{Gradient} = \frac{g(k) - g(-k)}{k - (-k)} = 0$$

Equation of the straight line

$$y - g(k) = 0(x - k)$$

$$y = g(k)$$

$$y = 18k^2 - 36k^4 \quad \mathbf{1A}$$

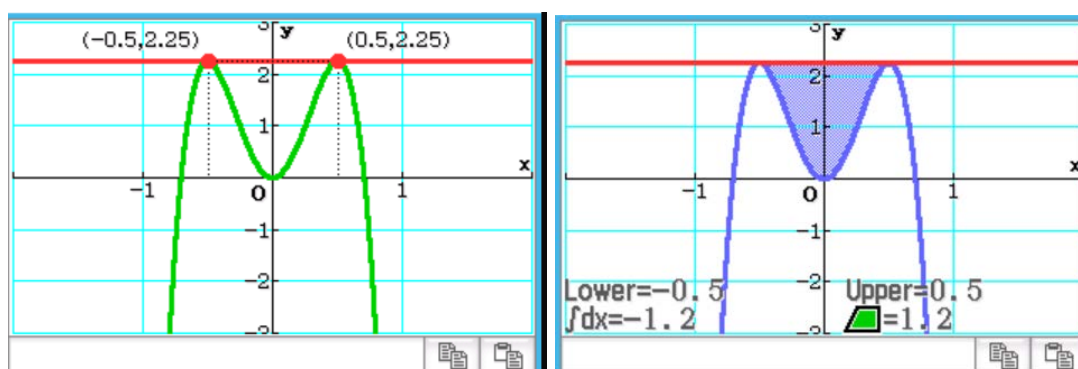


f.i. Area when $k = \frac{1}{2}$

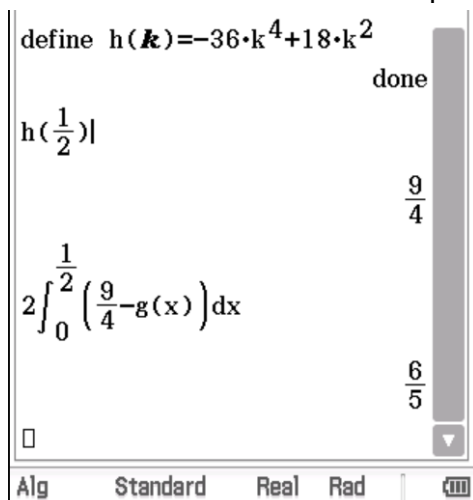
$$\text{Area} = 2 \int_0^{\frac{1}{2}} \left(\frac{9}{4} - g(x) \right) dx$$

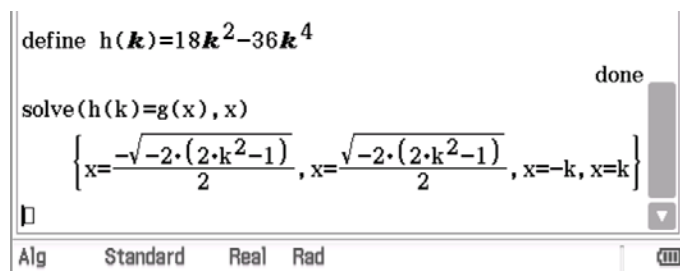
$$= \frac{6}{5} \text{ sq units}$$

1A

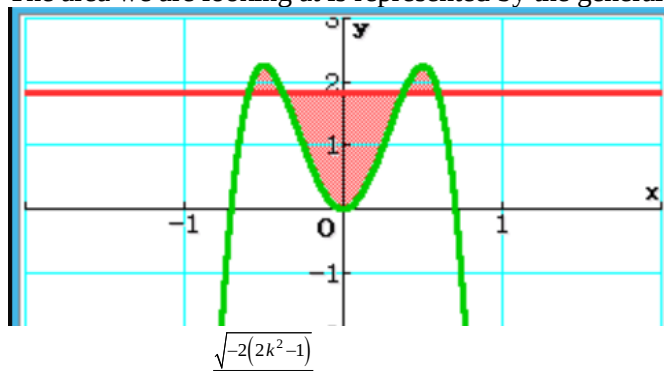


Graph showing intersection of $y = \frac{9}{4}$ and $g(x)$, then showing enclosed Area $= \frac{6}{5} = 1.2$



