

Year 2017

VCE

Mathematical Methods

Trial Examination 2

Solutions



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SECTION A

ANSWERS

1	A	B	C	D	E
2	A	B	C	D	E
3	A	B	C	D	E
4	A	B	C	D	E
5	A	B	C	D	E
6	A	B	C	D	E
7	A	B	C	D	E
8	A	B	C	D	E
9	A	B	C	D	E
10	A	B	C	D	E
11	A	B	C	D	E
12	A	B	C	D	E
13	A	B	C	D	E
14	A	B	C	D	E
15	A	B	C	D	E
16	A	B	C	D	E
17	A	B	C	D	E
18	A	B	C	D	E
19	A	B	C	D	E
20	A	B	C	D	E

SECTION A

Question 1

Answer D

The graph is the bottom half of a circle with centre at the origin and radius $2b$.

$$f: D \rightarrow R, f(x) = -\sqrt{4b^2 - x^2}$$

$$\text{solving } f(x) = 0 \Rightarrow x = \pm 2b$$

$$\text{solving } f(x) = -\sqrt{3}b$$

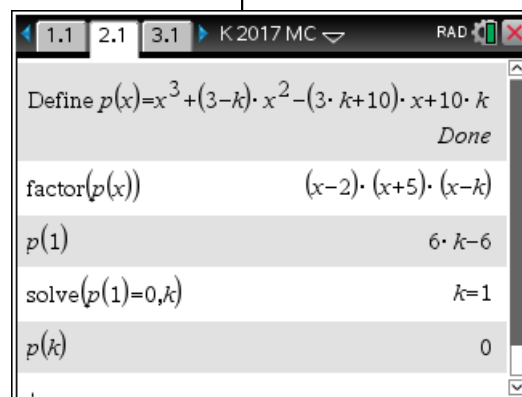
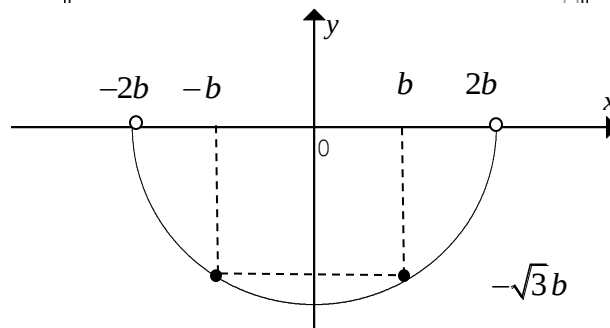
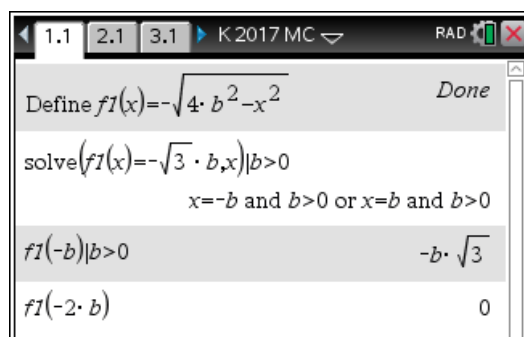
$$-\sqrt{4b^2 - x^2} = -\sqrt{3}b$$

$$4b^2 - x^2 = 3b^2$$

$$x^2 = b^2$$

$$x = \pm b$$

since the range is $[-\sqrt{3}b, 0)$ the domain could be $(-2b, -b]$ or $[b, 2b)$



Question 2

Answer B

$$p(x) = x^3 + (3-k)x^2 - (3k+10)x + 10k$$

$$= (x-2)(x+5)(x-k)$$

$$p(1) = 6k - 6, p(1) = 0 \Rightarrow k = 1$$

Since $(x-k)$ is a factor, then $p(k) = 0$

$(x-2)$ and $(x+5)$ are both factors

$(x+k)$ is not a factor.

Question 3

Answer A

$$f: (-\infty, b) \rightarrow R, f(x) = -x^4 + 2x^3$$

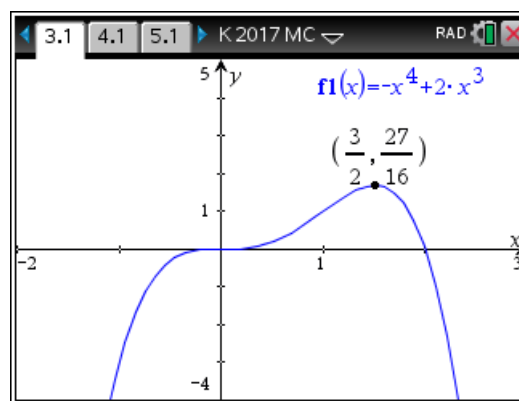
$$f'(x) = -4x^3 + 6x^2 = 2x^2(3-2x) = 0$$

The graph has an inflexion point at $x = 0$ the origin, and a maximum turning point at

$\left(\frac{3}{2}, \frac{27}{16}\right)$. The function is only a one-one

function, when its domain is restricted to

$(-\infty, b)$ where $b < \frac{3}{2}$.



