

Trial Examination 2017

VCE Mathematical Methods Units 3&4

Written Examination 2

Suggested Solutions

SECTION A – MULTIPLE-CHOICE QUESTIONS

1	A	B	C	D	E
2	A	B	C	D	E
3	A	B	C	D	E
4	A	B	C	D	E
5	A	B	C	D	E
6	A	B	C	D	E
7	A	B	C	D	E
8	A	B	C	D	E
9	A	B	C	D	E
10	A	B	C	D	E

11	A	B	C	D	E
12	A	B	C	D	E
13	A	B	C	D	E
14	A	B	C	D	E
15	A	B	C	D	E
16	A	B	C	D	E
17	A	B	C	D	E
18	A	B	C	D	E
19	A	B	C	D	E
20	A	B	C	D	E

Question 1 C

$2 - 3x > 0$ for f to exist

$$x < \frac{2}{3}$$

Question 2 B

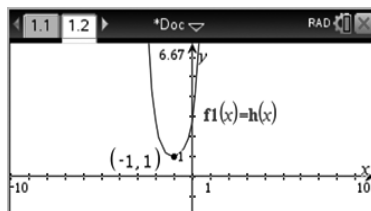
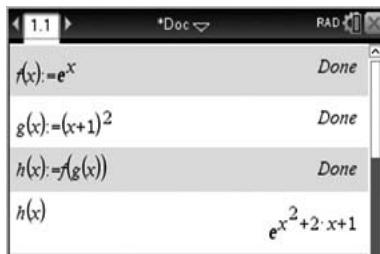
$$\text{average rate of change} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

**Question 3 C**

$$\text{period} = \frac{\pi}{n}$$

$$n = \frac{2}{3}$$

$$\begin{aligned} \therefore \text{period} &= \frac{\pi}{\frac{2}{3}} \\ &= \frac{3\pi}{2} \end{aligned}$$

Question 4 E**Question 5 C**

$$f'(x) = 3x^2 - 2bx + 3$$

$$\text{Let } f'(x) = 0 \rightarrow 3x^2 - 2bx + 3 = 0.$$

$$\Delta = (-2b)^2 - 4(3)(3)$$

$$= 4b^2 - 36$$

$$\Delta > 0 \rightarrow 4b^2 - 36 > 0$$

$$b \in (-\infty, -3) \cup (3, \infty)$$

Question 6 **A**

from the formula sheet: $E(X) = \int xf(x)dx$

$$= \int_0^1 x(-2x + 2)dx$$

A screenshot of a TI-Nspire calculator interface. The input field shows the integral $\int_0^1 (x \cdot (-2 \cdot x + 2)) dx$. The result on the right is $\frac{1}{3}$.

Question 7 **D**

$$\int_{-1}^0 \frac{4}{1-2x} dx = p$$

A screenshot of a TI-Nspire calculator interface. The input field shows the equation $\text{solve}\left(\int_{-1}^0 \frac{4}{1-2 \cdot x} dx = p, p\right)$. The result on the right is $p = 2 \cdot \ln(3)$.

$$p = \log_e(9)$$

$$\frac{e^p}{2} = \frac{e^{\log_e(9)}}{2}$$

$$= \frac{9}{2}$$

Question 8 E

$$\begin{aligned}\text{average value} &= \frac{1}{b-a} \int_a^b g(x) dx \\ &= \frac{1}{3-1} \int_1^3 \left(\frac{4}{x} - 2 \right) dx\end{aligned}$$



$$\text{average value} = 2(\log_e(3) - \log_e(e))$$

$$\begin{aligned}&= 2\log_e\left(\frac{3}{e}\right) \\ &= \log_e\left(\left(\frac{3}{e}\right)^2\right) \\ &= \log_e\left(\frac{9}{e^2}\right)\end{aligned}$$

Question 9 D

$$-f(2x+4) - 1$$

$$\Rightarrow -f(2(x+2)) - 1$$

- dilation factor of $\frac{1}{2}$ from the y -axis: $(6, 2) \rightarrow (3, 2)$
- reflection in the x -axis: $(3, 2)$ maximum $\rightarrow (3, -2)$ minimum
- translation of 2 units left and 1 unit down: $(3, -2) \rightarrow (1, -3)$

Question 10 C

If A and B are mutually exclusive, then $\Pr(A \cap B) = 0$.

