

Trial Examination 2017

MATHEMATICAL METHODS

Trial Written Examination 1 - SOLUTIONS

Question 1

a. $y = \tan(x^2 + 2)$

$$\frac{dy}{dx} = \sec^2(x^2 + 2) \times 2x \quad 1M$$

$$= 2x \sec^2(x^2 + 2) \quad 1A$$

OR

$$\frac{2x}{\cos^2(x^2 + 2)} \quad 1A$$

b. $f(x) = \frac{\log_e(x^2 - 1)}{x^2 - 1}$

$$f'(x) = \frac{(x^2 - 1) \times \frac{2x}{x^2 - 1} - \log_e(x^2 - 1) \times 2x}{(x^2 - 1)^2} \quad 1M$$

$$= \frac{2x - 2x \log_e(x^2 - 1)}{(x^2 - 1)^2} \quad 1M$$

$$\therefore f'(-2) = \frac{-4 - (-4) \log_e(3)}{(3)^2} \quad 1A$$
$$= \frac{-4 + 4 \log_e(3)}{9}$$

Question 2

$$d(t) = -5 \cos\left(\frac{\pi t}{8}\right) + 5$$

a. $d'(t) = \frac{5\pi}{8} \sin\left(\frac{\pi t}{8}\right)$

$$d'(t) = 0 \Rightarrow \sin\left(\frac{\pi t}{8}\right) = 0$$

$$\frac{\pi t}{8} = 0, \pi, \dots$$

$$t = 0, 8, \dots$$

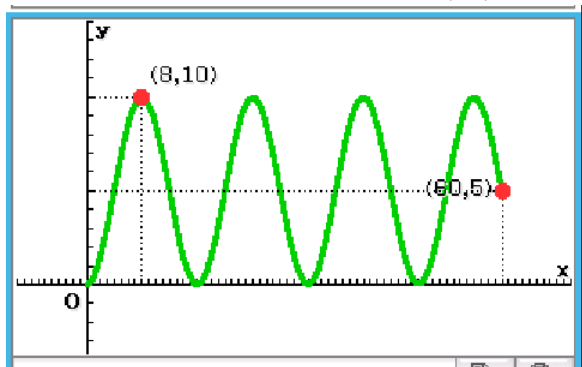
Understanding that the graph starts at (0, 0).

First at its maximum at $t = 8$.

1A

OR

Consider the graph of $d(t) = -5\cos\left(\frac{\pi t}{8}\right) + 5$, with amp = 5 and period = 16.



First at its maximum at $t = 8$

1A

b. Solve $d(t) > 2.5$

$$-5\cos\left(\frac{\pi t}{8}\right) + 5 = 2.5$$

$$\cos\left(\frac{\pi t}{8}\right) = \frac{1}{2}$$

For one cycle solutions are

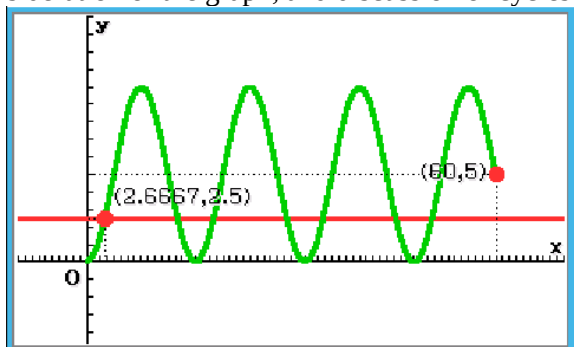
$$\frac{\pi t}{8} = \frac{\pi}{3}, 2\pi - \frac{\pi}{3}$$

$$\frac{\pi t}{8} = \frac{\pi}{3}, \frac{5\pi}{3}$$

$$\therefore t = \frac{8}{3}, \frac{40}{3}$$

1M

Consideration of the graph, and discussion of cycles



We have 3 complete cycles with $\frac{40}{3} - \frac{8}{3} = \frac{32}{3}$ giving a time fraction each of $\frac{\frac{32}{3}}{16} = \frac{2}{3}$.