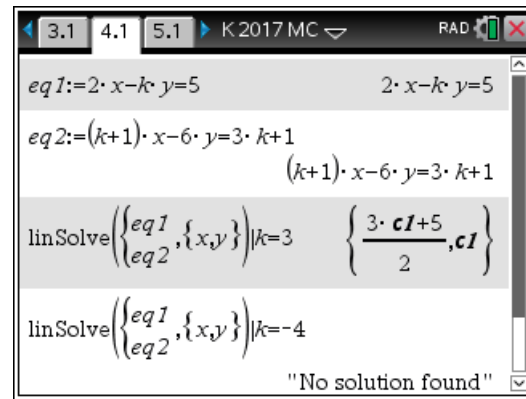
Question 4**Answer E**

(1) $2x - ky = 5$

$$ky = 2x - 5 \Rightarrow y = \frac{2x}{k} - \frac{5}{k}$$

(2) $(k+1)x - 6y = 3k+1$

$$6y = (k+1)x - (3k+1) \Rightarrow y = \frac{(k+1)x}{6} - \frac{3k+1}{6}$$



equating gradients, when the lines are parallel

$$\frac{2}{k} = \frac{k+1}{6} \Rightarrow k(k+1) = 12 \Rightarrow k^2 + k - 12 = (k+4)(k-3) = 0$$

There is a unique solution when $k \in \mathbb{R} \setminus \{-4, 3\}$

When $k = 3$ the equations become

$$\begin{aligned} 2x - 3y &= 5 \\ 4x - 6y &= 10 \end{aligned}$$

these lines are coincident, that is the same line, so there is infinite number of solutions when $k = 3$

When $k = -4$ the equations become

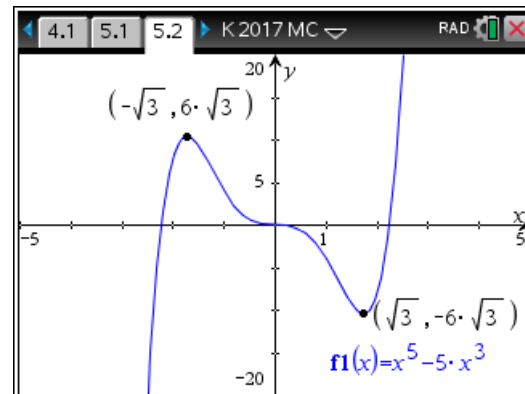
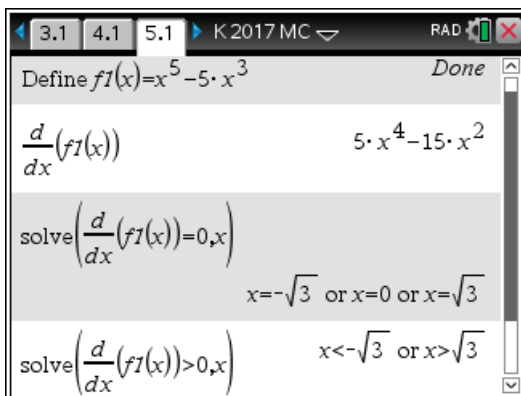
$$\begin{aligned} 2x + 4y &= 5 \\ -3x - 6y &= -9 \end{aligned}$$

these lines are parallel so there is no solution when $k = -4$ **Question 5****Answer C**

$$y = x^5 - 5x^3 \Rightarrow \frac{dy}{dx} = 5x^4 - 15x^2 = 5x^2(x^2 - 3), \text{ stationary points occur when } \frac{dy}{dx} = 0$$

$$\Rightarrow x = 0 \text{ the point of inflexion} \Rightarrow x = \pm\sqrt{3} \text{ the turning points.}$$

$$\text{The graph has a positive gradient } \frac{dy}{dx} > 0 \Rightarrow x \in (-\infty, -\sqrt{3}) \cup (\sqrt{3}, \infty)$$

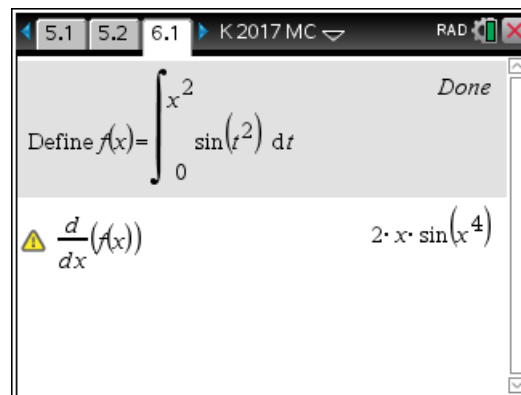


Question 6

Answer B

$$f(x) = \int_0^{x^2} \sin(t^2) dt$$

$$f'(x) = 2x \sin(x^4)$$

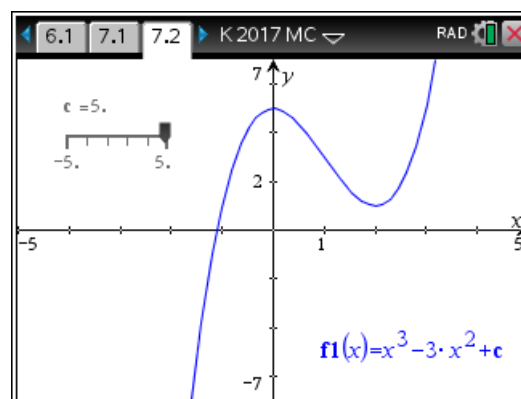
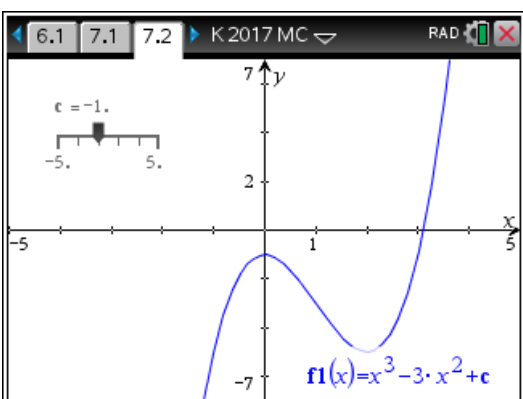
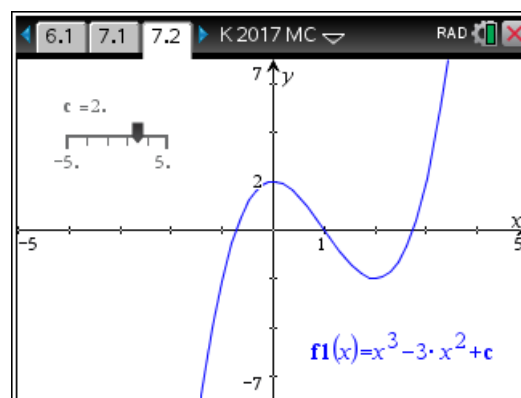
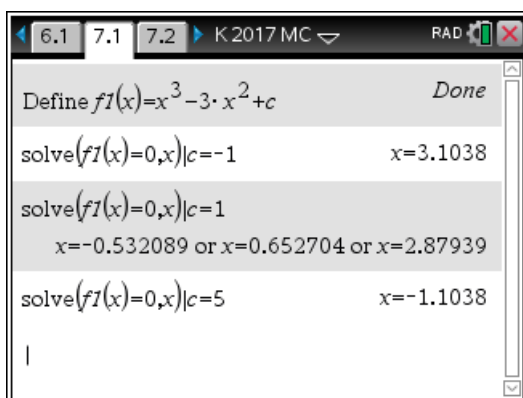


