

Year 2018

VCE

Mathematical Methods

Trial Examination 2

Solutions



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SECTION A

ANSWERS

1	A	B	C	D	E
2	A	B	C	D	E
3	A	B	C	D	E
4	A	B	C	D	E
5	A	B	C	D	E
6	A	B	C	D	E
7	A	B	C	D	E
8	A	B	C	D	E
9	A	B	C	D	E
10	A	B	C	D	E
11	A	B	C	D	E
12	A	B	C	D	E
13	A	B	C	D	E
14	A	B	C	D	E
15	A	B	C	D	E
16	A	B	C	D	E
17	A	B	C	D	E
18	A	B	C	D	E
19	A	B	C	D	E
20	A	B	C	D	E

SECTION A**Question 1****Answer C**

$$f(x) = 2x - a = a$$

$$2x = 2a \Rightarrow x = a \text{ included}$$

$$f(x) = 2x - a = -a$$

$$\Rightarrow x = 0 \text{ not included, so the domain is } (0, a]$$

Question 2**Answer E**

$$P(x) = (x+2)(x-4)Q(x) + a(x+2) + b$$

$$P(-2) = 6 \Rightarrow b = 6$$

$$P(4) = 12 \Rightarrow 6a + b = 12 \Rightarrow a = 1$$

$$P(x) = (x+2)(x-4)Q(x) + x + 8$$

$$\frac{P(x)}{(x+2)(x-4)} = Q(x) + \frac{x+8}{(x+2)(x-4)} \text{ the remainder is } x+8.$$

Question 3**Answer D**

All of the functions **A. B. C.** and **E.** join up when $x = 2$,

D. $\log_2(3x-5)$ when $x = 2$, gives $\log_2(1) = 0$

Question 4**Answer B**

$$3x - (k+2)y = 2 \Rightarrow y = \frac{3x}{k+2} - \frac{2}{k+2} \text{ and } -2kx + 10y = k-7 \Rightarrow y = \frac{kx}{5} + \frac{k-7}{10}$$

$$\text{equal gradients when } \frac{3}{k+2} = \frac{k}{5} \Rightarrow k(k+2) = 15 \Rightarrow k^2 + 2k - 15 = (k+5)(k-3) = 0$$

There is a unique solution when $k \in \mathbb{R} \setminus \{-5, 3\}$, Colin is correct.

When $k = 3$ the equations become $3x - 5y = 2$ and $-6x + 10y = -4$,

these are multiples, so there is an infinite number of solution when $k = 3$.

When $k = -5$ the equations become $3x + 3y = 2$ and $10x + 10y = -12$,

these are inconsistent, the lines are parallel with different y intercepts,

so there is no solution when $k = -5$, Ben is correct.

Question 5**Answer E**

$$h(x) = f(g(x)) \Rightarrow h(1) = f(g(1)) = f(2) = 3$$

$$h'(x) = g'(x) f'(g(x))$$

$$h'(1) = g'(1) f'(g(1)) = g'(1) f'(2) = -2 \times -4 = 8$$

Question 6**Answer D**

$f : [0, b] \rightarrow \mathbb{R}$, $f(x) = x(2b - x) = 2bx - x^2$ where $b > 0$.

$f(b) = b^2$ so the range is $[0, b^2]$ Alan is incorrect.

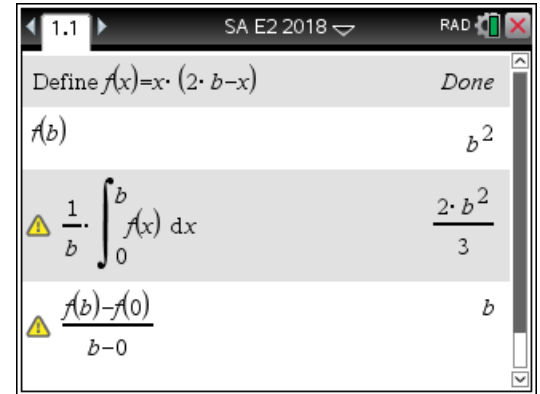
The average value of the function is

$$\bar{f} = \frac{1}{b-0} \int_0^b x(2b-x) dx = \frac{2b^2}{3}$$

The average rate of change of the function is

$$\frac{f(b) - f(0)}{b-0} = \frac{b^2}{b} = b, \text{ so Ben is correct.}$$

The gradient $f' : (0, b) \rightarrow \mathbb{R}$, $f'(x) = 2b - 2x = 2(b - x)$, since $0 < x < b$, $f'(x) > 0$ the function is always increasing and that the gradient at an end-point $x = b$ does not exist, David is correct.

**Question 7****Answer A**

6 B, 5R, total 11, at least one of each colour

$\Pr(RBB + BRB + BBR + BRR + RBR + RRB)$

$$= 3\Pr(RBB) + 3\Pr(BRR) = 3 \times \frac{5}{11} \times \frac{6}{10} \times \frac{5}{9} + 3 \times \frac{6}{11} \times \frac{5}{10} \times \frac{4}{9} = \frac{9}{11}$$

Question 8**Answer A**

The graph has a vertical asymptote at $x = -a < 0$ so $a > 0$.

The graph crosses the x-axis when $y = 0$, at a negative value

$$\Rightarrow \log_e(x+a) + b = 0 \Rightarrow \log_e(x+a) = -b \quad x+a = e^{-b}, \quad x = e^{-b} - a < 0 \Rightarrow a > e^{-b}$$

The graph crosses the y-axis when $x = 0$, at a positive value

$$\Rightarrow \log_e(a) + b > 0 \Rightarrow \log_e(a) > -b \quad a > e^{-b}$$

Question 9**Answer B**

$$\int_{-1}^2 (3x^2 - 2f(x) + g(x)) dx = 18$$

$$\int_{-1}^2 3x^2 dx - 2 \int_{-1}^2 f(x) dx + \int_{-1}^2 g(x) dx = 18$$

$$\left[x^3 \right]_{-1}^2 + 2 \int_2^{-1} f(x) dx - \int_2^{-1} g(x) dx = 18$$

$$(8 - (-1)) + 2 \int_2^{-1} f(x) dx - 3 = 18$$

$$2 \int_2^{-1} f(x) dx = 12$$

$$\int_2^{-1} f(x) dx = 6$$

Question 10

Answer D

$$f(x) = kx^n, \quad g(x) = \log_e(x)$$

$$f(a) = g(a) \Rightarrow ka^n = \log_e(a) \Rightarrow k = \frac{\log_e(a)}{a^n} \quad \text{A. is true}$$

$$f'(x) = nkx^{n-1}, \quad g'(x) = \frac{1}{x}$$

$$f'(a) = g'(a) \Rightarrow nka^{n-1} = \frac{1}{a} \Rightarrow nka^n = 1 \Rightarrow k = \frac{1}{na^n} \quad \text{B. is true}$$

$$a^n = \frac{1}{nk}, \quad \log_e(a) = k a^n = \frac{1}{n} \Rightarrow a = e^{\frac{1}{n}} \Rightarrow a^n = e \Rightarrow \frac{1}{nk} = e \Rightarrow k = \frac{1}{en} \quad \text{C. is true}$$

$$a^n = \frac{1}{k} \log_e(a) = \frac{1}{nk} \Rightarrow n = \frac{1}{\log_e(a)} = \log_a(e) \quad \text{E. is true} \quad \text{D. is false}$$