Construct a probability table as follows.

	$\Pr(A)$	$\Pr(A')$	
$\Pr(B)$	0	$\frac{1}{8}$	$\frac{1}{8}$
$\Pr(B')$	$\frac{1}{4}$		$\frac{7}{8}$
	$\frac{1}{4}$	$\frac{3}{4}$	1

$$\begin{aligned}\Pr(A' \cap B') &= \frac{7}{8} - \frac{1}{4} \\ &= \frac{5}{8}\end{aligned}$$

Question 11 B

As confidence interval is symmetrical about the sample proportion, then $\hat{p} = 0.15$.

$$\text{for 95\% confidence interval: } \left(\hat{p} - 1.96 \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}, \hat{p} + 1.96 \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \right)$$

$$1.96 \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} = 0.05$$

$$\begin{aligned}\Rightarrow 1.96 \sqrt{\frac{\hat{p}(1-\hat{p})}{4n}} &= \frac{0.05}{\sqrt{4}} \\ &= 0.025\end{aligned}$$

$$\begin{aligned}\text{new 95\% confidence interval} &= (0.15 - 0.025, 0.15 + 0.025) \\ &= (0.125, 0.175)\end{aligned}$$

Question 12 E

A function f is said to be strictly increasing when $a < b$ implies $f(a) < f(b)$ for all a and b in its domain. Remember to include the x -values at stationary points in the interval.

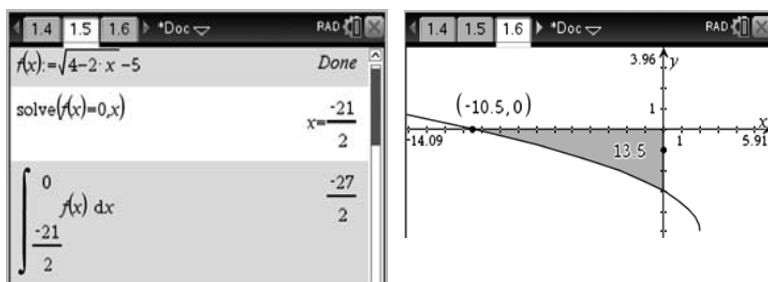
$$\therefore (-\infty, b] \cup [d, \infty)$$

$$= R \setminus (b, d)$$

Question 13 B

The required area is enclosed by the graph and the coordinate axes between $x = -\frac{21}{2}$ and $x = 0$.

$$\begin{aligned}\text{area} &= -\int_{-\frac{21}{2}}^0 (\sqrt{4-2x} - 5) dx \\ &= -\frac{27}{2}\end{aligned}$$



However, the area is a positive quantity.

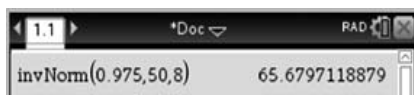
$$\therefore \text{area} = \frac{27}{2}$$

Question 14 C

$$X \sim N(50, 60)$$

$$\Pr(X > b) = 0.025$$

$$\therefore \Pr(X < b) = 0.975$$



$$b = 65.6797\dots$$

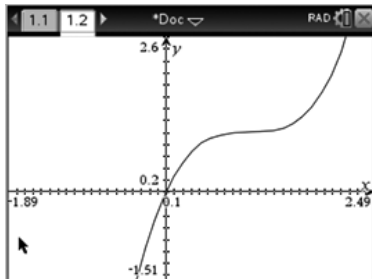
$$\frac{b}{2} = 32.8398\dots$$



$$\therefore \Pr\left(X > \frac{b}{2}\right) = 0.9840$$

Question 15 D

Gradient function graph indicates the original function is always strictly increasing, with a stationary point of inflection located on the x -axis at an x -coordinate corresponding to the turning point of the gradient function. A function of the form $f(x) = (x - a)^3 + c$ would fit this profile, and when $c = 0$, $f(x)$ would pass through the origin as shown below.

**Question 16 D**

application of the chain rule:

$$\frac{dy}{dx} = f(x)g'(x)e^{g(x)} + f'(x)e^{g(x)}$$

$$\therefore \frac{dy}{dx} = e^{g(x)}(f'(x) + f(x)g'(x))$$

Question 17 A

Use a CAS calculator to find the equation of the tangent. The tangent has the equation $y = 2ax + 1$.