Question 7

Answer C

$$y = x^3 - 3x^2 + c \quad \frac{dy}{dx} = 3x^2 - 6x = 3x(x - 2) = 0 \quad \text{turning points occur when } x = 0, 2$$

when $x = 0$, $y = c$, when $x = 2$, $y = 8 - 12 + c = c - 4$, turning points at $(0, c)$ and $(2, c - 4)$. The graph crosses the x -axis three times when $c > 0$ and $c - 4 < 0$, that is when $0 < c < 4$.



Question 8**Answer A**

$$f(x) = \frac{\log_e(3x)}{g(x)} \text{ using the quotient rule}$$

$$f'(x) = \frac{g(x) \frac{d}{dx} [\log_e(3x)] - g'(x) \log_e(3x)}{[g(x)]^2} = \frac{\frac{g(x)}{x} - g'(x) \log_e(3x)}{[g(x)]^2}$$

$$f'(2) = \frac{\frac{g(2)}{2} - g'(2) \log_e(6)}{[g(2)]^2} = \frac{\frac{4}{2} - 3 \log_e(6)}{4^2} = \frac{2 - 3 \log_e(6)}{16}$$

Question 9**Answer E**

$$F(x) = \int_0^x f(t) dt$$

Note that $y = f(t)$ may also be modelled by

$$y = 4 \sin\left(\frac{\pi t}{2}\right)$$

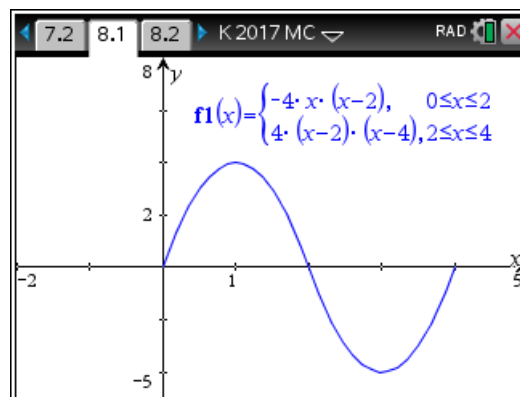
A. $F(0) = 0 = \int_0^0 f(t) dt$, is true, by properties of definite integrals

B. $F(4) = 0 = \int_0^4 f(t) dt$, is true, the value of the definite integral is zero.

C. $F(2) = \int_0^2 f(t) dt = 2F(1) = 2 \int_0^1 f(t) dt$ is true, by symmetry, area between 0 to 2 is double the area from 0 to 1.

D. $F(2) = \int_0^2 f(t) dt = 2F(3) = 2 \int_0^3 f(t) dt$, is true, similar to **C**.

E. $F(3) + F(1) = 0$, is false, $F(3) = \int_0^3 f(t) dt = F(1) = \int_0^1 f(t) dt$.



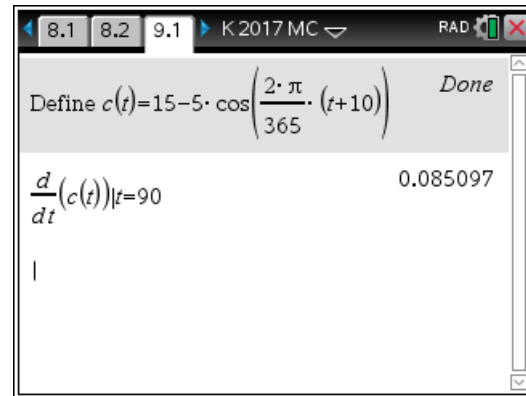
Define $f(x) = \int_0^x f(t) dt$	Done
$f(0)$	0
$f(1)$	$\frac{8}{3}$
$f(2)$	$\frac{16}{3}$
$f(3)$	$\frac{8}{3}$
$f(4)$	0