Question 11

Answer C

$$y_1 = kx + 1 \text{ and } y_2 = x^2 - kx + 2$$

$$y_1 = y_2 \Rightarrow kx + 1 = x^2 - kx + 2$$

$$x^2 - 2kx + 1 = 0$$

$$\Delta = 4k^2 - 4 = 4(k^2 - 1)$$

$$\text{do not intersect when } \Delta < 0 \Rightarrow k^2 - 1 < 0 \Rightarrow -1 < k < 1$$



Question 12

Answer E

$$f: (-\infty, a) \rightarrow \mathbb{R}, \quad f(x) = 2ax - x^2 = x(2a - x)$$

$$f: \quad y = 2ax - x^2$$

$$f^{-1} \quad x = 2ay - y^2$$

$$y^2 - 2ay + a^2 = a^2 - x$$

$$(y - a)^2 = a^2 - x$$

$$y - a = \pm \sqrt{a^2 - x} \quad \text{take negative since } \text{dom } f^{-1} = \text{ran } f = (-\infty, a^2)$$

$$y = f^{-1}(x) = a - \sqrt{a^2 - x}$$

$$f^{-1}: (-\infty, a^2) \rightarrow \mathbb{R}, \quad f^{-1}(x) = a - \sqrt{a^2 - x}$$

Question 13

Answer B

If $f(x)$ is a non-zero even function, then $f(x) = f(-x)$, the graph is symmetrical about the y-axis so $\int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx$. If $g(x)$ is a non-zero odd function, then $g(x) = -g(-x)$,

then $\int_{-a}^a g(x) dx = 0$, so $\int_{-a}^a (f(x) + g(x)) dx = 2 \int_0^a f(x) dx$

Question 14

Answer B

$$\frac{dy}{dx} = 1 - 4 \sin\left(\frac{x}{2}\right)$$

$$y = \int \left(1 - 4 \sin\left(\frac{x}{2}\right)\right) dx = x + 8 \cos\left(\frac{x}{2}\right) + c$$

to find c , $x = 2\pi$, $y = 0$

$$\Rightarrow 0 = 2\pi + 8 \cos(\pi) + c \Rightarrow c = 8 - 2\pi$$

$$y = x + 8 \cos\left(\frac{x}{2}\right) + 8 - 2\pi$$

Question 15

Answer E

$$f(x) = \frac{1}{1+x^2} \quad \text{A. B. C. D. are all true}$$

E. is false

$$\left(f(\sqrt{x})\right)^2 = \left(\frac{1}{1+x}\right)^2 = \frac{1}{(1+x)^2} \neq f(x) = \frac{1}{1+x^2}$$

Calculator screenshot showing the integration of $1 - 4 \sin\left(\frac{x}{2}\right)$ to get $x + 8 \cos\left(\frac{x}{2}\right) + c$. Then, it solves for c using the condition $y(2\pi) = 0$, resulting in $c = 8 - 2\pi$.

Calculator screenshot showing the definition of $f(x) = \frac{1}{1+x^2}$ and testing various properties: $f\left(\frac{1}{x}\right) = x^2 \cdot f(x)$, $f\left(\frac{1}{x}\right) + f(x) = 1$, and $f\left(\frac{1}{x}\right) \cdot f(x) = x^2 \cdot (f(x))^2$.

Question 16

Answer A

The graph of $y = \cos(x)$ is transformed into the graph of

$$y = -3 \sin(x) = -3 \cos\left(\frac{\pi}{2} - x\right) = -3 \cos\left(x - \frac{\pi}{2}\right) \quad \text{since } \cos(-x) = \cos(x) \text{ by}$$

A dilation by a scale factor of 3 parallel to the y -axis, a reflection in the x -axis,

and a translation of $\frac{\pi}{2}$ to the right parallel to the x -axis.

Question 17

Answer C

$$\Pr(A) = \sqrt{p}, \quad \Pr(B) = 10p, \quad \Pr(A|B) = \frac{1}{4} = \frac{\Pr(A \cap B)}{\Pr(B)} = \frac{\Pr(A \cap B)}{10p} \Rightarrow \Pr(A \cap B) = \frac{5p}{2}$$

	A	A'	
B	$\frac{5p}{2}$	$10p - \frac{5p}{2}$	$10p$
B'	$\sqrt{p} - \frac{5p}{2}$	$\Pr(A' \cap B')$	$1 - 10p$
	$\sqrt{p}$	$1 - \sqrt{p}$	

Calculator screenshot showing the solving of the equation $1 - \sqrt{p} - \left(10p - \frac{5p}{2}\right) = 1 - 10p$ to find $p = 0.04$.

$$\Pr(A' \cap B') = \frac{1}{2} = 1 - \sqrt{p} - \left(10p - \frac{5p}{2}\right) = 1 - 10p - \left(\sqrt{p} - \frac{5p}{2}\right) = 1 - \sqrt{p} - \frac{15p}{2}$$

$$\text{solving } 1 - \sqrt{p} - \frac{15p}{2} = \frac{1}{2} \quad \text{gives } p = 0.04$$

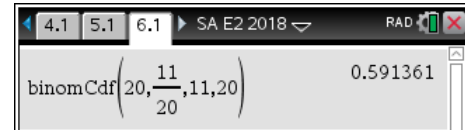
Question 18**Answer D**

55% have blue eyes. $p = \frac{11}{20} = 0.55$

$$\hat{P} = \frac{X}{20} \quad X \stackrel{d}{=} \text{Bi}\left(n = 20, p = \frac{11}{20}\right)$$

$$\Pr(\hat{P} > 0.5) = \Pr(X > 10)$$

$$= \Pr(X \geq 11) = 0.5914$$

**Question 19****Answer C**

Given $\Pr(a < Z < b) = P$ and $\Pr(Z > a) = A \Rightarrow \Pr(Z < a) = 1 - A$

since $b > a$, $\Pr(Z < a) + \Pr(a < Z < b) + \Pr(Z > b) = 1$, then

$$\Pr(Z > b) = 1 - (\Pr(a < Z < b) + \Pr(Z < a)) = 1 - P - (1 - A) = A - P$$

$$\Pr(Z > b | Z > a) = \frac{\Pr(Z > b)}{\Pr(Z > a)} = \frac{A - P}{A} = 1 - \frac{P}{A}$$