Question 18 D

Use a quadratic formula for an algebraic solution, or potentially set up a slider with k on a graph page.

$$\begin{aligned}\sin(x) &= \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(k)}}{2(1)} \\ &= \frac{2 \pm \sqrt{4 - 4k}}{2} \\ &= \frac{2 \pm 2\sqrt{1 - k}}{2} \\ &= 1 \pm \sqrt{1 - k}\end{aligned}$$

Two considerations with this equation, firstly:

$1 - k < 0$ gives no solutions.

$\therefore k > 1$ is first part of solution.

Students also need to consider valid values for $\sin(x)$ as: $-1 \leq \sin(x) \leq 1$.

The positive solution from the quadratic formula will always result in $\sin(x) \geq 1$.

So then students need to investigate values from the negative part of the quadratic formula that produce no solutions.

$$\sin(x) < -1$$

$$1 - \sqrt{1 - k} < -1$$

$$\sqrt{1 - k} > 2$$

$$1 - k > 4$$

$$k < -3$$

The two parts of the solution combined gives $k \in (-\infty, -3) \cup (1, \infty)$.

Question 19 A

$$\Pr(X = 3 | X > 1) = \frac{1}{2p + 1}$$

$$\Pr(X = 3 | X > 1) = \frac{\Pr(X = 3 \cap X > 1)}{\Pr(X > 1)}$$

$$\Pr(X = 3 | X > 1) = \frac{\Pr(X = 3)}{\Pr(X > 1)}$$

$$\Pr(X > 1) = 0.2p + q$$

$$\Pr(X = 3 | X > 1) = \frac{q}{0.2p + q}$$

$$\therefore \frac{q}{0.2p + q} = \frac{1}{2p + 1}$$

TI-Nspire calculator screen showing the equation $\text{solve}\left(\frac{q}{0.2 \cdot p + q} = \frac{1}{2 \cdot p + 1}, q\right)$ with $q=0.1$.

Note: Solving for p and q will give $p = 0$ and a parameter for q (see below), but this solution does not keep the probability distribution valid. It is also the incorrect solution path for option C.

TI-Nspire calculator screen showing the equation $\text{solve}\left(\frac{q}{0.2 \cdot p + q} = \frac{1}{2 \cdot p + 1}, q, p\right)$ with solutions $q=0.1$ and $p=0.5$.

Given $q = 0.1$, we need to find p . This can be done through trial and error with options **A** and **B** or algebraically:

$$\Sigma \Pr(X = x) = 1$$

$$\therefore 0.4p^2 + (1 - 0.6p) + 0.2p + 0.1 = 1$$

TI-Nspire calculator screen showing the equation $\text{solve}(0.4 \cdot p^2 + 1 - 0.6 \cdot p + 0.2 \cdot p + 0.1 = 1, p)$ with $p=0.5$.

Question 20 B

$$x' = \frac{1}{2}x + \frac{\pi}{2}$$

$$y' = -y + 3$$

$$\Rightarrow -y + 3 = 2 \sin\left(\frac{1}{2}x + \frac{\pi}{2} + \frac{\pi}{2}\right) - 3$$

$$-y = 2 \sin\left(\frac{1}{2}x + \pi\right) - 6$$

$$y = -2 \sin\left(\frac{1}{2}(x + 2\pi)\right) + 6$$

$$y = 2 \sin\left(\frac{1}{2}x\right) + 6$$

$$\therefore a = 2, b = \frac{1}{2}, c = 6$$

SECTION B**Question 1** (8 marks)

a. $f'(x) = -\frac{2}{x^2}$ A1

b. i. $y = -\frac{2}{3}x + \frac{3}{a^2}$ A1

ii. let $-\frac{2}{3}x + \frac{3}{a^2} = \frac{1}{x^2}$

$$\therefore x = -\frac{a}{2} \text{ or } x = a$$

but $x > 0$ and $a > 0$

M1

$$\therefore x = -\frac{a}{2}$$

$$\begin{aligned} \text{at } x = -\frac{a}{2}, f\left(-\frac{a}{2}\right) &= \frac{1}{\left(-\frac{a}{2}\right)^2} \\ &= \frac{4}{a^2} \end{aligned}$$