Question 10**Answer D**

$$C(t) = 15 - 5 \cos\left(\frac{2\pi}{365}(t+10)\right)$$

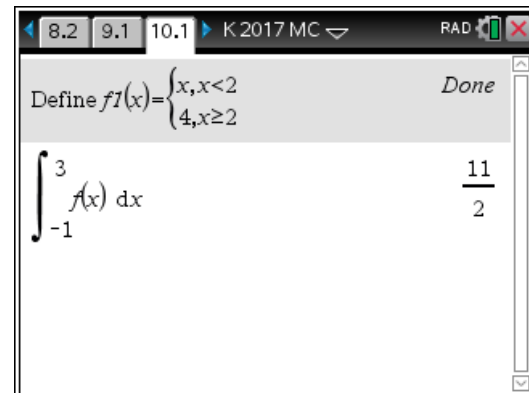
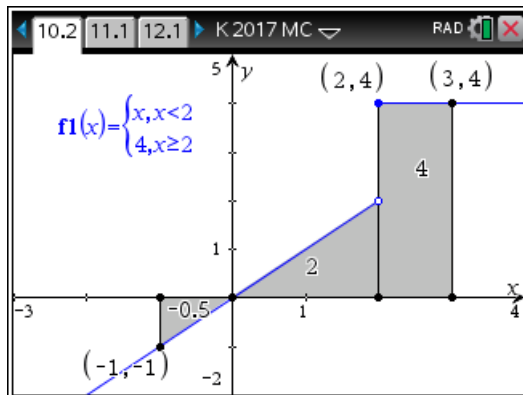
$$\frac{dC}{dt} = \frac{5 \times 2\pi}{365} \sin\left(\frac{2\pi}{365}(t+10)\right)$$

$$\left.\frac{dC}{dt}\right|_{t=90} = \frac{10\pi}{365} \sin\left(\frac{200\pi}{365}\right) \approx 0.085$$

**Question 11****Answer D**

$$f(x) = \begin{cases} x & \text{for } x < 2 \\ 4 & \text{for } x \geq 2 \end{cases} \quad \text{Then} \quad \int_{-1}^3 f(x) dx = 5\frac{1}{2}$$

Although the function is not continuous at $x = 2$, the definite integral can still be calculated as the area of the rectangle and the area of two triangles. The smaller triangle below the x -axis has a value of -0.5 , it is not an area, the value of the definite integral is $4 + 2 - 0.5 = 5\frac{1}{2} = \frac{11}{2}$

**Question 12****Answer C**

Let $y = f^{-1}(x) = g(x)$ then $x = f(y)$ now differentiate with respect to y ,

$$\frac{dx}{dy} = f'(y), \text{ inverting } \frac{dy}{dx} = \frac{d}{dx}[g(x)] = g'(x) = \frac{1}{f'(y)} = \frac{1}{f'(g(x))}$$

$$\text{Now } f(1) = 2 \text{ so that } g(2) = 1, \text{ and } g'(2) = \frac{1}{f'(g(2))} = \frac{1}{f'(1)} = \frac{1}{-4} = -\frac{1}{4}$$

Question 13**Answer A**

From the graphs $f(1) = 1$, $f(2) = 0$, $g(1) = 2$

$$f(g(1)) = f(2) = 0$$

$$g(f(1)) = g(1) = 2$$

Question 14**Answer E**

$$\frac{dN}{dt} = 5e^{0.2t} = 5e^{\frac{t}{5}}$$

$$N = \int_0^{10} 5e^{\frac{t}{5}} dt = \left[25e^{\frac{t}{5}} \right]_0^{10}$$

$$= 25 \left(e^{\frac{10}{5}} - e^0 \right) = 25(e^2 - 1)$$

Calculator screenshot showing the integral calculation for Question 14. The expression entered is $\int_0^{10} \left(5 \cdot e^{\frac{t}{5}} \right) dt$. The result displayed is $25 \cdot (e^2 - 1)$.

Question 15**Answer D**

$$\Pr(RBW) + \Pr(RWB) + \Pr(WRB) + \Pr(WBR) + \Pr(BWR) + \Pr(BRW)$$

There are 6 different ways, the total number of balls is $(r + b + w)$ when one ball is drawn there is $(r + b + w - 1)$ and when two balls are drawn there are $(r + b + w - 2)$ balls remaining.

The total number of ways of drawing balls of different colours is

$$\frac{6rbw}{(r + b + w)(r + b + w - 1)(r + b + w - 2)}$$

Question 16**Answer B**

The graph of $y = \cos(x)$ crosses the x -axis at $x = \pm \frac{\pi}{2}$.

The area of the rectangle is $A(x) = 2x \cos(x)$, solving $\frac{dA}{dx} = 0$ for x , with $0 < x < \frac{\pi}{2}$, gives $x = 0.8603$. The maximum area of the rectangle is $A(0.8603) = 1.1222$.

The area under the cosine wave is $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(x) dx = 2$, so the minimum shaded area is

$$2 - 1.122 = 0.878$$

Calculator screenshot showing the area calculation for Question 16. The steps shown are:

- Define $a(x) = 2 \cdot x \cdot \cos(x)$
- Solve $\frac{d}{dx}(a(x)) = 0, x$ for $0 < x < \frac{\pi}{2}$, resulting in $x = 0.860334$.
- Calculate $a(x)$ at $x = 0.86033$, resulting in 1.12219 .
- Calculate the integral $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(x) dx$, resulting in 2 .
- Calculate the final result $2 - 1.1221$, resulting in 0.8779 .

Question 17

Answer C

$$p = 0.35, n = 200, 95\% \quad z = 1.96, \sigma_{\hat{p}} = \sqrt{\frac{p(1-p)}{n}} = \sqrt{\frac{0.35 \times 0.65}{200}} = 0.0334$$

$$\text{The width of a 95\% confidence interval is } 2z \sqrt{\frac{p(1-p)}{n}} = 2 \times 1.96 \times 0.0334 = 0.132$$

Question 18

Answer B

$$Z \stackrel{d}{=} N(\mu = 0, \sigma^2 = 1), \text{ given that } \Pr(-c < Z < c) = C \text{ by symmetry} \Rightarrow \Pr(0 < Z < c) = \frac{C}{2}$$

$$\Pr(Z < c | Z > 0) = \frac{\Pr(0 < Z < c)}{\Pr(Z > 0)} = \frac{\frac{C}{2}}{0.5} = C$$