Question 20**Answer A**

$$\sum \Pr(X = x) = 1 \quad \Rightarrow \quad a + b + c = 1 \quad (1)$$

$$E(X) = \sum x \Pr(X = x) = \frac{1}{2} \quad \Rightarrow \quad b + 2c = \frac{1}{2} \quad (2)$$

$$E(X^2) = \sum x^2 \Pr(X = x) \Rightarrow E(X^2) = b + 4c$$

$$\text{var}(X) = E(X^2) - (E(X))^2 = b + 4c - \left(\frac{1}{2}\right)^2 = b + 4c - \frac{1}{4} = \frac{1}{2} \quad \Rightarrow \quad b + 4c = \frac{3}{4} \quad (3)$$

$$(3) - (2) \quad 2c = \frac{3}{4} - \frac{1}{2} = \frac{1}{4} \quad \Rightarrow \quad c = \frac{1}{8} \quad \text{into}$$

$$(2) \quad b = \frac{1}{2} - 2c = \frac{1}{2} - \frac{1}{4} = \frac{1}{4} \quad (1) \quad a = 1 - b - c = 1 - \frac{1}{4} - \frac{1}{8} = \frac{5}{8}$$

$$a = \frac{5}{8}, \quad b = \frac{1}{4}, \quad c = \frac{1}{8}$$

END OF SECTION A SUGGESTED ANSWERS

SECTION B

Question 1

- a. the maximum on $f(x) = 5 + \sin\left(\frac{\pi x}{6}\right)$ is when $\sin\left(\frac{\pi x}{6}\right) = 1 \Rightarrow x = 3$

$$f(3) = 6 \quad N(3, 6), \text{ centre of the circle } C(4, 3) \quad \text{A1}$$

$$d(CN) = \sqrt{(4-3)^2 + (3-6)^2} = \sqrt{10} \quad \text{A1}$$

- b.i. $g(x) = 2\sin^2\left(\frac{\pi x}{4}\right) \quad g(5) = 2\sin^2\left(\frac{5\pi}{4}\right) = 2 \times \left(-\frac{1}{\sqrt{2}}\right)^2 = 1 \quad S(5, 1)$

$$g'(x) = \pi \sin\left(\frac{\pi x}{4}\right) \cos\left(\frac{\pi x}{4}\right)$$

$$g'(5) = \pi \sin\left(\frac{5\pi}{4}\right) \cos\left(\frac{5\pi}{4}\right) = \pi \times \left(-\frac{1}{\sqrt{2}}\right)^2 = \frac{\pi}{2} \quad \text{A1}$$

$$T: \quad y - 1 = \frac{\pi}{2}(x - 5)$$

$$T: \quad y = \frac{\pi x}{2} + 1 - \frac{5\pi}{2} = \frac{\pi x}{2} + \frac{2 - 5\pi}{2} \quad \text{A1}$$

Define $f1(x) = 5 + \sin\left(\frac{\pi \cdot x}{6}\right)$	Done
Define $f2(x) = 2 \cdot \left(\sin\left(\frac{\pi \cdot x}{4}\right)\right)^2$	Done
$f2(5)$	1
$\frac{d}{dx}(f2(x))$	$\pi \cdot \sin\left(\frac{\pi \cdot x}{4}\right) \cdot \cos\left(\frac{\pi \cdot x}{4}\right)$
$\frac{d}{dx}(f2(x)) _{x=5}$	$\frac{\pi}{2}$
tangentLine($f2(x), x, 5$)	$\frac{\pi \cdot x}{2} - \frac{5 \cdot \pi - 2}{2}$

ii. solving $\frac{\pi x}{2} + \frac{2-5\pi}{2} = 5 + \sin\left(\frac{\pi x}{6}\right)$ with $0 < x < 12$ M1

gives $x = 7.178$ $f(7.178) = 4.421$

$B(7.178, 4.421)$ A1

iii. $d(BS) = \sqrt{(7.178-5)^2 + (4.421-1)^2}$
 $= 4.056$ A1

Define $t(x) = \text{tangentLine}(f2(x), x, 5)$	Done
 solve($t(x) = f1(x), x$) $0 < x < 12$	$x = 7.17819$
$t(7.178189371571)$	4.42149
$f1(7.178189371571)$	4.42149
$\sqrt{(7.17819-5)^2 + (1-4.4215)^2}$	4.056

c.i. The circular island has a radius of 1 and an area of π .
 The area of water in the pool is

$\int_0^{12} \left(5 + \sin\left(\frac{\pi x}{6}\right) - 2 \sin^2\left(\frac{\pi x}{4}\right) \right) dx - \pi$ A1

ii. $\int_0^{12} \left(5 + \sin\left(\frac{\pi x}{6}\right) - 2 \sin^2\left(\frac{\pi x}{4}\right) \right) dx - \pi = 48 - \pi$
 $V = \frac{3}{2}(48 - \pi) = 72 - \frac{3\pi}{2} \text{ m}^3$ A1

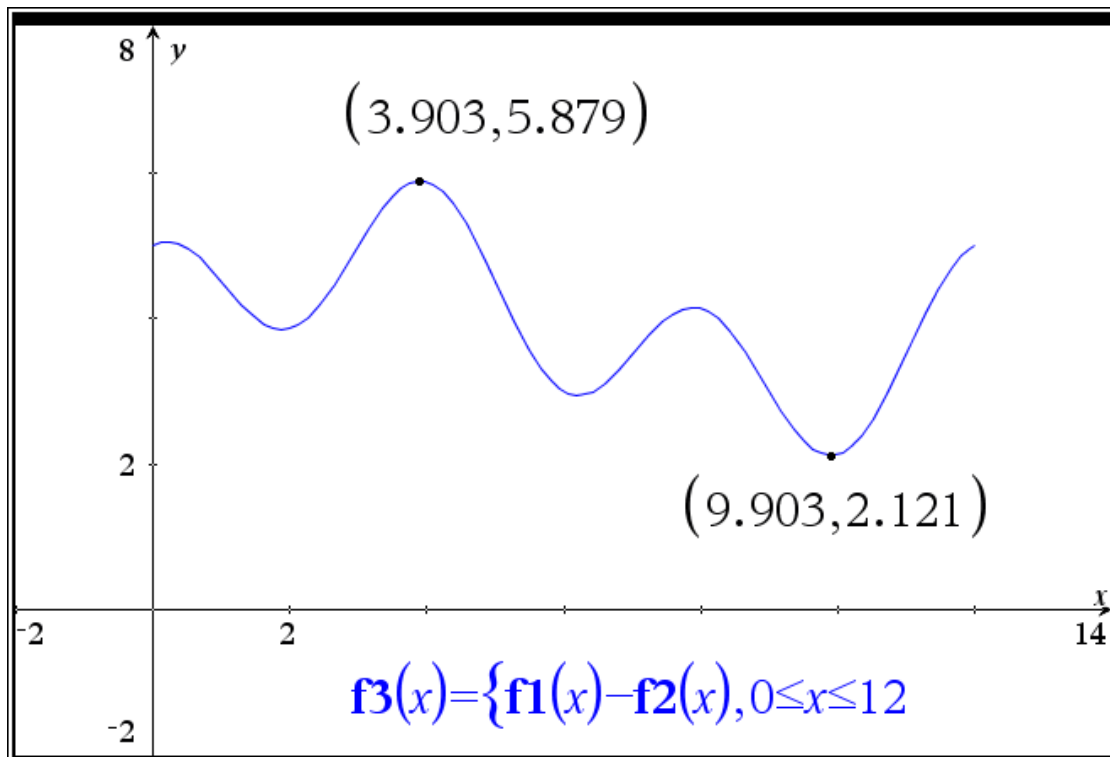
$\int_0^{12} (f1(x) - f2(x)) dx$	48
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d. $s(x) = f(x) - g(x) = 5 + \sin\left(\frac{\pi x}{6}\right) - 2\sin^2\left(\frac{\pi x}{4}\right)$