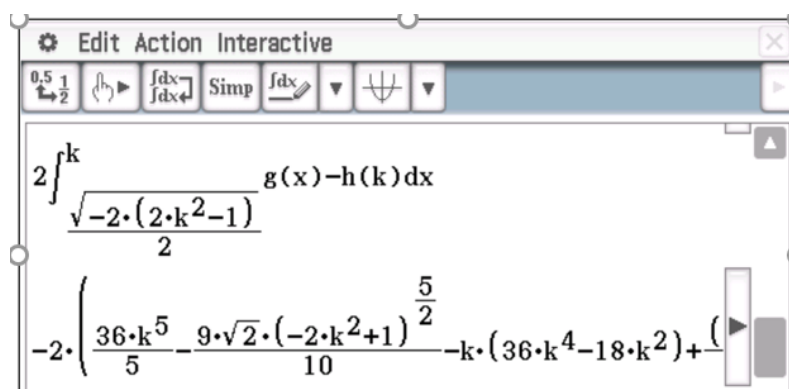
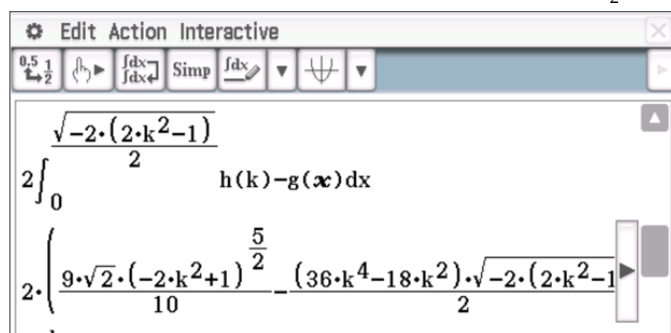


The area we are looking at is represented by the general case shown below.

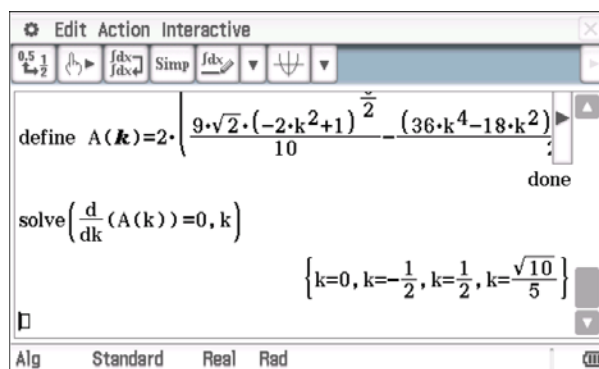


$$\text{Area} = A(k) = 2 \int_0^{\frac{\sqrt{-2(2k^2-1)}}{2}} (h(k) - g(x)) dx + 2 \int_{\frac{\sqrt{-2(2k^2-1)}}{2}}^k (g(x) - h(k)) dx$$

1M



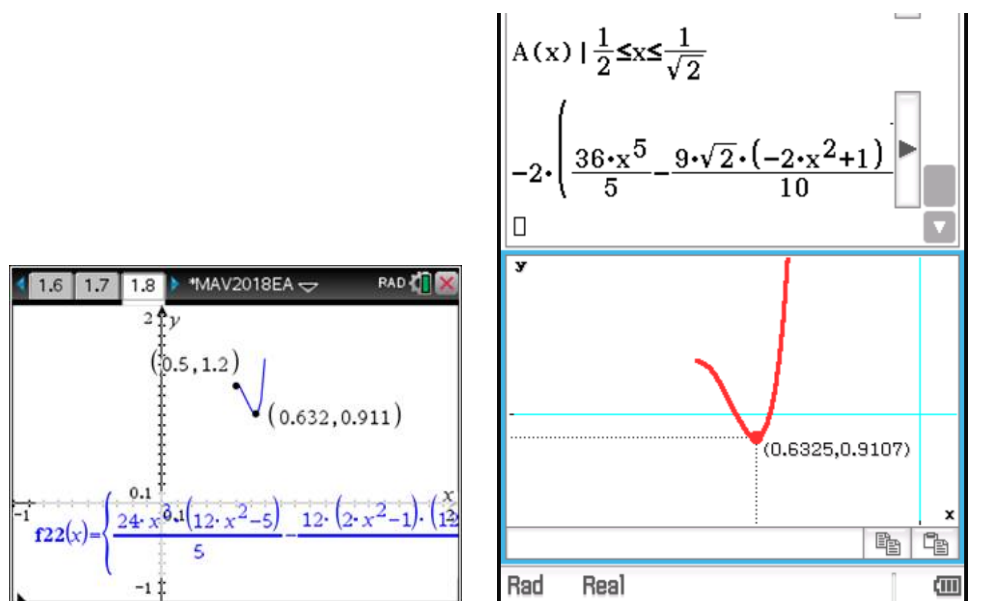
$$A'(k) = 0 \text{ gives } k = 0, \pm \frac{1}{2}, \frac{\sqrt{10}}{5}$$