We have an additional 12 minutes with $\frac{180}{3} - \frac{152}{3} = \frac{28}{3}$ giving a time fraction of $\frac{\frac{28}{3}}{12} = \frac{7}{9}$. **1M**

Time for $0 \leq t \leq 60$ minutes when the depth of water is more than 2.5 metres equals

$$\frac{2}{3} \times 48 + \frac{7}{9} \times 12 = \frac{124}{3} \text{ minutes.}$$

$$\text{Giving the fraction of time} = \frac{\frac{124}{3}}{60} = \frac{124}{180} = \frac{31}{45}$$

1A

Question 3

$$P(x) = 5x^3 - x^2 + x + 7.$$

a. Using the factor theorem.

$$P(1) = 5 - 1 + 1 + 7 \neq 0$$

$$P(-1) = -5 - 1 - 1 + 7 = 0$$

$\therefore (x+1)$ is a factor **1A**

b. Using a form of division

$$\begin{array}{r|rrrr} & 5 & -1 & 1 & 7 \\ -1 & & -5 & 6 & -7 \\ \hline & 5 & -6 & 7 & 0 \end{array}$$

$$Q(x) = 5x^2 - 6x + 7 \quad \mathbf{1A}$$

c. $Q(x) = 5x^2 - 6x + 7$

$$\Delta = (-6)^2 - 4 \times 5 \times 7 = 36 - 140 = -104$$

$\Delta < 0$ giving no linear factors for $Q(x)$. **1A**

d. $y = P(x)$ has no stationary points, giving $\frac{dy}{dx} \neq 0$ over the domain R

$$\frac{dy}{dx} = 15x^2 - 2x + 1 \quad \mathbf{1A}$$

Minimum is at the turning point

$$x = \frac{-b}{2a} = \frac{1}{15} \quad \mathbf{1A}$$

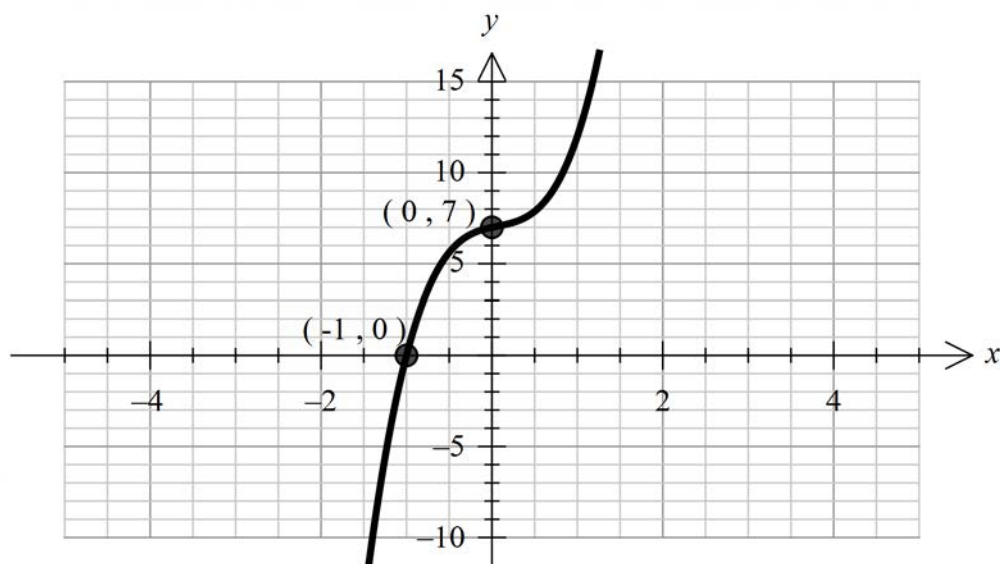
OR

$$\frac{d^2y}{dx^2} = 30x - 2 = 0$$

$$x = \frac{1}{15} \quad \mathbf{1A}$$

e. Shape **1A**

Intercepts **1A**

**Question 4**

$$f(x) = x \sin(x).$$

a. Average rate of change for $\left[\frac{\pi}{2}, \pi\right]$ equals $\frac{f(\pi) - f\left(\frac{\pi}{2}\right)}{\pi - \frac{\pi}{2}}$

$$= \frac{\pi \sin(\pi) - \frac{\pi}{2} \sin\left(\frac{\pi}{2}\right)}{\pi - \frac{\pi}{2}}$$

1M

$$= \frac{0 - \frac{\pi}{2}}{\frac{\pi}{2}}$$

Average rate of change = -1

1A

b. $f'(x) = \sin(x) \times 1 + x \cos(x) = \sin(x) + x \cos(x)$ **1A**

c. If $f'(x) = \sin(x) + x \cos(x)$ then $\int (\sin(x) + x \cos(x)) dx = x \sin(x) + c$ **1M**