Question 19

Answer A

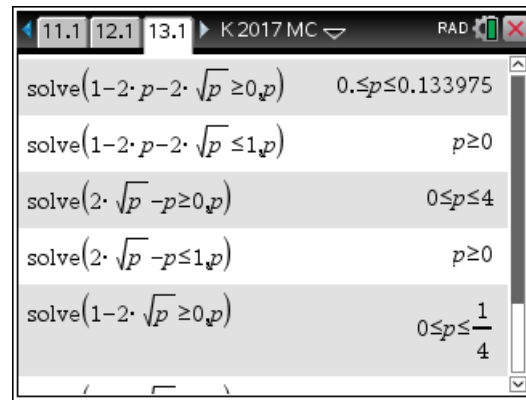
$$\Pr(A) = 3p, \Pr(B) = 2\sqrt{p} \text{ and } \Pr(A \cap B) = p$$

	A	A'	
B	p	$2\sqrt{p} - p$	$2\sqrt{p}$
B'	$2p$	$1 - 2p - 2\sqrt{p}$	$1 - 2\sqrt{p}$
	$3p$	$1 - 3p$	

For valid probabilities, all probabilities must be between 0 and 1. That is $0 \leq 3p \leq 1$,

$$0 \leq 1 - 2\sqrt{p} \leq 1, 0 \leq 2\sqrt{p} - p \leq 1, \text{ etc}$$

solving $0 \leq 1 - 2p - 2\sqrt{p} \leq 1$ gives $p = 0.13398$, any value greater than this will result in negative probabilities, which is not possible.



Question 20

Answer E

$$X \stackrel{d}{=} \text{Bi}(n = ?, p = ?)$$

$$\Pr(\text{more than one}) = \Pr(X > 1) = 1 - [\Pr(X = 0) + \Pr(X = 1)] = 1 - (0.65^8 + 8(0.35)(0.65)^7)$$

$$\text{Now } \Pr(X = 0) = q^n \text{ and } \Pr(X = 1) = npq^{n-1}$$

$$n = 8, q = 0.65 \text{ and } p = 0.35$$

$$8 \text{ trials and } p = \Pr(\text{success}) = 0.35$$

END OF SECTION A SUGGESTED ANSWERS

SECTION B**Question 1**

a. $f(x) = (x^2 - p^2)(x - 3p) = x^3 - 3px^2 - p^2x + 3p^3$

$$f'(x) = 3x^2 - 6px - p^2$$

At P, $x = 2p$, $f(2p) = -3p^3$ $f'(2p) = -p^2$

A1

$$T: y + 3p^3 = -p^2(x - 2p) = -p^2x + 2p^3$$

A1

$$t1(x) = y = -p^2x - p^3 \quad m_1 = -p^2, \quad c_1 = -p^3$$

Define $f(x) = (x^2 - p^2) \cdot (x - 3 \cdot p)$	Done
expand($f(x)$)	$x^3 - 3 \cdot p \cdot x^2 - p^2 \cdot x + 3 \cdot p^3$
$\frac{d}{dx}(f(x))$	$3 \cdot x^2 - 6 \cdot p \cdot x - p^2$
$f(2 \cdot p)$	$-3 \cdot p^3$
$\frac{d}{dx}(f(x)) _{x=2 \cdot p}$	$-p^2$
tangentLine($f(x), x, 2 \cdot p$)	$-p^2 \cdot x - p^3$
Define $t1(x) = -p^2 \cdot x - p^3$	Done

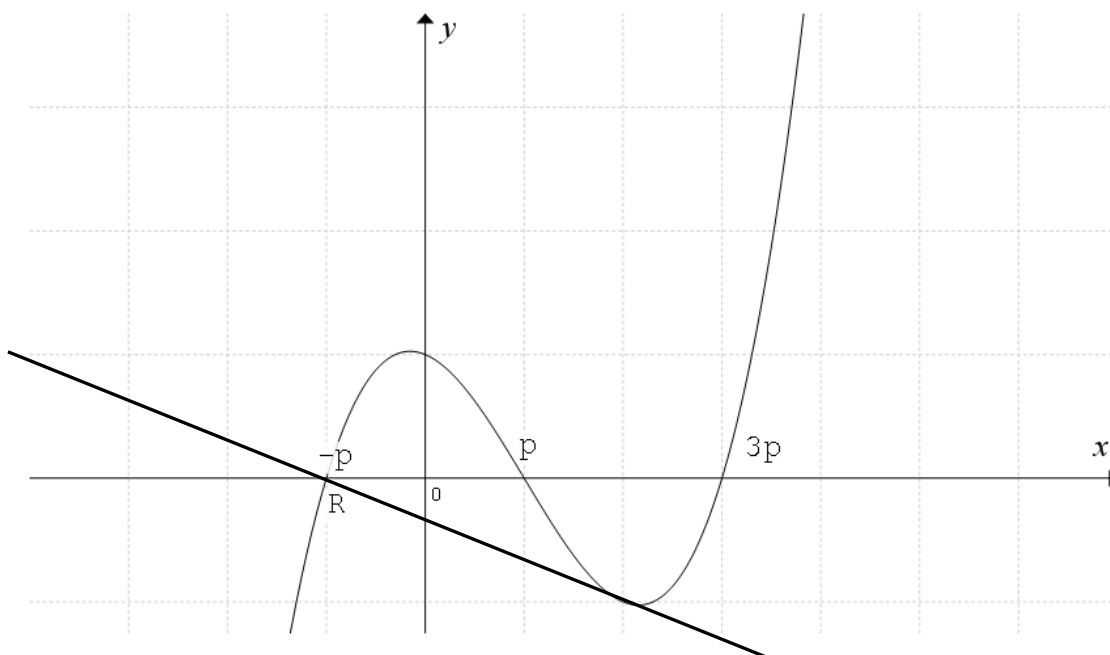
b. $t1(x) = y = -p^2x - p^3 = -p^2(x + p)$, $f(x) = (x + p)(x - p)(x - 3p)$