Within domain $\frac{1}{2} \leq k \leq \frac{1}{\sqrt{2}}$, minimum area occurs at

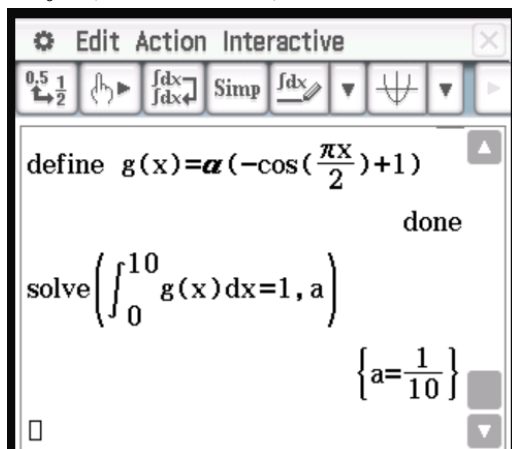
$$k = \frac{\sqrt{10}}{5} \quad \mathbf{1A}$$

Or to visualise the answer trace value as below.

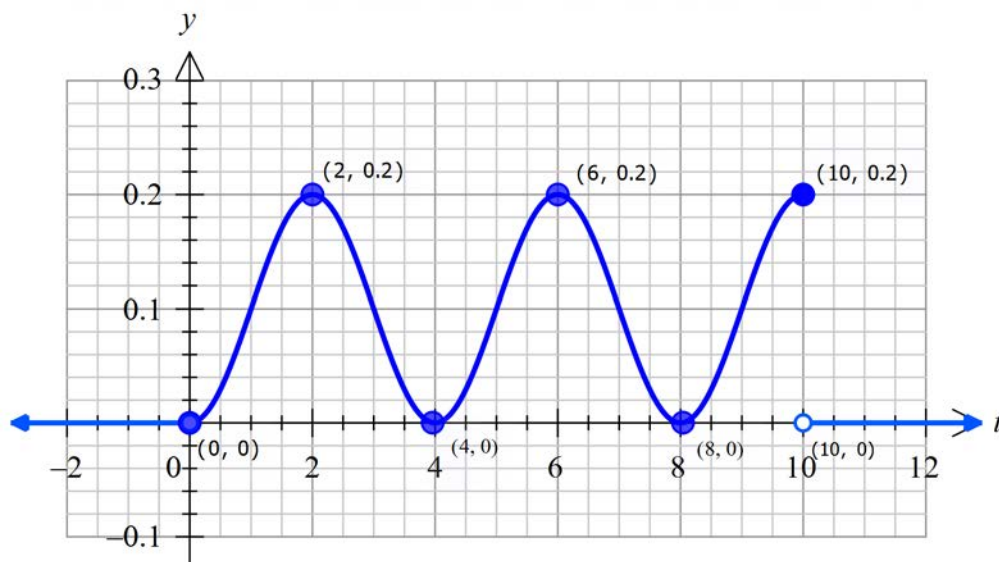


Question 2

a. $\int_0^{10} a \left(-\cos\left(\frac{\pi t}{2}\right) + 1 \right) dt = 1$

1A

b. shape **1A**, drawing along the axis **1A**, correct coordinates **1A**



c.i. mean $= \frac{1}{10} \int_0^{10} t \left(-\cos\left(\frac{\pi t}{2}\right) + 1 \right) dt$

1M

$= \frac{4}{5\pi^2} + 5$ hours

1A

define $g(x) = \frac{1}{10}(-\cos(\frac{\pi x}{2}) + 1)$

done

$\int_0^{10} xg(x) dx$

$\frac{4}{5 \cdot \pi^2} + 5$

c.ii. Median, m , solve $\frac{1}{10} \int_0^m \left(-\cos\left(\frac{\pi t}{2}\right) + 1 \right) dt = 0.5$ **1M**

