

Trial Examination 2018

MATHEMATICAL METHODS

Trial Written Examination 1 - SOLUTIONS

Question 1

a. $y = 2x \log_e(2x).$

$$\frac{dy}{dx} = \left(2x \times \frac{1}{x} \right) + (\log_e(2x) \times 2) \quad 1M$$

$$\frac{dy}{dx} = 2 + 2 \log_e(2x) \quad 1A$$

b. $f(x) = \frac{e^{x^2}}{e^x + 1}.$

$$f(x) = \frac{(e^x + 1) \times e^{x^2} \times 2x - e^{x^2} \times e^x}{(e^x + 1)^2} \quad 1M$$

$$f'(0) = \frac{(e^0 + 1) \times e^0 \times 0 - e^0 \times e^0}{(e^0 + 1)^2} \quad 1M$$

$$f'(0) = \frac{0 - 1}{(2)^2} = -\frac{1}{4} \quad 1A$$

Question 2

a. $f(x) = 18x^2 - ax^4$

$$f'(x) = 36x - 4ax^3$$

$$36x - 4ax^3 = 4x(9 - ax^2) = 0 \text{ for stationary point} \quad 1M$$

$$\text{Gives } x = 0, x = \pm \frac{3}{\sqrt{a}}$$

$$\text{Given stationary point at } x = -\frac{3}{\sqrt{5}} \text{ where } a \in R^+$$

$$a = 5 \text{ as required} \quad 1M \text{ (Show that)}$$

b. Coordinates

$$(0, 0) \quad 1A$$

$$\left(\frac{3}{\sqrt{5}}, \frac{81}{5} \right), \left(-\frac{3}{\sqrt{5}}, \frac{81}{5} \right) \quad 1A$$

Question 3

a. $g: \left[\frac{1}{2}, \infty\right) \rightarrow \mathbb{R}, g(x) = \sqrt{2x-1}$

$$g(x) = (2x-1)^{\frac{1}{2}}$$

$$g(1) = (2-1)^{\frac{1}{2}} = 1$$

$$g'(x) = 2 \times \frac{1}{2} (2x-1)^{-\frac{1}{2}} = \frac{1}{\sqrt{2x-1}} \text{ using the chain rule} \quad \mathbf{1A}$$

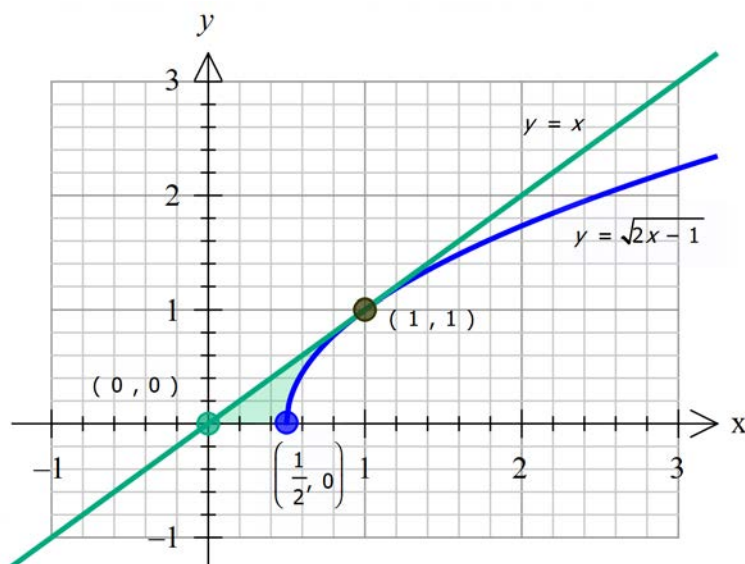
$$m = g'(1) = \frac{1}{\sqrt{2-1}} = 1$$

$$y-1 = 1 \times (x-1)$$

$$y = x$$

1M (Show that)

b.



$$A = \int_0^{\frac{1}{2}} (x) dx + \int_{\frac{1}{2}}^1 (x - \sqrt{2x-1}) dx \quad \mathbf{1M}$$

$$= \left[\frac{x^2}{2} \right]_0^{\frac{1}{2}} + \left[\frac{x^2}{2} - \frac{(2x-1)^{\frac{3}{2}}}{3} \right]_{\frac{1}{2}}^1$$

$$= \frac{1}{8} + \left(\frac{1}{2} - \frac{1}{3} \right) - \left(\frac{1}{8} - 0 \right)$$

$$= \frac{1}{6} \quad \mathbf{1A}$$

OR

$$A = \text{Area of the triangle} = \int_{\frac{1}{2}}^1 (\sqrt{2x-1}) dx \quad \mathbf{1M}$$

$$= \frac{1}{2} \left[\frac{(2x-1)^{\frac{3}{2}}}{\frac{3}{2}} \right]_{\frac{1}{2}}^1$$