A1

$$\therefore Q\left(-\frac{a}{2}, \frac{4}{a^2}\right) \text{ as required}$$

$$\begin{aligned}
 \text{c. } d(a) &= \sqrt{(x_P - x_Q)^2 + (y_P - y_Q)^2} \\
 &= \sqrt{\left(\frac{3a}{2}\right)^2 + \left(\frac{3}{a^2}\right)^2} \\
 &= \sqrt{\frac{9a^2}{4} + \frac{9}{a^4}} \\
 &= 3\sqrt{\frac{a^2}{4} + \frac{1}{a^4}} \\
 &= 3\left(\frac{a^4}{4} + a^{-4}\right)^{\frac{1}{2}}
 \end{aligned}$$

A1

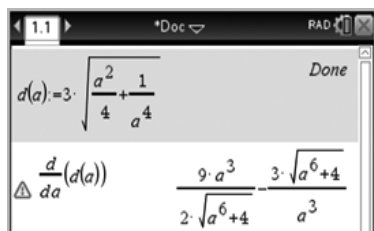
chain rule where $u = \left(\frac{a^4}{4} + a^{-4}\right)$

$$\begin{aligned}
 d'(a) &= \frac{du}{da} \times \frac{dd}{du} \\
 &= 3 \times \frac{1}{2} \times \left(\frac{a^4}{4} + a^{-4}\right)^{-\frac{1}{2}} \times \left(\frac{a}{2} - 4a^{-5}\right) \\
 &= \frac{3\left(\frac{a}{2} - \frac{4}{a^5}\right)}{2\sqrt{\frac{a^2}{4} + \frac{1}{a^4}}} \text{ or equivalent form}
 \end{aligned}$$

not required to simplify M1

OR

accept CAS form as below



The screenshot shows a CAS calculator interface. At the top, it says "1.1" and "RAD". Below that, the expression $d(a) = 3 \cdot \sqrt{\frac{a^2}{4} + \frac{1}{a^4}}$ is entered. Below this, the derivative $\frac{d}{da}(d(a))$ is calculated, resulting in $\frac{9 \cdot a^3}{2 \cdot \sqrt{\frac{a^2}{4} + \frac{1}{a^4}}} - \frac{3 \cdot \sqrt{\frac{a^2}{4} + \frac{1}{a^4}}}{a^3}$. The word "Done" is visible in the top right corner of the calculator window.

(solution continued on next page)

For minimum/maximum, let $d'(a) = 0$.

$$\frac{3\left(\frac{a}{2} - \frac{4}{a^5}\right)}{2\sqrt{\frac{a^2}{4} + \frac{1}{a^4}}} = 0$$

$$\therefore 3\left(\frac{a}{2} - \frac{4}{a^5}\right) = 0 \text{ (numerator must be zero)}$$

$$\frac{a}{2} - \frac{4}{a^5} = 0$$

$$\frac{a^6}{2} - 4 = 0 \text{ as } a \neq 0$$

$$a^6 = 8$$

$$a = \pm\sqrt[6]{8}$$

but $a > 0$

$$\therefore a = \sqrt[6]{8}$$

$$\text{For minimum distance, } d(\sqrt[6]{8}) = \frac{3\sqrt{3}}{2}.$$

correct a value A1

exact value of d A1

Question 2 (11 marks)

a. period = $\frac{2\pi}{n}$

$$= \frac{2\pi}{\frac{\pi}{6}}$$

$$= 12 \text{ months}$$

A1

amplitude = 200

A1

b. maximum population = $1000 + 200$

$$= 1200 \text{ worms}$$

A1

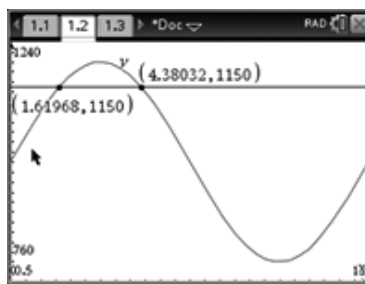
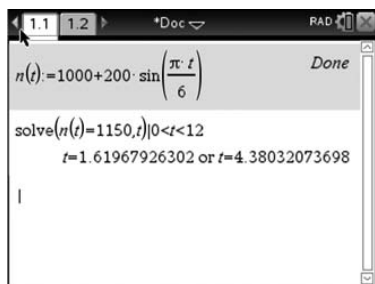
c. $n(t) > 1150$