$m = 5.5$ hours **1A**

define $g(x) = \frac{1}{10}(-\cos(\frac{\pi x}{2}) + 1)$

done

solve $\left(\int_0^m g(x) dx = 0.5 \mid 0 \leq m \leq 10, m \right)$

$\{m=5.470516209\}$

Alg Standard Real Rad

d. $\Pr(T \leq 6) = \frac{3}{5}$ **1A**

$\int_0^6 g(x) dx$

$\frac{3}{5}$

Alg Standard Real Rad

e. $\Pr(T > 3 \mid T < 6) = \frac{\Pr(3 < T < 6)}{\Pr(T < 6)}$ **1M**

$= 0.3939$ **1A**

f. $(0.4)^3 = 0.064$ OR $\frac{8}{125}$ **1A**

g. $X \sim \text{Bi}(5, 0.4)$

$$\Pr(X = 5 | X \geq 3) = \frac{\Pr(X = 5)}{\Pr(X = 3) + \Pr(X = 4) + \Pr(X = 5)} \quad \textbf{1M}$$

$$= \frac{0.4^5}{{}^5C_3 0.4^3 0.6^2 + {}^5C_4 0.4^4 0.6^1 + 0.4^5} \quad \textbf{1M}$$

$$= \frac{1}{31} \quad \textbf{1A}$$

h. Standard deviation of $\hat{P} = \sqrt{\frac{p(1-p)}{n}}$ where $p = 0.55, n = 400$

$$= \sqrt{\frac{0.55(1-0.55)}{400}}$$

$$= 0.0249 \quad \textbf{1A}$$

i. $\frac{60}{100} \times 400 = 240$

$$X \sim \text{Bi}(400, 0.55)$$

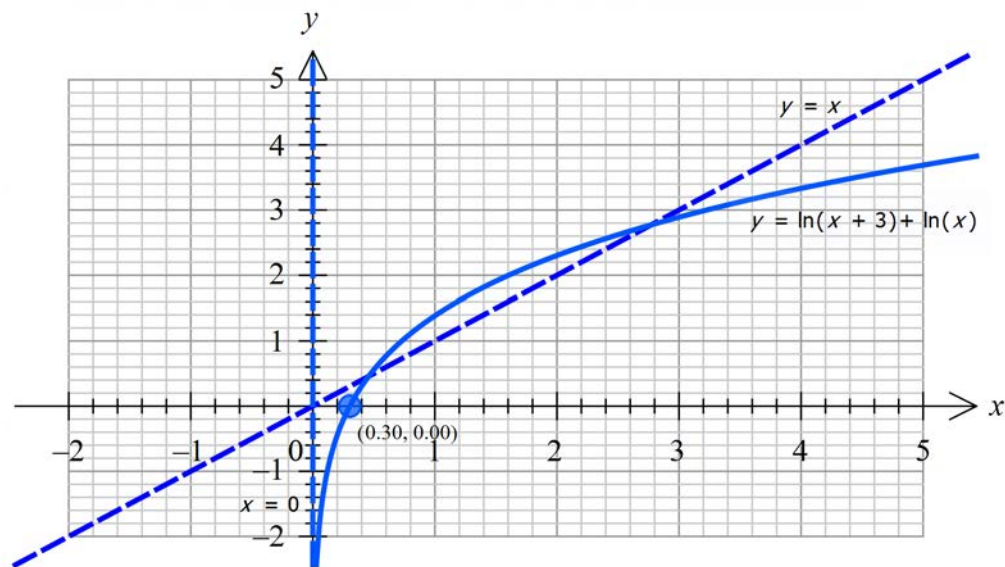
$$\Pr(X > 240) = \Pr(X \geq 241) \quad \textbf{1M}$$