$$= \frac{1}{2} - \frac{1}{3}$$

$$= \frac{1}{6} \quad \mathbf{1A}$$

Question 4

$$\mathbf{a.} \quad \hat{p} = \frac{0.132 + 0.024}{2} = \frac{0.156}{2} = 0.078 \quad \mathbf{1A}$$

$$\mathbf{b.} \quad 0.132 - 0.078 = 0.054$$

$$2\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} = 2\sigma_{\bar{x}} = 0.054 \quad \mathbf{1M}$$

$$\sigma_{\bar{x}} = 0.027 \quad \mathbf{1A}$$

Question 5

$$\mathbf{a.} \quad \sqrt{3} \sin(2x) = \cos(2x)$$

$$\frac{\sqrt{3} \sin(2x)}{\cos(2x)} = 1$$

$$\tan(2x) = \frac{1}{\sqrt{3}} \quad \mathbf{1A}$$

$$2x = \frac{\pi}{6} + n\pi, n \in \mathbb{Z}$$

$$x = \frac{\pi}{12} + \frac{n\pi}{2}, n \in \mathbb{Z} \quad \mathbf{1A}$$

$$\mathbf{b.} \quad \frac{d}{dx}(\tan(2x)) = \frac{2}{\cos^2(2x)}$$

$$y = \sqrt{3} \tan(2x)$$

$$\frac{dy}{dx} = \frac{2\sqrt{3}}{\cos^2(2x)}$$

For stationary point

$$\frac{dy}{dx} = \frac{2\sqrt{3}}{\cos^2(2x)} = 0 \text{ gives no solution for } x.$$

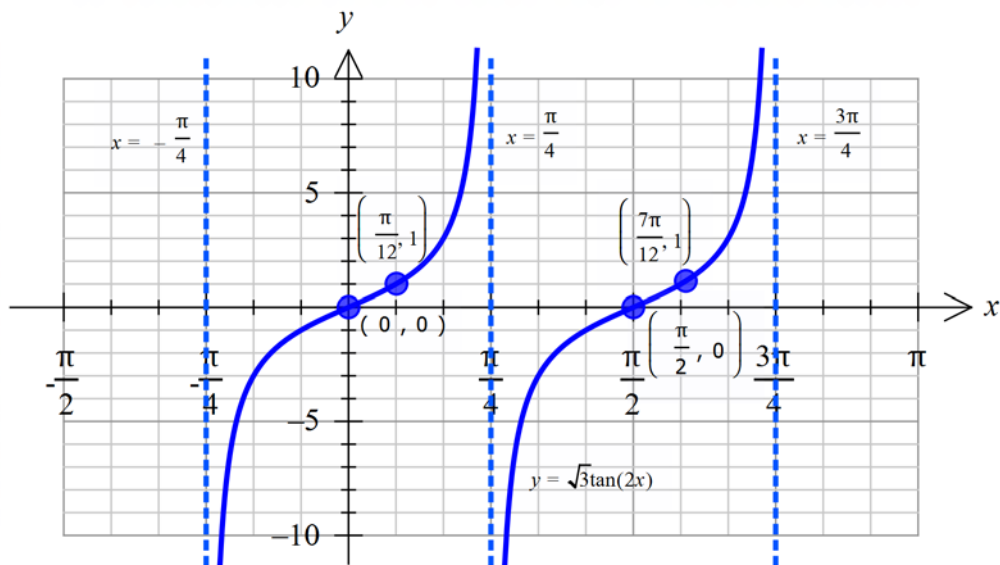
So no stationary point.

1M (Show that)

c. Shape 1A

Vertical asymptotes at $x = -\frac{\pi}{4}, \frac{\pi}{4}, \frac{3\pi}{4}$ **1A**

Points $\left(\frac{\pi}{12}, 1\right), \left(\frac{7\pi}{12}, 1\right), (0, 0), \left(\frac{\pi}{2}, 0\right)$ **1A**

**Question 6**

$$2\log_2(x-2) + \log_2(x) = 0$$

$$\log_2(x-2)^2 + \log_2(x) = 0$$

$$\log_2(x(x-2)^2) = 0 \quad \mathbf{1A}$$

$$x(x-2)^2 = 2^0 = 1$$

$$x(x^2 - 4x + 4) = 1$$

$$x^3 - 4x^2 + 4x - 1 = 0 \quad \mathbf{1A}$$

$$(x-1)(x^2 - 3x + 1) = 0 \quad \mathbf{1A}$$

$$x \neq 1, x \neq \frac{3-\sqrt{5}}{2} \text{ as } x > 2$$

$$x = \frac{3+\sqrt{5}}{2} \text{ only} \quad \mathbf{1A}$$

Question 7

$$\text{Average value} = \frac{1}{12-0} \int_0^{12} \left(-20 \sin\left(\frac{\pi t}{12} + 10\right) + 20 \right) dt \quad \mathbf{1M}$$

$$= \frac{20}{12} \int_0^{12} \left(-\sin\left(\frac{\pi t}{12} + 10\right) + 1 \right) dt$$

$$= \frac{5}{3} \int_0^{12} \left(-\sin\left(\frac{\pi t}{12} + 10\right) + 1 \right) dt$$

$$= \frac{5}{3} \left[\frac{12}{\pi} \cos\left(\frac{\pi t}{12} + 10\right) + t \right]_0^{12} \quad \mathbf{1M}$$

$$= \frac{5}{3} \left[\left(\frac{12}{\pi} \cos(\pi + 10) + 12 \right) - \left(\frac{12}{\pi} \cos(0 + 10) + 0 \right) \right]$$