solving $t1(x) = f(x) \Rightarrow x = -p$ R $(-p, 0)$

A1

tangent must pass through $(-p, 0)$

G1



c. At R, $x = -p$, $f(-p) = 0$ $f'(-p) = 8p^2$ A1

$T: y - 0 = 8p^2(x + p) = 8p^2x + 8p^3$ A1

$t2(x) = y = 8p^2x + 8p^3$ $m_2 = 8p^2$, $c_2 = 8p^3$

<code>solve(t1(x)=f(x),x)</code>	$x=2 \cdot p$ or $x=-p$
<code>f(p)</code>	0
<code>$\frac{d}{dx}(f(x)) _{x=p}$</code>	$8 \cdot p^2$
<code>tangentLine(f(x),x,p)</code>	$8 \cdot p^2 \cdot x + 8 \cdot p^3$
Define <code>t2(x)=8 \cdot p^2 \cdot x + 8 \cdot p^3</code>	Done

d. solving $f(x) = t2(x)$

$x^3 - 3px^2 - p^2x + 3p^3 = 8p^2x + 8p^3$
 $\Rightarrow x = -p$ or $x = 5p$, $f(5p) = 48p^3$ M1

S $(5p, 48p^3)$ A1

<code>solve(t2(x)=f(x),x)</code>	$x=5 \cdot p$ or $x=-p$
<code>f(5 \cdot p)</code>	$48 \cdot p^3$

e. $A_1 = \int_{-p}^{2p} (f(x) - t1(x)) dx$ M1

$A_1 = \int_{-p}^{2p} (x^3 - 3px^2 + 4p^3) dx$ A1

f. $A_2 = \int_{-p}^{5p} (t2(x) - f(x)) dx$ M1

$A_2 = \int_{-p}^{5p} (-x^3 + 3px^2 + 9p^2x + 5p^3) dx$ A1

g. $A_1 = \frac{27p^4}{4}$, $A_2 = 108p^4$, $\frac{A_2}{A_1} = 16$ A1

<code>f(x)-t1(x)</code>	$x^3 - 3 \cdot p \cdot x^2 + 4 \cdot p^3$
<code>a1:=$\int_{-p}^{2 \cdot p} (x^3 - 3 \cdot p \cdot x^2 + 4 \cdot p^3) dx$</code>	$\frac{27 \cdot p^4}{4}$
<code>t2(x)-f(x)</code>	$-x^3 + 3 \cdot p \cdot x^2 + 9 \cdot p^2 \cdot x + 5 \cdot p^3$
<code>a2:=$\int_{-p}^{5 \cdot p} (-x^3 + 3 \cdot p \cdot x^2 + 9 \cdot p^2 \cdot x + 5 \cdot p^3) dx$</code>	$108 \cdot p^4$
<code>$\frac{a2}{a1}$</code>	16

Question 2

- a.i.** $H \stackrel{d}{=} N(\mu = 58, \sigma^2 = 6^2)$
 $\Pr(H > 50) = 0.9088$ A1
- ii.** $Y \stackrel{d}{=} \text{Bi}(n = 12, p = 1 - 0.9088)$
 $\Pr(Y \geq 3) = 0.0894$ A1
- iii.** $X \stackrel{d}{=} \text{Bi}(n = ?, p = 0.9088)$
 $\Pr(X \geq 5) \geq 0.95$ to find the value of n , using trial and error
 $n = 7$ A1

normCdf(50, ∞, 58, 6)	0.908789
p := 0.90878871813013	0.908789
binomCdf(12, 1-p, 3, 12)	0.089416
binomCdf(n, p, 5, n) n=5	0.61989
binomCdf(n, p, 5, n) n=6	0.902595
binomCdf(n, p, 5, n) n=7	0.979952

- b.i.** $E \stackrel{d}{=} N(\mu = ?, \sigma^2 = ?^2)$
jumbo eggs, $J = \Pr(E > 66.7) = 0.023 \Rightarrow \Pr(E \leq 66.7) = 0.977$ M1
(1) $\frac{66.7 - \mu}{\sigma} = 1.9954$
medium eggs, $M = \Pr(E < 50) = 0.09$ A1
(2) $\frac{50 - \mu}{\sigma} = -1.34076$
solving (1) and (2) $\mu = 56.7$, $\sigma = 5.0$ A1

invNorm(1-0.023)	1.99539
invNorm(0.09)	-1.34076
$\frac{66.7 - m}{s} = 1.995393311222$	$\frac{66.7 - m}{s} = 1.99539$
$\frac{50 - m}{s} = -1.3407550347445$	$\frac{50 - m}{s} = -1.34076$
solve($\frac{66.7 - m}{s} = 1.9954$ and $\frac{50 - m}{s} = -1.34076, \{m, s\}$)	$s = 5.00576$ and $m = 56.7115$

b.ii. $\Pr(E \geq w) = \frac{35.2 + 2.3}{100} = 0.375$, $\Pr(E \leq w) = 0.625$

minimum weight for extra large egg is $w = 58.3$

A1

$\text{invNorm}\left(\frac{53.5+9}{100}, 56.7, 5\right)$	58.2932
--	---------

iii. completing the table

A1

	medium	large	extra large	jumbo
probability	0.09	0.535	0.352	0.023
egg weight grams	< 50	50 to < 58.3	58.3 to ≤ 66.7	> 66.7
revenue cents	5	15	20	25

expected revenue $E(R) = 0.09 \times 5 + 0.535 \times 15 + 0.352 \times 20 + 0.023 \times 25 = 16.09$
 $= 16$ cents