$$= 0.0194 \quad \textbf{1A}$$

j. $Y \sim N(0.55, (0.02487\dots)^2)$

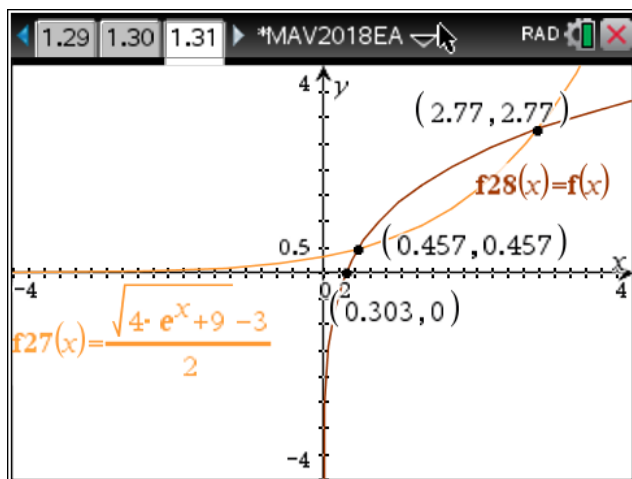
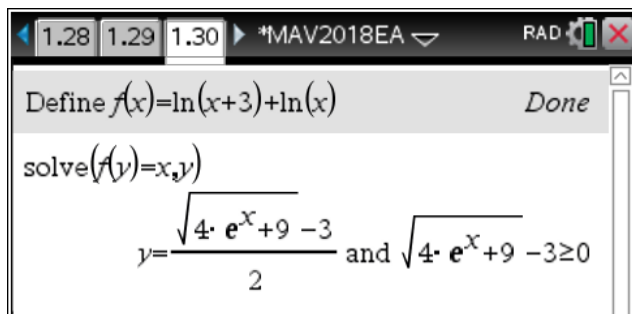
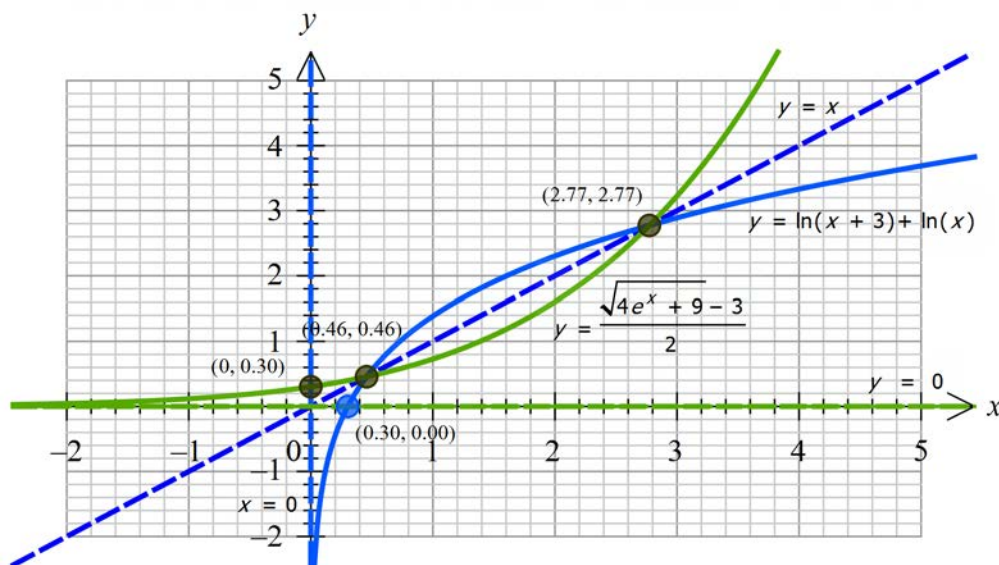
$$\Pr(Y > 0.6) = 0.022 \quad \textbf{1A}$$

binomialCdf(241, 400, 400, 0.55)	0.01935532016
normCdf(0.6, 1, 0.02487, 0.55)	0.02219156088
Alg Standard Real Rad	

Question 3**a. Shape 1A**Asymptote $x = 0$ and axial intercept $(0.30, 0.00)$ **1A**