Rearrange to get

$$\int (\sin(x)) dx + \int (x \cos(x)) dx = x \sin(x) + c$$

$$\int (x \cos(x)) dx = x \sin(x) - \int (\sin(x)) dx + c$$

We need $\int_0^{\frac{\pi}{2}} x \cos(x) dx$

$$\int_0^{\frac{\pi}{2}} (x \cos(x)) dx = \left[x \sin(x) \right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} (\sin(x)) dx \quad \mathbf{1M}$$

$$= \frac{\pi}{2} \sin\left(\frac{\pi}{2}\right) - 0 - \left[-\cos(x)\right]_0^{\frac{\pi}{2}}$$

$$= \frac{\pi}{2} + \cos\left(\frac{\pi}{2}\right) - \cos(0)$$

$$\text{Giving } \int_0^{\frac{\pi}{2}} x \cos(x) dx = \frac{\pi}{2} - 1 \quad \mathbf{1A}$$

Question 5

$$\log_2((2x-2)^2) - 4\log_2(1-x) = 1$$

$$\log_2\left(\frac{(2x-2)^2}{(1-x)^4}\right) = 1 \quad \mathbf{1A}$$

$$\frac{(2x-2)^2}{(1-x)^4} = 2$$

$$\frac{4(x-1)^2}{(x-1)^4} = 2$$

$$\frac{4}{(x-1)^2} = 2$$

$$2x^2 - 4x - 2 = 0 \quad \mathbf{1A}$$

$$x = \frac{2 \pm \sqrt{8}}{2}$$

$$x = 1 \pm \sqrt{2}$$

$$\text{As } x < 1$$

$$x = 1 - \sqrt{2} \quad \mathbf{1A}$$

Question 6

$$\left(\hat{p} - 2\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}, \hat{p} + 2\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \right)$$

$$= \left(\frac{1}{5} - 2\sqrt{\frac{\frac{1}{5} \times \frac{4}{5}}{10\,000}}, \frac{1}{5} + 2\sqrt{\frac{\frac{1}{5} \times \frac{4}{5}}{10\,000}} \right) \quad \mathbf{1M}$$

$$= \left(\frac{1}{5} - 2\sqrt{\frac{4}{250\,000}}, \frac{1}{5} + 2\sqrt{\frac{4}{250\,000}} \right)$$

$$= \left(\frac{1}{5} - \frac{1}{125}, \frac{1}{5} + \frac{1}{125} \right) \quad \mathbf{1A}$$

$$= \left(\frac{24}{125}, \frac{26}{125} \right) \quad \mathbf{1A}$$

Question 7

$$\begin{aligned}
 \text{a. } & \int_0^1 (x) dx \\
 &= \left[\frac{x^2}{2} \right]_0^1 \\
 &= \frac{1}{2} \\
 & \int_1^a \left(\frac{1}{x^2} \right) dx = \frac{1}{2} \quad \mathbf{1A} \\
 & \left[-\frac{1}{x} \right]_1^a = \frac{1}{2} \\
 & -\frac{1}{a} + 1 = \frac{1}{2} \\
 & -\frac{1}{a} = -\frac{1}{2} \\
 & a = 2 \text{ as required} \quad \mathbf{1M}
 \end{aligned}$$

$$\begin{aligned}
 \text{b. } E(X) &= \int_0^1 (x^2) dx + \int_1^2 \left(\frac{1}{x} \right) dx \quad \mathbf{1M} \\
 &= \left[\frac{x^3}{3} \right]_0^1 + \left[\log_e(x) \right]_1^2 \quad \mathbf{1M} \\
 &= \frac{1}{3} + \log_e(2) - \log_e(1) \\
 &= \frac{1}{3} + \log_e(2) \quad \mathbf{1A}
 \end{aligned}$$

Question 8

$$\begin{aligned}
 \text{a. } f(x) &= 3\sqrt{4-2x} + 1 \\
 f(x) &= 3\sqrt{-2(x-2)} + 1
 \end{aligned}$$

Translate 2 units left and 11 units up.