A1

iv. $\Pr(50 \leq E \leq 66.7 | E \leq 66.7) = \frac{\Pr(50 \leq E \leq 66.7)}{\Pr(E \leq 66.7)} = \frac{0.535 + 0.352}{1 - 0.023}$
 $= 0.908$

A1

$\frac{0.535 + 0.352}{1 - 0.023}$	0.907881
$0.09 \cdot 5 + 0.535 \cdot 15 + 0.352 \cdot 20 + 0.023 \cdot 25$	16.09

c.i. $E(\hat{p}) = \frac{x}{n} = \frac{5}{95} = 0.0526$

$\text{Var}(\hat{p}) = \frac{\hat{p}(1 - \hat{p})}{n} = \frac{0.0526 \times (1 - 0.0526)}{95} = 0.000525$

A1

ii. $Sd(\hat{P}) = \sqrt{0.000525} = 0.0229$

$$\Pr(0.0526 - 2 \times 0.0229 \leq P \leq 0.0526 + 2 \times 0.0229) = \Pr(0.0068 \leq P \leq 0.098)$$

for 95 eggs, we require

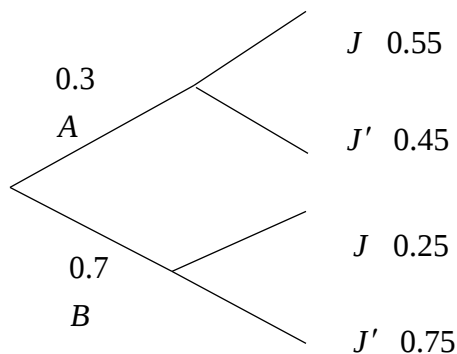
$$\Pr(0.0068 \times 95 \leq J \leq 0.098 \times 95) = \Pr(0.64 \leq J \leq 9.35) \quad \text{M1}$$

Since it is discrete, $J \stackrel{d}{=} \text{Bi}\left(n = 95, p = \frac{5}{95}\right)$

$$\Pr(1 \leq J \leq 9) = 0.966 \quad \text{A1}$$

$\frac{5}{95}$	0.052632
$\frac{5}{95} \cdot \frac{90}{95}$	0.000525
$\sqrt{5.2485785099868E-4}$	0.02291
$0.0526 + 2 \cdot 0.02291$	0.09842
$0.0526 - 2 \cdot 0.02291$	0.00678
$0.00678 \cdot 95$	0.6441
$0.09842 \cdot 95$	9.3499
$\text{binomCdf}\left(95, \frac{5}{95}, 1, 9\right)$	0.966125

d. tree diagram



$$\Pr(J') = \Pr(J' \cap A) + \Pr(J' \cap B) = 0.3 \times 0.45 + 0.7 \times 0.75 = 0.66 \quad \text{M1}$$

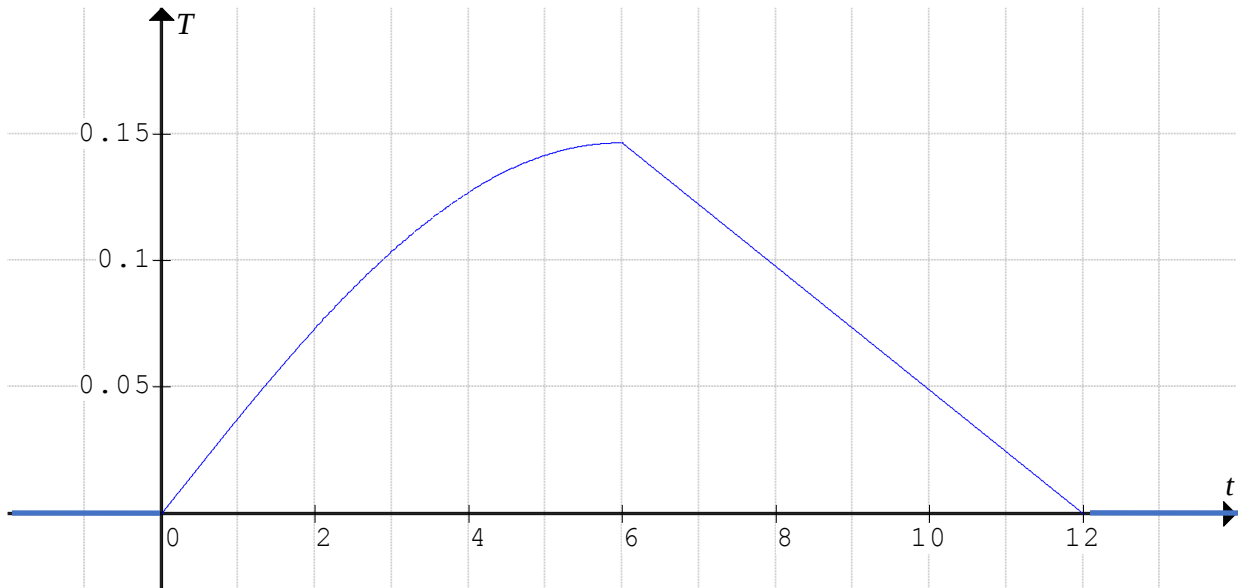
$$\Pr(B | J') = \frac{0.7 \times 0.75}{0.66} = 0.795 \quad \text{A1}$$

e.i. The function is continuous at $t = 6$, so that

$$\sin\left(\frac{\pi}{2}\right) = 1 = b\left(1 - \frac{6}{12}\right) = \frac{b}{2} \Rightarrow b = 2 \quad \text{M1}$$

ii. $\frac{\pi}{3(\pi+4)} \approx 0.147$, sine curve, graph joins up with straight line at $(6, 0.147)$