$$\text{total time} = 4.380 - 1.619$$

correct times A1

$$= 2.76 \text{ months (to two decimal places)}$$

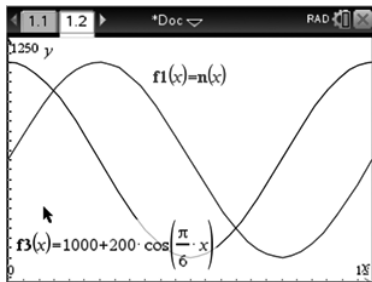
A1



- d. i. Graphically, a translation of 3 units right, or 9 units to the left to match the graphs is required.

As $d > 0$, translation must be to the left, therefore $d = 9$ is the smallest possible value.

A1



- ii. Starting with a translation of 9 units left and then multiples of the period (12 months) gives the following general solution for d .

$$d = 9 + 12k \text{ where } k \in \mathbb{Z}^+ \cup \{0\}$$

expression for d A1positive integers for k A1

e. $w(t) = 1000 + 200 \sin\left(\frac{\pi t}{6}\right) + q(t - 12)$

$$w'(t) = \frac{100\pi}{3} \cos\left(\frac{\pi t}{6}\right) + q$$

A1

$$w'(t) > 0$$

$$\frac{100\pi}{3} \cos\left(\frac{\pi t}{6}\right) + q > 0$$

minimum value for $\cos\left(\frac{\pi t}{6}\right) = -1$

M1

$$-\frac{100\pi}{3} + q > 0$$

$$q > \frac{100\pi}{3}$$

A1

Question 3 (19 marks)

a. i. $X \sim Bi(n, p)$

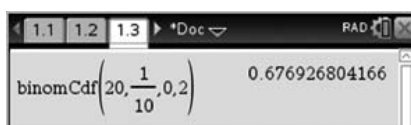
$$X \sim Bi\left(20, \frac{1}{10}\right)$$

M1

$$\Pr(X < 3) = \Pr(X \leq 2)$$

$$= 0.6769$$

A1



ii. $X \sim Bi\left(20, \frac{1}{10}\right)$

$$\Pr(X = 0 | X \leq 2) = \frac{\Pr(X = 0 \cap X \leq 2)}{\Pr(X \leq 2)}$$

$$= \frac{\Pr(X = 0)}{\Pr(X \leq 2)}$$

$$= \frac{0.121576...}{0.676926...}$$

$$= 0.1796$$

M1

A1

b. $X \sim Bi\left(n, \frac{1}{10}\right) \quad n = ?$

$$\Pr(X > 0) \geq 0.8$$

$$1 - \Pr(X = 0) \geq 0.8$$

$$1 - 0.9^n \geq 0.8$$

$$n \geq 15.27$$

$$n = 16$$

M1

A1

c. i. $E(\hat{P}) = p$
 $= \frac{1}{10}$

A1

ii. $\text{sd}(\hat{P}) = \sqrt{\frac{p(1-p)}{n}}$
 $= \sqrt{\frac{\frac{1}{10}\left(1 - \frac{1}{10}\right)}{20}}$

$$\approx 0.0671 \text{ (to four decimal places)}$$

A1

iii. $\text{sd}(\hat{P}) \propto \frac{1}{\sqrt{n}}$

To halve the standard deviation, increase n by a factor of 4.

$$n = 4 \times 20$$

$$= 80$$

A1

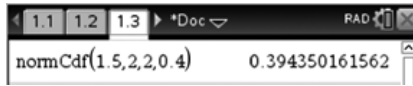
d. i. $Y \sim N(\mu, \sigma^2)$

$$Y \sim N(2, 0.4^2)$$

M1

$$\Pr(1.5 < Y < 2) \approx 0.3944 \text{ (to four decimal places)}$$

M1



ii. $\Pr(Y \geq 2 | Y \geq 1.5) = \frac{\Pr(Y > 2)}{\Pr(Y > 1.5)}$

A1

$$= \frac{\Pr(Y > 2)}{\Pr(1.5 < Y < 2) + \Pr(Y > 2)}$$

M1

Note: Students must use this step as it is a 'hence' question.

$$= \frac{0.5}{0.3944 + 0.5}$$

$$= 0.5591$$

A1

e. $\hat{p} = \frac{20}{100}$

$$= 0.2$$

An approximate 95% confidence interval for p is given by:

$$\hat{p} - z \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} < p < \hat{p} + z \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

Where $\hat{p} = 0.2$, $z = 2$, $n = 100$.