$$\text{b. } f^{-1}(x) = \frac{\sqrt{(4e^x + 9)} - 3}{2} \quad \text{1A}$$

 $(0.46, 0.46)$ and $(2.77, 2.77)$ **1A**Everything else correct **1A**



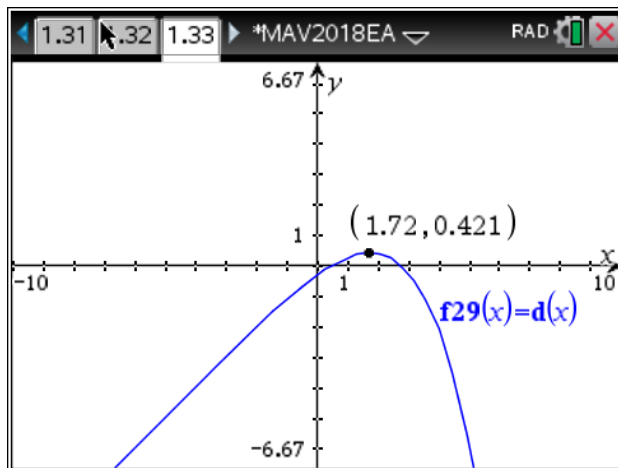
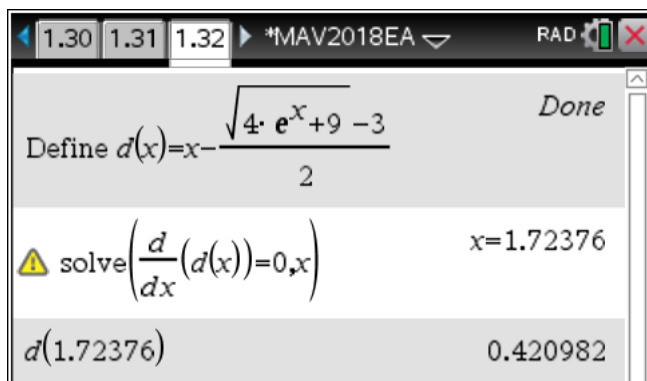
c. Translation of $-k$ units to the right. 1A

d. $d(x) = x - \frac{\sqrt{(4e^x + 9)} - 3}{2}$ 1A

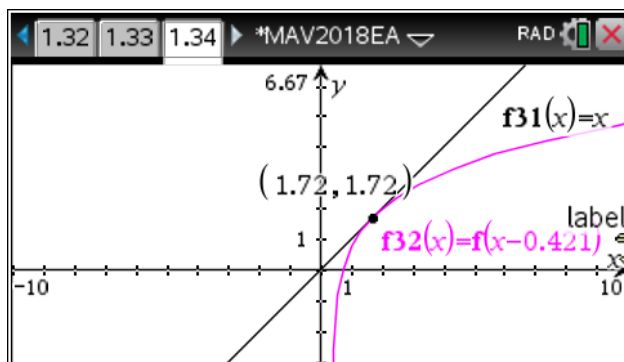
e. Solve $d'(x) = 0$ or graph d 1M

Maximum $(1.72..., 0.421...)$

$k = -0.421$ 1A



Check with the graph



f. Dilation by a factor of a from the y -axis. **1A**

g. Solve $f_2'(x) = 1$ for x .

$$x = \frac{\sqrt{9a^2 + 4} - 3a + 2}{2} \quad \mathbf{1A}$$

Define $f_2(x) = \ln\left(\frac{x}{a} + 3\right) + \ln\left(\frac{x}{a}\right)$ Done

solve $\left(\frac{d}{dx}(f_2(x)) = 1, x\right)$

$\frac{\sqrt{9a^2 + 4} + 3a - 2}{2}$ or $x = \frac{\sqrt{9a^2 + 4} - 3a + 2}{2}$

h. Solve $f_2(x) = x$ and $x = \frac{\sqrt{9a^2 + 4} - 3a + 2}{2}$ for a . **1M**

$a = 1.395\dots$

$= 1.40$ correct to two decimal places **1A**

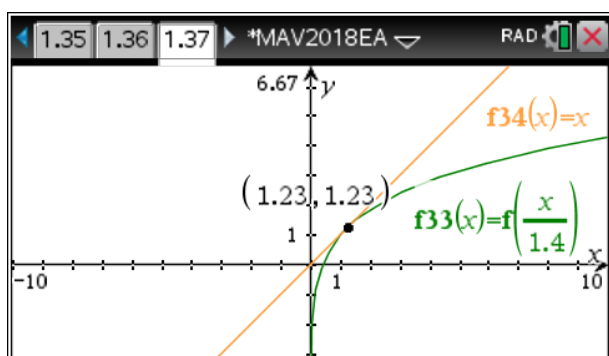
solve $\left(\frac{d}{dx}(f_2(x)) = 1, x\right)$