graphically find the maximum and minimum of s , or using $\frac{ds}{dx} = 0$ M1

the maximum value of s occurs when $x = 3.903$ and the maximum width is
 $s(3.903) = 5.879$ m A1

the minimum value of s occurs when $x = 9.903$ and the minimum width is
 $s(9.903) = 2.121$ m A1



Question 2

a.i. $f(x) = x^3 - 6x^2 + cx + d$

$$f'(x) = 3x^2 - 12x + c$$

$$\Delta = 144 - 12c > 0 \quad \text{for two stationary points}$$

$$c < 12$$

A1

ii. Given that $c = 9$

$$f'(x) = 3x^2 - 12x + 9 = 3(x^2 - 4x + 3) = 3(x-3)(x-1) = 0$$

for two stationary points $x = 1, x = 3$

A1

$$f(1) = 1 - 6 + 9 + d = 4 + d$$

M1

$$f(3) = 27 - 54 + 27 + d = d$$

so stationary points at $(1, 4 + d)$ is the maximum turning

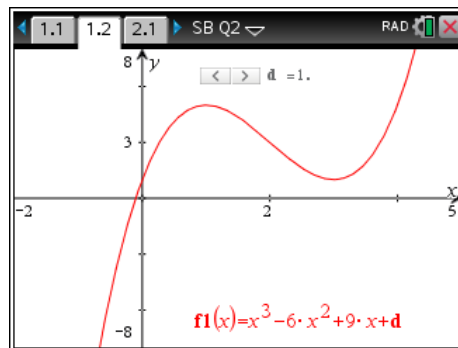
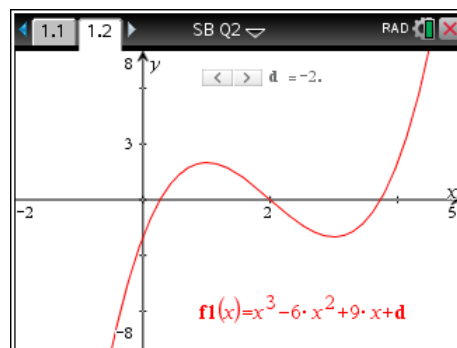
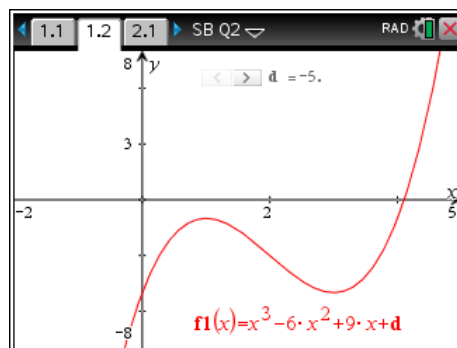
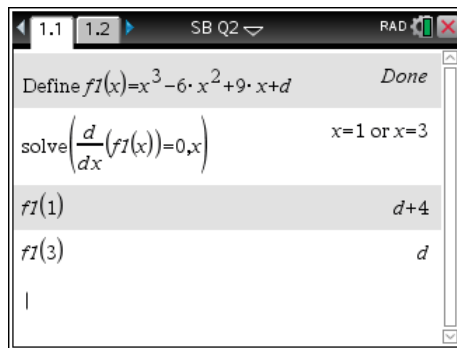
and $(3, d)$ is the minimum turning point.

The graph crosses the x -axis three times if

$$4 + d > 0 \quad \text{and} \quad d < 0 \quad \text{that is}$$

$$-4 < d < 0$$

A1



iii. If the stationary point is at $x = 1$ then from **a.ii.** $c = 9$

one stationary point is $(1, 4 + d) = (1, 9)$ so

$d = 5$ and the other stationary point is $(3, 5)$

A1

b.i. $f'(x) = 3x^2 - 12x + c$

$$\Delta = 144 - 12c = 0 \quad \text{for one stationary point}$$

$$c = 12$$

A1

ii. $f(x) = x^3 - 6x^2 + cx + d$

$$f(x+p) = x^3$$

$$f(x+p) = (x+p)^3 - 6(x+p)^2 + c(x+p) + d$$

$$= x^3 + 3x^2p + 3xp^2 + p^3 - 6x^2 - 12xp - 6p^2 + cx + cp + d$$

$$= x^3 + (3p-6)x^2 + (c+3p^2-12p)x + cp + d + p^3 - 6p^2$$

$$= x^3$$

M1

equation coefficients

$$(1) \quad 3p - 6 = 0 \Rightarrow p = 2$$

$$(2) \quad c + 3p^2 - 12p = 0$$

$$c = 12p - 3p^2 = 24 - 12$$

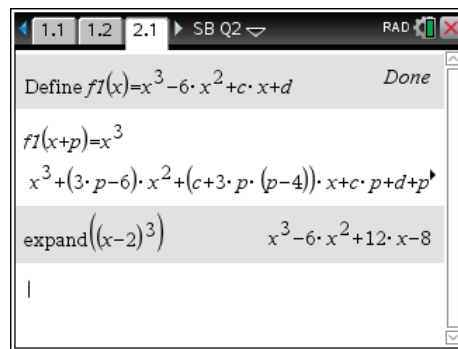
$$\Rightarrow c = 12$$

$$(3) \quad cp + d + p^3 - 6p^2 = 0$$

$$\Rightarrow d = 6p^2 - p^3 - cp = 24 - 8 - 24 = -8$$

$$p = 2, c = 12, d = -8$$

$$\text{alternatively } f(x) = (x-2)^3 = x^3 - 6x^2 + 12x - 8$$



A1

c. crosses the y-axis at $y = -4 \Rightarrow d = -4$, $f(x) = x^3 - 6x^2 + cx - 4$

$$f(a) = a^3 - 6a^2 + ac - 4 \quad \text{the tangent is } t(x) = -3x + 4$$

$$\text{so } t(a) = 4 - 3a \quad \text{and } t(a) = f(a)$$

$$(1) \quad a^3 - 6a^2 + ac - 4 = 4 - 3a$$

$$f'(x) = 3x^2 - 12x + c \quad \text{the tangent has a gradient of } -3, \text{ therefore}$$