M1

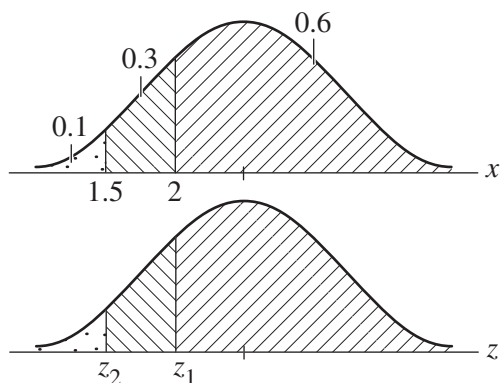
$$\left(0.2 - 2 \sqrt{\frac{0.2 \times 0.8}{100}}, 0.2 + 2 \sqrt{\frac{0.2 \times 0.8}{100}} \right) = \left(\frac{3}{25}, \frac{7}{25} \right)$$

$$a = 3, b = 7$$

A1

f. $\Pr(X > 2) = \frac{12}{20}$
 $= 0.6$

$\Pr(1.5 < X < 2) = \frac{6}{20}$
 $= 0.3$



invNorm(0.4, 0, 1)	-0.253347101143
invNorm(0.1, 0, 1)	-1.28155156658

$$z = \frac{x - \mu}{\sigma}$$

$$-0.25334 = \frac{2 - \mu}{\sigma}$$

$$-1.28155 = \frac{1.5 - \mu}{\sigma}$$

solving simultaneously gives:

$$\mu = 2.1232$$

$$\sigma = 0.4863$$

correct simultaneous equations M1

A1

A1

Question 4 (7 marks)

- a. dilation factor of $\frac{1}{4}$ from the x -axis or alternatively a dilation factor of 2 from the y -axis
 translation of 5 in the negative y -direction

both correct A1

A1

b. $\frac{1}{4}x^2 - 5 = a^2$

finding correct x -coordinates M1

$$\frac{1}{4}x^2 = a^2 + 5$$

$$x^2 = 4(a^2 + 5)$$

$$x = 2\sqrt{a^2 + 5} \text{ as } x > 0$$

$$B \text{ is } (2\sqrt{a^2 + 5} - a, a^2).$$

c. i. Let $f(a) = \text{area}$

$$\begin{aligned}
 &= \int_{x_1}^{x_2} (y - g(x)) dx \\
 &= \int_a^{2\sqrt{a^2+5}} \left(a^2 - \left(\frac{1}{4}x^2 - 5 \right) \right) dx \\
 &= 4(a^2 + 5)^{\frac{3}{2}} - \frac{a(11a^2 + 60)}{12}
 \end{aligned}$$

correct integral M1

correct area function A1

ii. For minimum area, let $f'(a) = 0$.

$$4a\sqrt{a^2+5} - \frac{11a^2}{4} - 5 = 0$$

M1

$$a = \frac{2}{3}$$

$$a = \frac{2}{3}, y = \left(\frac{2}{3} \right)^2 = \frac{4}{9}$$

$$\text{coordinates for minimum area} = \left(\frac{2}{3}, \frac{4}{9} \right)$$

A1

Question 5 (15 marks)

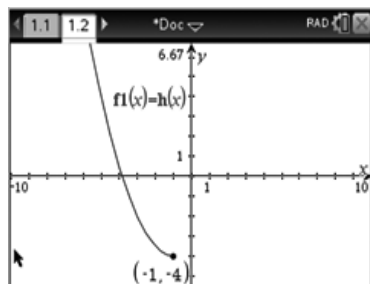
a. i. $h(x) = \frac{1}{2}(x^2 + 2x - 7)$

$$= \frac{1}{2}(x+1)^2 - 4$$

M1

$$k = -1$$

A1



ii. Let $y = \frac{1}{2}(x+1)^2 - 4$.