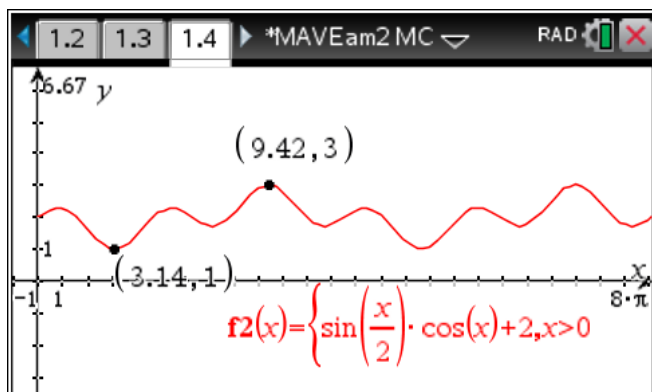
$\frac{\sqrt{9a^2 + 4} + 3a - 2}{2}$ or $x = \frac{\sqrt{9a^2 + 4} - 3a + 2}{2}$

solve $\left(f_2(x) = x \text{ and } x = \frac{\sqrt{9a^2 + 4} - 3a + 2}{2}, a\right)$

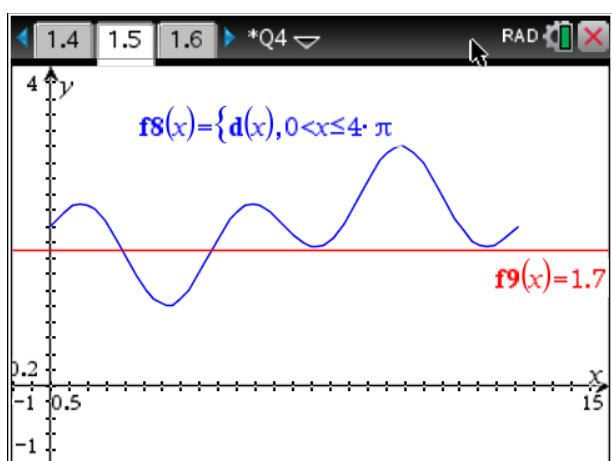
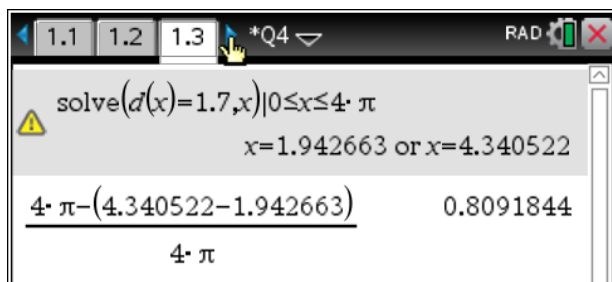
$a = 1.39546$ and $x = 1.2266$

Check with the graph.



Question 4a. period is 4π **1A**range is $[1, 3]$ **1A**b. Solve $d(t) = 1.7$ for t .

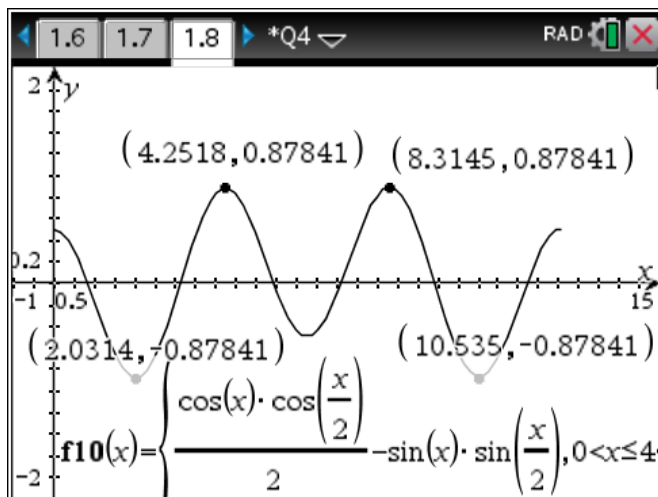
$$\frac{4\pi - (4.340\dots - 1.942\dots)}{4\pi} \quad \mathbf{1M}$$

 $= 0.80918\dots = 0.809$ correct to three decimal places **1A**

$$\mathbf{c.} \quad d'(t) = \frac{1}{2} \cos(t) \cos\left(\frac{t}{2}\right) - \sin(t) \sin\left(\frac{t}{2}\right) \quad \mathbf{1A}$$

$$t = 2.03, 4.25, 8.31, 10.54 \quad \mathbf{1A}$$

Calculator screen showing the definition of a function $d(x) = \sin\left(\frac{x}{2}\right) \cdot \cos(x) + 2$ and its derivative $\frac{d}{dx}(d(x)) = \frac{\cos(x) \cdot \cos\left(\frac{x}{2}\right)}{2} - \sin(x) \cdot \sin\left(\frac{x}{2}\right)$.