M1

$$(2) \quad f'(a) = 3a^2 - 12a + c = -3$$

$$\text{solving (1) and (2) gives } a = 2 \text{ and } c = 9 \quad \text{or } a = -1 \text{ and } c = -18$$

A1

both pairs of answers are acceptable.

$$\text{tangentLine}(f1(x), x, a) \quad (3 \cdot a^2 - 12 \cdot a + c) \cdot x - 2 \cdot (a^3 - 3 \cdot a^2 + 2)$$

$$\text{solve}(3 \cdot a^2 - 12 \cdot a + c = -3 \text{ and } -2 \cdot (a^3 - 3 \cdot a^2 + 2) = 4, \{a, c\})$$

$$a = -1 \text{ and } c = -18 \text{ or } a = 2 \text{ and } c = 9$$

Define $f1(x)=x^3-6\cdot x^2+c\cdot x-4$	Done
$f1(a)$	$a^3-6\cdot a^2+a\cdot c-4$
Define $t(x)=4-3\cdot x$	Done
$t(a)$	$4-3\cdot a$
$f1(a)=t(a)$	$a^3-6\cdot a^2+a\cdot c-4=4-3\cdot a$
$eq1:=a^3-6\cdot a^2+a\cdot c-4=13-3\cdot a$	$a^3-6\cdot a^2+a\cdot c-4=13-3\cdot a$
$\frac{d}{dx}(f1(x)) _{x=a}$	$3\cdot a^2-12\cdot a+c$
$eq2:=3\cdot a^2-12\cdot a+c=-3$	$3\cdot a^2-12\cdot a+c=-3$
$solve(eq1 \text{ and } eq2, \{a, c\})$	$a=-1 \text{ and } c=-18 \text{ or } a=2 \text{ and } c=9$

d.i. $f(x) = x^3 - 6x^2 + cx + d$, $A = \int_0^2 (x^3 - 6x^2 + cx + d) dx$

using four equally spaced intervals, $h = \frac{1}{2}$

$$(1) \quad L = \frac{1}{2} \left[f(0) + f\left(\frac{1}{2}\right) + f(1) + f\left(\frac{3}{2}\right) \right] = 15\frac{1}{4} = \frac{61}{4} \quad A1$$

$$\Rightarrow 6c + 8d - 33 = 61$$

$$(2) \quad R = \frac{1}{2} \left[f\left(\frac{1}{2}\right) + f(1) + f\left(\frac{3}{2}\right) + f(2) \right] = 16\frac{1}{4} = \frac{65}{4} \quad A1$$

$$\Rightarrow 10c + 8d - 65 = 65$$

$$\text{alternatively } (2) - (1) \Rightarrow \frac{1}{2} (f(2) - f(0)) = 1$$

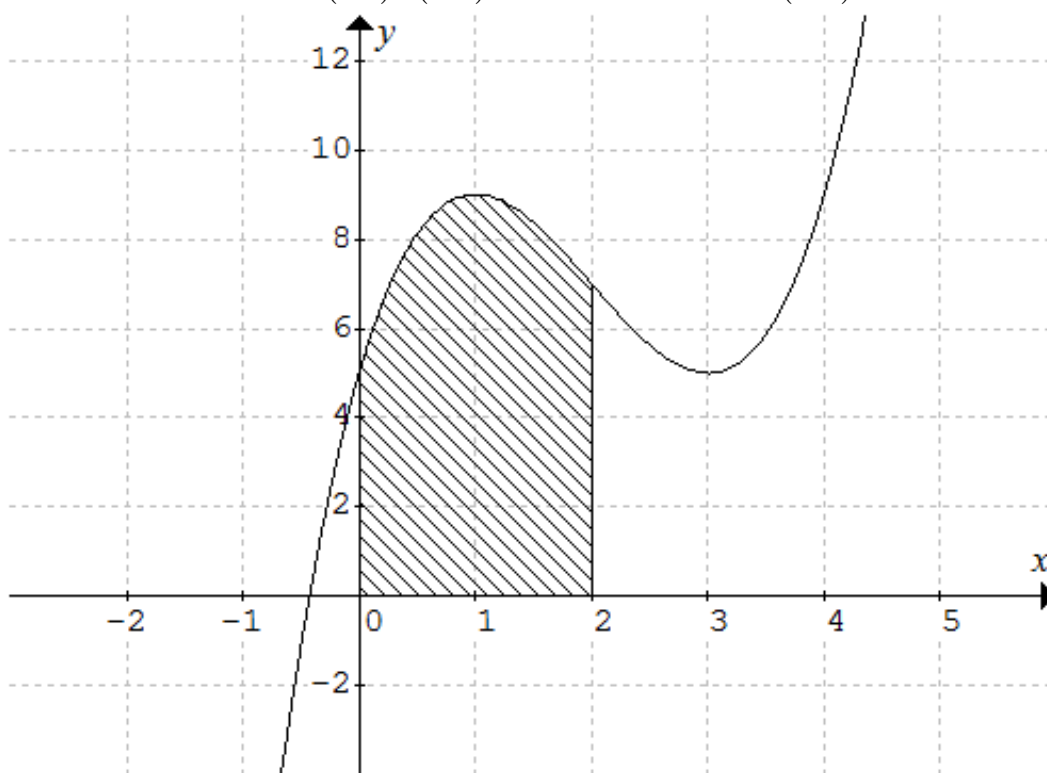
$$f(2) - f(0) = 8 - 24 + 2c + d - d = 2c - 16 = 2$$

or solving (1), (2)

$$c = 9, \quad d = 5, \quad A = \int_0^2 (x^3 - 6x^2 + 9x + 5) dx = 16 \quad A1$$

- ii. stationary points $(1,9)$, $(3,5)$ and the y-intercept at $(0,5)$

G1



Define $f(x)=x^3-6 \cdot x^2+c \cdot x+d$		Done
$\frac{1}{2} \cdot \left(f(0)+f\left(\frac{1}{2}\right)+f(1)+f\left(\frac{3}{2}\right) \right) = \frac{61}{4}$	$\frac{6 \cdot c+8 \cdot d-33}{4} = \frac{61}{4}$	
$eq1:=\left(\frac{6 \cdot c+8 \cdot d-33}{4} = \frac{61}{4} \right) \cdot 4$	$6 \cdot c+8 \cdot d-33=61$	
$\frac{1}{2} \cdot \left(f\left(\frac{1}{2}\right)+f(1)+f\left(\frac{3}{2}\right)+f(2) \right) = \frac{65}{4}$	$\frac{10 \cdot c+8 \cdot d-65}{4} = \frac{65}{4}$	
$eq2:=\left(\frac{10 \cdot c+8 \cdot d-65}{4} = \frac{65}{4} \right) \cdot 4$	$10 \cdot c+8 \cdot d-65=65$	
$\text{solve}(eq1 \text{ and } eq2, \{c,d\})$	$c=9 \text{ and } d=5$	
$\int_0^2 f(x) dx c=9 \text{ and } d=5$	16	