For inverse, swap x and y .

$$x = \frac{1}{2}(y+1)^2 - 4$$

M1

$$y = -\sqrt{2(x+4)} - 1 \text{ (negative } \sqrt{\text{ as using LHS of parabola)}$$

$$h^{-1}(x) = -\sqrt{2(x+4)} - 1$$

A1

iii. domain: $[-4, \infty)$

A1

range: $(-\infty, -1]$

A1

- b. Let $h(x) = h^{-1}(x)$ or $h(x) = x$ or $h^{-1}(x) = x$.

$$\frac{1}{2}(x+1)^2 - 4 = -\sqrt{2(x+4)} - 1$$

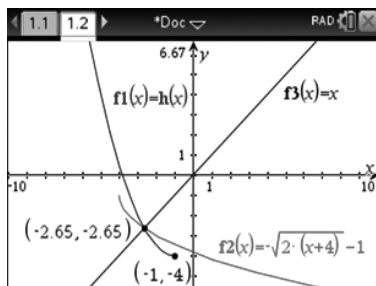
M1

$$x = -\sqrt{7} \text{ as } x < -1$$

point of intersection where $x = y$

$$\text{point of intersection} = (-\sqrt{7}, -\sqrt{7})$$

A1



- c. i. $g(x) = h(x+b)$

$$= \frac{1}{2}(x+b+1)^2 - 4$$

A1

- ii. As $g(x)$ is a transformation of $h(x)$ ' b ' units left, then $g^{-1}(x)$ will be a transformation of $h^{-1}(x)$ ' b ' units down.

M1

$$g^{-1}(x) = -\sqrt{2(x+4)} - 1 - b$$

A1

- d. For intersection at origin, $g^{-1}(x)$ must pass through $(0, 0)$.

$$g^{-1}(0) = 0$$

M1

$$b = -2\sqrt{2} - 1$$

A1

$$\text{check } g(x) \rightarrow g(-2\sqrt{2}-1) = 0$$

$$g(x) \text{ and } g^{-1}(x) \text{ intersect at } (0, 0) \text{ when } b = -2\sqrt{2} - 1$$

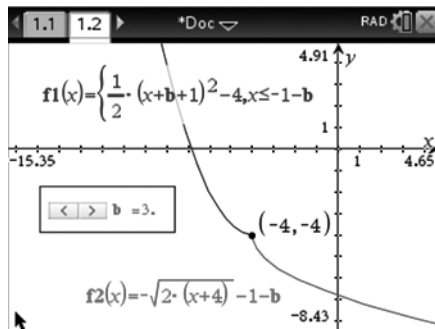
e. Let $g(x) = g^{-1}(x)$. M1

$$\frac{1}{2}(x+b+1)^2 - 4 = -\sqrt{2(x+4)} - 1 - b \quad \text{A1}$$

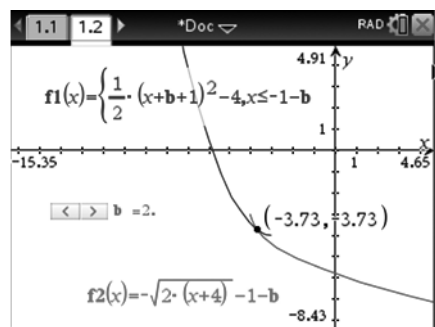
However, solving this equation either by hand or with CAS introduces questionable solutions (resulting from squaring both sides of an equation) and a graphical solution may be the simplest method.

Graphically:

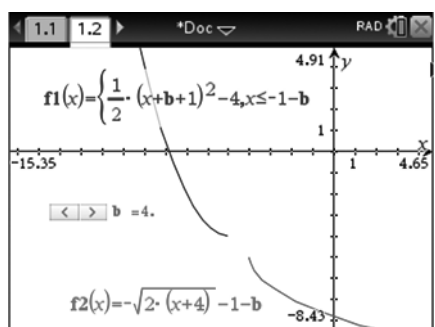
If $b = 3$, the function and its inverse intersect once at $(-4, -4)$. *method for identifying $b = 3$* M1



If $b < 3$, the function and its inverse still intersect.



If $b > 3$, the function and its inverse do not intersect.



So graphs have no intersection points for $b > 3$.

A1