Calculator screen showing the solution to the equation $\frac{d}{dx}\left(\frac{d}{dx}(d(x))\right) = 0$ for $0 < x \leq 4\pi$. The solutions are $x = 2.03135$ or $x = 4.251835$ or $x = 6.283185$.

Calculator screen showing the solution to the equation $\frac{d}{dx}\left(\frac{d}{dx}(d(x))\right) = 0$ for $0 < x \leq 4\pi$. The solutions are $x = 8.314536$ or $x = 10.53502$ or $x = 12.566$.

d. $\theta = \tan^{-1}\left(\frac{25}{2}\right)$

$\approx 85.4^\circ$

1A

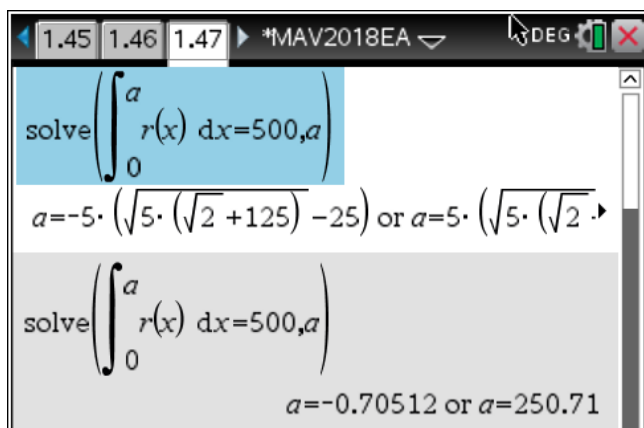
Calculator screen showing the value of $\tan^{-1}\left(\frac{25}{2}\right) \approx 85.426$.

e. Total Volume = $\frac{25 \times 10}{2}(1+3) \text{ m}^3$ **1M**

= 500 m^3 **1A**

f. Solve $\int_0^a \left(\frac{dV}{dt} \right) dt = 500$ for a . **1M**

$a = 250.71 \text{ min}$ **1A**



g. $1 \times 25 \times 10 = 250 \text{ m}^3$ out

Hence depth 2 m **1A**

h. $V_{\text{pumped}} = \int \left(\frac{dV}{dt} \right) dt$
 $= 2t + \frac{2}{2t + \sqrt{2}} + c$, substitute (0, 0)

Hence $c = -\frac{2}{\sqrt{2}} = -\sqrt{2}$ **1M**

$V_{\text{remaining}} = 500 - \left(2t + \frac{2}{2t + \sqrt{2}} - \frac{2}{\sqrt{2}} \right)$ **1H**

Average value = $\frac{1}{250.705...} \int_0^{250.705...} \left(500 - \left(2t + \frac{2}{2t + \sqrt{2}} - \frac{2}{\sqrt{2}} \right) \right) dt$ **1H**

= 250.69 m^3 correct to two decimal places **1A**

Handheld calculator screen showing the integration of a function. The display shows the integral of $r(x)$ with respect to x . The function $r(x)$ is defined as $\frac{2}{2 \cdot x + \sqrt{2}} + 2 \cdot x$. The calculator is set to RAD (Radian mode). The result of the integration is shown as $\frac{1}{250.705} \cdot \int_0^{250.705} \left(500 - \left(\frac{2}{2 \cdot x + \sqrt{2}} + 2 \cdot x - \frac{2}{\sqrt{2}} \right) dx \right)$.

Handheld calculator screen showing the evaluation of the integral. The display shows the integral of $r(x)$ with respect to x . The function $r(x)$ is defined as $\frac{2}{2 \cdot x + \sqrt{2}} + 2 \cdot x$. The calculator is set to RAD (Radian mode). The result of the integration is shown as $\frac{1}{250.705} \cdot \int_0^{250.705} \left(500 - \left(\frac{2}{2 \cdot x + \sqrt{2}} + 2 \cdot x - \frac{2}{\sqrt{2}} \right) dx \right)$. The final result is displayed as 250.6858.

END OF SOLUTIONS