Question 3

a. Let the burning time in minutes be $B \stackrel{d}{=} N(60, 10^2)$

i. $\Pr(45 \leq B \leq 55) = 0.2417$ A1

ii. $\Pr(B \leq 45) = 0.0668$ A1

iii. For the cake $C \stackrel{d}{=} \text{Bi}(n=10, p=0.0668)$

$$\begin{aligned} \Pr(C \geq 1) &= 1 - \Pr(C = 0) = 1 - (1 - 0.0668)^{10} \\ &= 0.4991 \end{aligned}$$
A1

$\text{normCdf}(45, 55, 60, 10)$	0.24173
$\text{normCdf}(-\infty, 45, 60, 10)$	0.066807
$p := 0.06680722872027$	0.066807
$\text{binomCdf}(10, p, 1, 10)$	0.499143
$1 - (1 - 0.0668)^{10}$	0.499104

b.i. $\hat{P} = \frac{S}{80}$ $S \stackrel{d}{=} \text{Bi}(n=80, p=0.0668)$

$$\hat{P} = 0.1 \Rightarrow S = 8, \hat{P} = 0.05 \Rightarrow S = 4$$
M1

$$\Pr(\hat{P} \geq 0.1 | \hat{P} \geq 0.05)$$

$$= \Pr(S \geq 8 | S \geq 4) = \frac{\Pr(S \geq 8)}{\Pr(S \geq 4)} = \frac{0.16486}{0.789845}$$

$$= 0.2087$$
A1

$\text{binomCdf}(80, p, 8, 80)$	0.164864
$\text{binomCdf}(80, p, 4, 80)$	0.789845
$\frac{0.16486440850829}{0.7898451707682}$	0.20873

ii. $\hat{p} = \frac{16}{80} = \frac{1}{5} = 0.2 \quad z_{0.9} = 1.645$

$$\begin{aligned} & \hat{p} \pm z \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \\ &= 0.2 \pm 1.645 \sqrt{\frac{0.2 \times 0.8}{80}} \\ &= (0.126, 0.274) \end{aligned}$$

A1

invNorm(0.95,0,1)	1.64485												
$0.2 + 1.6448 \cdot \sqrt{\frac{0.2 \cdot 0.8}{80}}$	0.273558												
$0.2 - 1.6448 \cdot \sqrt{\frac{0.2 \cdot 0.8}{80}}$	0.126442												
zInterval_1Prop 16,80,0.9: stat.results													
<table> <tr> <td>"Title"</td><td>"1-Prop z Interval"</td></tr> <tr> <td>"CLower"</td><td>0.12644</td></tr> <tr> <td>"CUpper"</td><td>0.27356</td></tr> <tr> <td>"p"</td><td>0.2</td></tr> <tr> <td>"ME"</td><td>0.07356</td></tr> <tr> <td>"n"</td><td>80.</td></tr> </table>		"Title"	"1-Prop z Interval"	"CLower"	0.12644	"CUpper"	0.27356	"p"	0.2	"ME"	0.07356	"n"	80.
"Title"	"1-Prop z Interval"												
"CLower"	0.12644												
"CUpper"	0.27356												
"p"	0.2												
"ME"	0.07356												
"n"	80.												

c. long life candles $L \stackrel{d}{=} N(\mu = ?, \sigma^2 = ?)$

$$\Pr(L > 252) = 0.25 \Rightarrow \Pr(L < 252) = 0.75$$

M1

$$(1) \frac{252 - \mu}{\sigma} = 0.675$$

$$\Pr(L < 246) = 0.09$$

$$(2) \frac{246 - \mu}{\sigma} = -1.3408$$

A1

$$\text{solving (1), (2) gives } \mu = 250, \sigma = 3 \text{ minutes}$$

A1

$\text{invNorm}(0.75,0,1)$	0.67449
$\text{invNorm}(0.09,0,1)$	-1.34076
$\frac{246-m}{s} = -1.3408$	$\frac{246-m}{s} = -1.3408$
$\frac{252-m}{s} = 0.675$	$\frac{252-m}{s} = 0.675$
$\text{solve}\left(\frac{246-m}{s} = -1.3408 \text{ and } \frac{252-m}{s} = 0.675, \{m, s\}\right)$	
$s = 2.97649 \text{ and } m = 249.991$	

d. I ice-cream cake, C chocolate cake

$$I \rightarrow I = 0.65 \Rightarrow I \rightarrow C = 0.35$$

$$C \rightarrow C = 0.55 \Rightarrow C \rightarrow I = 0.45$$

$$\Pr(I I I C) + \Pr(I I C I) + \Pr(I C I I) \quad \text{M1}$$

$$= 0.65^2 \times 0.35 + 0.65 \times 0.35 \times 0.45 + 0.35 \times 0.45 \times 0.65$$

$$= 0.3526 \quad \text{A1}$$

$$\text{e.i} \quad \Pr(T > 2 | T > 1) = \frac{\Pr(T > 2)}{\Pr(T > 1)} = \frac{\frac{9}{16} \int_2^\infty t e^{-\frac{3t}{4}} dt}{\frac{9}{16} \int_1^\infty t e^{-\frac{3t}{4}} dt} \quad \text{M1}$$

$$= \frac{10}{7} e^{-\frac{3}{4}} \quad \text{A1}$$

$$\text{ii.} \quad E(T) = \frac{9}{16} \int_0^\infty t^2 e^{-\frac{3t}{4}} dt = \frac{8}{3} \quad \text{A1}$$

$$E(T^2) = \frac{9}{16} \int_0^\infty t^3 e^{-\frac{3t}{4}} dt = \frac{32}{3}$$

$$\text{var}(T) = E(T^2) - (E(T))^2 = \frac{32}{3} - \left(\frac{8}{3}\right)^2$$

$$\text{var}(T) = \frac{32}{9} = 3\frac{5}{9} \quad \text{A1}$$

$$\text{iii.} \quad \text{Solving } \frac{9}{16} \int_0^m t e^{-\frac{3t}{4}} dt = 0.5 \text{ gives } m = 2.238 \quad \text{A1}$$

Done

Define $f(t) = \frac{9}{16} \cdot t \cdot e^{\frac{-3 \cdot t}{4}}$

$\int_0^{\infty} f(t) dt$	1
$\int_2^{\infty} f(t) dt$	$\frac{5 \cdot e^{\frac{-3}{2}}}{2}$
$\int_1^{\infty} f(t) dt$	$\frac{7 \cdot e^{\frac{-3}{4}}}{4}$

$\frac{5 \cdot e^{\frac{-3}{2}}}{2}$	$\frac{10 \cdot e^{\frac{-3}{4}}}{7}$
$\frac{7 \cdot e^{\frac{-3}{4}}}{4}$	

$\int_0^{\infty} (t \cdot f(t)) dt$	$\frac{8}{3}$
$\int_0^{\infty} (t^2 \cdot f(t)) dt$	$\frac{32}{3}$
$\frac{32}{3} - \left(\frac{8}{3}\right)^2$	$\frac{32}{9}$