As the graph has to be dilated by a factor of $\frac{1}{3}$ from the x-axis,

the vertical translation can be worked by solving $\frac{1+d}{3} = 4, d = 11$.

$$\begin{aligned}
 f_1(x) &= 3\sqrt{-2x} + 12 \\
 c = -2, d = 11 \quad \mathbf{1A}
 \end{aligned}$$

Dilate by a factor of $\frac{1}{3}$ from the x-axis.

$$f_2(x) = \sqrt{-2x} + 4$$

Dilate by a factor of 2 from the y-axis.

$$f_3(x) = \sqrt{-x} + 4$$

Reflect in the y-axis.

$$g(x) = \sqrt{x} + 4$$

$$a = -2, b = \frac{1}{3} \quad \mathbf{2A}$$

OR

$$x' = a(x + c) = -2(x - 2)$$

$$a = -2, c = -2 \quad \mathbf{1A}$$

$$y' = b(y + d) = \frac{y-1}{3} + 4 = \frac{1}{3}(y + 11) \quad \mathbf{1M}$$

$$b = \frac{1}{3}, d = 11 \quad \mathbf{1A}$$

OR

$$x' = a(x + c), x = \frac{x'}{a} - c$$

$$y' = b(y + d), y = \frac{y'}{b} - d \quad \mathbf{1M}$$

$$\frac{y'}{b} - d = 3\sqrt{4 - 2\left(\frac{x'}{a} - c\right)} + 1$$

$$y' = 3b\sqrt{-\frac{2x'}{a} + 4 + 2c + (1 + d)b}$$

$$3b = 1, b = \frac{1}{3}$$

$$(1 + d)b = 4, d = 11$$

$$-\frac{2}{a} = 1, a = -2$$

$$4 + 2c = 0, c = -2$$

1A 2 correct
2A all correct

b. $y = 3\sqrt{4 - 2x} + 1$

Inverse swap x and y

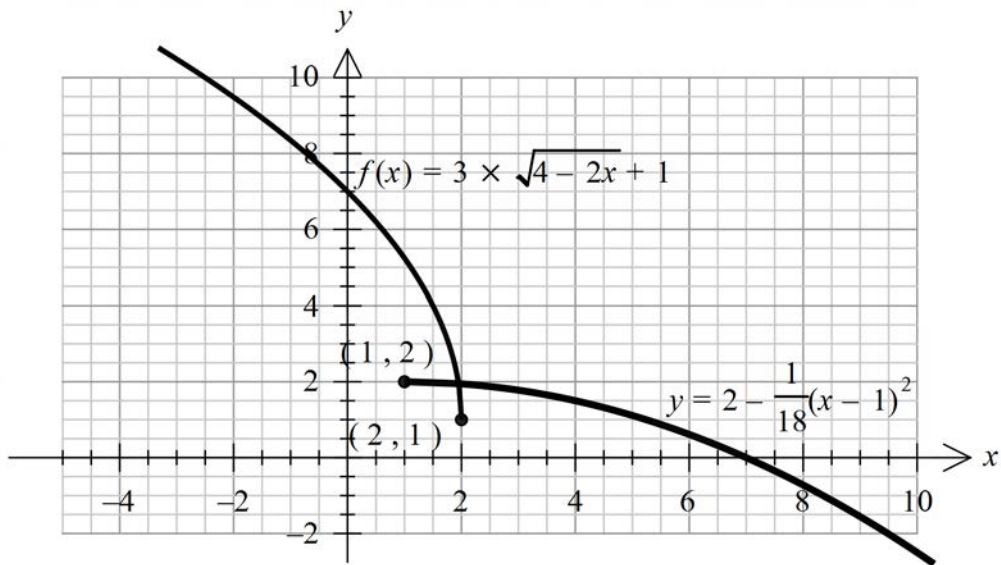
$$x = 3\sqrt{4 - 2y} + 1$$

$$\frac{x-1}{3} = \sqrt{4 - 2y}$$

$$f^{-1}(x) = 2 - \frac{(x-1)^2}{18} \quad \mathbf{1A}$$

Domain $[1, \infty)$ **1A**

$$f^{-1} : [1, \infty) \rightarrow \mathbb{R}, f^{-1}(x) = -\frac{(x-1)^2}{18} + 2$$



c. $g(f(x)) = \sqrt{3\sqrt{4-2x+1}+4}$ **1A**

Range $[5, \infty)$ **1A**