Average height

$$= \frac{5}{3} \left[\left(\frac{12}{\pi} \cos(\pi + 10) + 12 \right) - \left(\frac{12}{\pi} \cos(10) \right) \right]$$

$$= \frac{20}{\pi} \cos(\pi + 10) - \frac{20}{\pi} \cos(10) + 20 \quad \mathbf{1A}$$

$$\cos(\pi + x) = -\cos(x)$$

So the expression simplifies to

$$= -\frac{20}{\pi} \cos(10) - \frac{20}{\pi} \cos(10) + 20$$

$$= -\frac{40}{\pi} \cos(10) + 20 \quad \mathbf{1A}$$

Question 8

a.i. $0.1 + a + b + 0.1 = 1$

$$a + b = 0.8, \quad b = 0.8 - a \quad \mathbf{1A}$$

ii. $E(X) = a + 2b + 0.3$

$$= a + 1.6 - 2a + 0.3$$

$$= -a + 1.9 \quad \mathbf{1M (Show that)}$$

b. $\text{Var}(X) = E(X^2) - (E(X))^2$

$$a + 4b + 0.9 - (-a + 1.9)^2 = 0.56 \quad \mathbf{1M}$$

$$a + 4b + 0.9 - (a^2 - 3.8a + 3.61) = 0.56$$

$$a^2 - 0.8a + 0.07 = 0 \quad \mathbf{1A}$$

$$(a - 0.1)(a - 0.7) = 0$$

$$a = 0.1, \quad b = 0.7 \quad \text{OR} \quad a = 0.7, \quad b = 0.1 \quad \mathbf{1A}$$

Question 9

a. Let $y = f(x) = 2e^{1-x}$

Inverse swap x and y .

$$x = 2e^{1-y} \quad \mathbf{1M}$$

$$\frac{x}{2} = e^{1-y}$$

$$1 - y = \log_e \left(\frac{x}{2} \right)$$

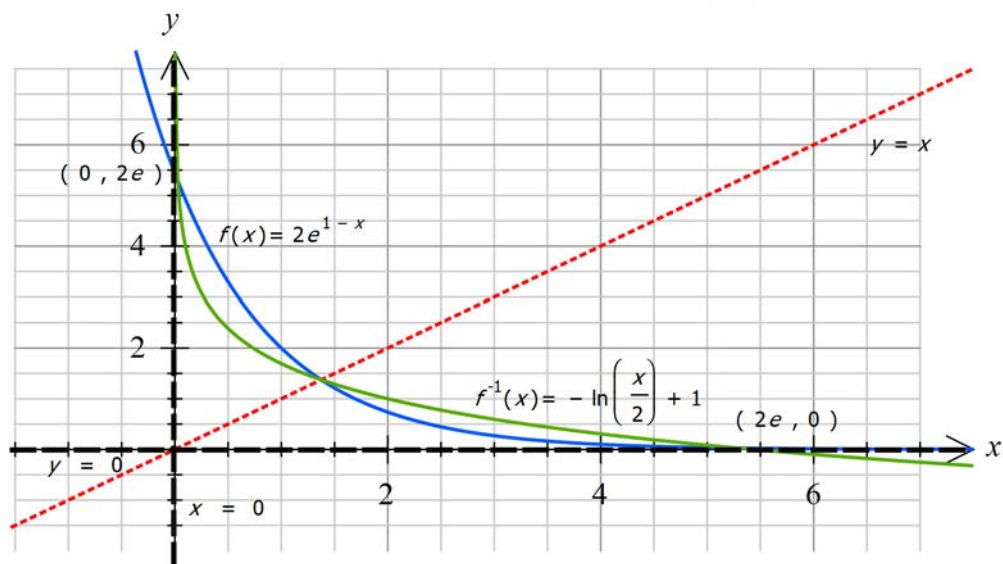
$$y = -\log_e \left(\frac{x}{2} \right) + 1$$

$$f^{-1}(x) = -\log_e \left(\frac{x}{2} \right) + 1 \quad \mathbf{1A}$$

Range is R $\mathbf{1A}$

b. Shape and three points of intersection $\mathbf{1A}$

Asymptote $x = 0$ and intercept $(2e, 0)$ $\mathbf{1A}$



END OF SOLUTIONS