⚠ solve $\left(\int_0^m f(t) dt = 0.5, m \right) | m > 0$ $m = 2.2378$

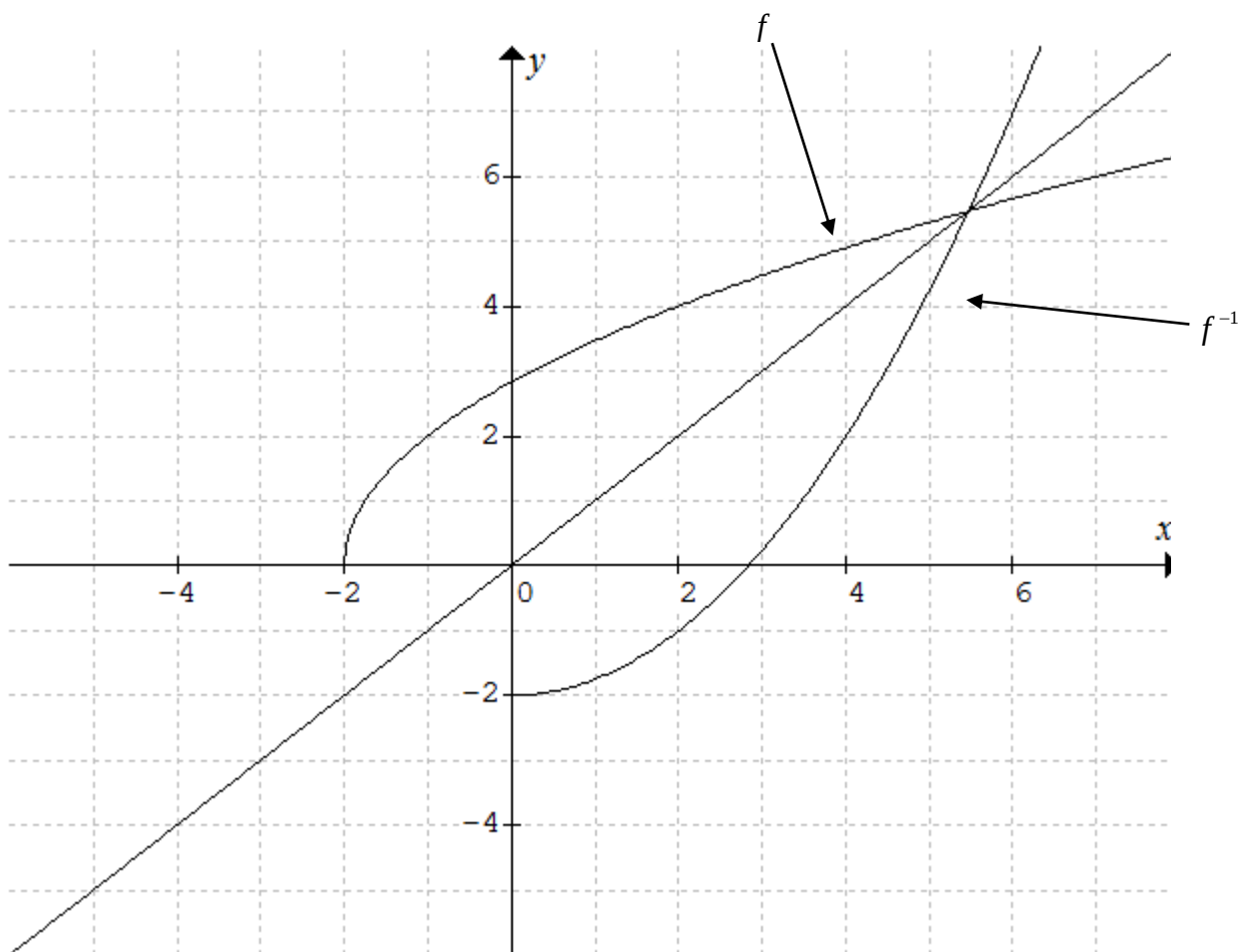
Question 4

- a.** maps onto $y' = 2\sqrt{x'+2}$ $\frac{y'}{2} = \sqrt{x'+2}$
 $x'+2 = x \Rightarrow x' = x-2$ M1
 $\frac{y'}{2} = y \Rightarrow y' = 2y$
 $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -2 \\ 0 \end{bmatrix}$ $a=1, b=2, h=-2, k=0$ A1
- b.** $x+2 \geq 0 \Rightarrow x \geq -2$
 $B = [-2, \infty)$ A1
- c.** $f: y = 2\sqrt{x+2}$
 $f^{-1}: x = 2\sqrt{y+2}$
 $\sqrt{y+2} = \frac{x}{2}$
 $y+2 = \frac{x^2}{4}$ M1
 $y = \frac{x^2}{4} - 2$
 but $\text{dom } f = [-2, \infty)$ $\text{ran } f = [0, \infty) = \text{dom } f^{-1}$
 $f^{-1}: [0, \infty) \rightarrow \mathbb{R}, f^{-1}(x) = \frac{x^2}{4} - 2$ A1
- d.** solving $f^{-1}(u) = f(u)$ or $f(u) = u$ or $f^{-1}(u) = u$, with $u > 0$
 it is easiest to solve $f(u) = u$
 $2\sqrt{u+2} = u$
 $4(u+2) = u^2$
 $u^2 - 4u + 4 = 4 + 8 = 12$
 $(u-2)^2 = 12$ so $u = 2 \pm 2\sqrt{3}$ but $u > 0$
 $u = 2 + 2\sqrt{3}$ A1

Define $f1(x)=2\cdot\sqrt{x+2}$	Done
solve($f1(y)=x,y$)	$y=\frac{x^2}{4}-2$ and $x\geq 0$
Define $f2(x)=\frac{x^2}{4}-2$	Done
solve($f1(x)=x,x$)	$x=2\cdot(\sqrt{3}+1)$
solve($f2(x)=x,x$) $ x>0$	$x=2\cdot(\sqrt{3}+1)$
solve($f2(x)=f1(x),x$) $ x>0$	$x=2\cdot(\sqrt{3}+1)$
$u:=2\cdot(\sqrt{3}+1)$	$2\cdot(\sqrt{3}+1)$

e.

G2



$$\begin{aligned} \text{f.i.} \quad A &= \int_{-2}^u (f(x) - x) dx = \int_{-2}^{2+2\sqrt{3}} (2\sqrt{x+2} - x) dx \\ &= 4\sqrt{3} + \frac{22}{3} \end{aligned} \quad \text{A1}$$

$$\begin{aligned} \text{ii.} \quad A &= \int_0^u (x - f^{-1}(x)) dx = \int_0^{2+2\sqrt{3}} \left(x - \frac{x^2}{4} + 2 \right) dx \\ &= 4\sqrt{3} + \frac{16}{3} \end{aligned} \quad \text{A1}$$

Handwritten solutions for parts f.i. and f.ii. Part f.i. shows the integral of $(f(x) - x)$ from -2 to u , resulting in $4 \cdot \sqrt{3} + \frac{22}{3}$. Part f.ii. shows the integral of $(x - f^{-1}(x))$ from 0 to u , resulting in $\frac{4 \cdot (3 \cdot \sqrt{3} + 4)}{3}$.

$$\begin{aligned} \text{g.} \quad f'(x) &= \frac{1}{\sqrt{x+2}} \\ f'(u) &= f'(2+2\sqrt{3}) = m_1 = \frac{1}{\sqrt{4+2\sqrt{3}}} \end{aligned} \quad \text{A1}$$

$$(f^{-1})'(x) = \frac{x}{2}$$

$$(f^{-1})'(u) = (f^{-1})'(2+2\sqrt{3}) = m_2 = 1 + \sqrt{3}$$

$$\tan(\theta) = \frac{m_2 - m_1}{1 + m_1 m_2} = \frac{3 + \sqrt{3}}{4}$$

$$\text{or alternatively } \theta_1 = \tan^{-1}(m_1), \theta_2 = \tan^{-1}(m_2)$$

$$\theta = \theta_2 - \theta_1 = \tan^{-1}\left(\frac{3 + \sqrt{3}}{4}\right) = 49.8^\circ \quad \text{A1}$$

$m1 := \frac{d}{dx}(f1(x)) _{x=u}$	$\frac{\left(\sqrt{\frac{3}{2}} - \sqrt{\frac{1}{2}}\right) \cdot \sqrt{2}}{2}$
$m2 := \frac{d}{dx}(f2(x)) _{x=u}$	$\sqrt{3} + 1$
$\tan^{-1}\left(\frac{m2-m1}{1+m1 \cdot m2}\right)$	49.7922
$\theta1 := \tan^{-1}(m1)$	$\tan^{-1}\left(\frac{\sqrt{3}-1}{2}\right)$
$\theta2 := \tan^{-1}(m2)$	$\tan^{-1}(\sqrt{3}+1)$
$\theta2 - \theta1$	49.7922

- h.** a translation of k units to the left parallel to the x -axis, or away from the y -axis
and a dilation by a factor of k , parallel to the y -axis or away from the x -axis. A1

i. $g: y = k\sqrt{x+k}$
 $g^{-1}: x = k\sqrt{y+k}$

$$\sqrt{y+k} = \frac{x}{k}$$

$$y+k = \frac{x^2}{k^2}$$

$$y = \frac{x^2}{k^2} - k$$

but $D = \text{dom } g = [-k, \infty)$, $\text{ran } g = [0, \infty) = \text{dom } g^{-1}$

$$g^{-1}: [0, \infty) \rightarrow \mathbb{R}, g^{-1}(x) = \frac{x^2}{k^2} - k \quad \text{A1}$$

- j. solving $g^{-1}(c) = g(c)$ or $g(c) = c$ or $g^{-1}(c) = c$, with $c > 0$ gives

$$c = \frac{k^2 + \sqrt{k^3(k+4)}}{2}$$

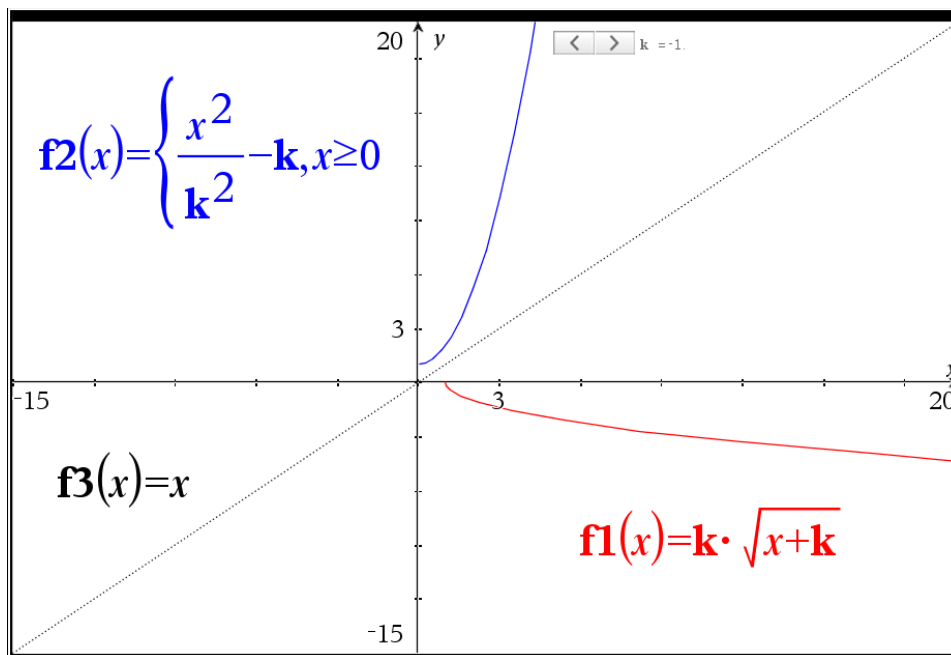
A1

The calculator screen shows the following steps:

- Define $g(x) = k \cdot \sqrt{x+k}$ (Done)
- Define $gi(x) = \frac{x^2}{k^2} - k$ (Done)
- solve($g(x)=x, x$)
- Result: $x = \frac{-\left(\sqrt{k^3 \cdot (k+4)} - k^2\right)}{2}$ or $x = \frac{\sqrt{k^3 \cdot (k+4)} + k^2}{2}$
- Assignment: $c := \frac{\sqrt{k^3 \cdot (k+4)} + k^2}{2}$

- k. $k < 0$

A1



1. Let $m_1 = \frac{d}{dx}(g(x))$ when $x=c$ and $m_2 = \frac{d}{dx}(g^{-1}(x))$ when $x=c$ M1

given $\theta = 45^\circ$ $\tan(45^\circ) = 1$ solving $\frac{m_2 - m_1}{1 + m_1 m_2} = 1$ for k , with $k > 0$

as the graphs do not intersect when $k < 0$ gives

$k = 4$

A1

or alternatively $\theta_1 = \tan^{-1}(m_1)$, $\theta_2 = \tan^{-1}(m_2)$, solving $\theta_2 - \theta_1 = 45^\circ$ for k

$m1 := \frac{d}{dx}(g(x))|_{x=c}$
 $m2 := \frac{d}{dx}(g^{-1}(x))|_{x=c}$
 $\text{solve}\left(\frac{m2-m1}{1+m1 \cdot m2} = 1, k\right)$
 $\theta1 := \tan^{-1}(m1)$
 $\theta2 := \tan^{-1}(m2)$
 $\text{solve}(\theta2 - \theta1 = 45, k)$
 $\theta2 - \theta1 |_{k=4}$

END OF SECTION B SUGGESTED ANSWERS