correct scaling, shape, zero elsewhere G2



$$\text{iii. } T(t) = \frac{\pi}{3(\pi+4)} \begin{cases} \sin\left(\frac{\pi t}{12}\right) & 0 \leq t \leq 6 \\ 2\left(1 - \frac{t}{12}\right) & 6 \leq t \leq 12 \\ 0 & \text{otherwise} \end{cases}$$

$$E(T) = \frac{\pi}{3(\pi+4)} \left[\int_0^6 t \sin\left(\frac{\pi t}{12}\right) dt + 2 \int_6^{12} t \left(1 - \frac{t}{12}\right) dt \right] = 5.6586$$

$$E(T^2) = \frac{\pi}{3(\pi+4)} \left[\int_0^6 t^2 \sin\left(\frac{\pi t}{12}\right) dt + 2 \int_6^{12} t^2 \left(1 - \frac{t}{12}\right) dt \right] = 38.3625 \quad \text{M1}$$

$$\text{Var}(T) = 38.3625 - (5.6586)^2 = 6.343 \quad \text{A1}$$

iv. Since $\frac{\pi}{3(\pi+4)} \int_0^6 \sin\left(\frac{\pi t}{12}\right) dt = 0.56$, the median m , satisfies

$$\frac{\pi}{3(\pi+4)} \int_0^m \sin\left(\frac{\pi t}{12}\right) dt = 0.5, \text{ solving gives } m = 5.589 \quad \text{A1}$$

Done

Define $fI(x) = \frac{\pi}{3 \cdot (\pi+4)} \cdot \begin{cases} \sin\left(\frac{\pi \cdot x}{12}\right), & 0 \leq x \leq 6 \\ 2 \cdot \left(1 - \frac{x}{12}\right), & 6 \leq x \leq 12 \end{cases}$

$\int_0^{12} fI(x) dx$	1.
$\int_0^{12} (x \cdot fI(x)) dx$	5.65863
$\int_0^{12} (x^2 \cdot fI(x)) dx$	38.3625
$38.3625 - (5.6586)^2$	6.34275
$\int_0^6 fI(x) dx$	0.560099
$\Delta \text{ solve } \left(\int_0^m fI(x) dx = 0.5, m \right) 0 < m < 6$	$m = 5.58935$

Question 3

a. $LHS = [C(x)]^2 - [S(x)]^2$

$$= \left[\frac{1}{2}(e^x + e^{-x}) \right]^2 - \left[\frac{1}{2}(e^x - e^{-x}) \right]^2$$

$$= \left[\frac{1}{4}(e^{2x} + 2 + e^{-2x}) \right] - \left[\frac{1}{4}(e^{2x} - 2 + e^{-2x}) \right] \quad \text{M1}$$

$$= \frac{1}{4}[(e^{2x} + 2 + e^{-2x}) - (e^{2x} - 2 + e^{-2x})] \quad \text{M1}$$

$$= 1 = RHS$$

b. $\frac{d}{dx}(S(kx)) = \frac{d}{dx} \left[\frac{1}{2}(e^{kx} - e^{-kx}) \right]$

$$= \frac{1}{2} \left[\frac{d}{dx}(e^{kx}) - \frac{d}{dx}(e^{-kx}) \right] \quad \text{M1}$$

$$= \frac{1}{2}(ke^{kx} + ke^{-kx})$$

$$= \frac{k}{2}(e^{kx} + e^{-kx}) = k \left[\frac{1}{2}(e^{kx} + e^{-kx}) \right] \quad \text{M1}$$

$$= kC(kx)$$

Also $\frac{d}{dx}(C(kx)) = \frac{d}{dx} \left[\frac{1}{2}(e^{kx} + e^{-kx}) \right] = \frac{1}{2} \left[\frac{d}{dx}(e^{kx}) + \frac{d}{dx}(e^{-kx}) \right]$

$$= \frac{1}{2}(ke^{kx} - ke^{-kx}) = \frac{k}{2}(e^{kx} - e^{-kx}) = k \left[\frac{1}{2}(e^{kx} - e^{-kx}) \right] = kS(kx)$$

c. $y = T(kx) = \frac{S(kx)}{C(kx)} = \frac{u}{v}$

$u = S(kx)$ and $v = C(kx)$ using the quotient rule M1

$$\frac{du}{dx} = kC(kx) \text{ and } \frac{dv}{dx} = kS(kx)$$

$$\frac{dy}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2} = \frac{k[C(kx)]^2 - k[S(kx)]^2}{[C(kx)]^2} \quad \text{M1}$$

$$\frac{dy}{dx} = \frac{d}{dx}[T(kx)] = \frac{k([C(kx)]^2 - [S(kx)]^2)}{[C(kx)]^2} = \frac{k}{[C(kx)]^2} \text{ from Question 3a} \quad \text{M1}$$

- d.** $S(x) = \frac{3}{4}$ that is $\frac{1}{2}(e^x - e^{-x}) = \frac{3}{4}$
 $e^x - e^{-x} = \frac{3}{2}$ Let $u = e^x$ $e^{-x} = \frac{1}{u}$
 $u - \frac{1}{u} = \frac{3}{2}$ multiply both sides by u , transposing $u^2 - \frac{3u}{2} = 1$
 $\left(u^2 - \frac{3u}{2} + \frac{9}{16}\right) = 1 + \frac{9}{16}$ completing the square M1
 $\left(u - \frac{3}{4}\right)^2 = \frac{25}{16}$
 $u - \frac{3}{4} = \pm \frac{5}{4}$
 $u = e^x = 2, -\frac{1}{4}$ but $e^x > 0$ reject the negative
 $x = \log_e(2)$ A1
- e.** The graph of $y = S(x)$ is a one-one function, so it has an
inverse which is a function. The domain and range are both R . A1
If $y = S(x) = \frac{1}{2}(e^x - e^{-x})$ then the inverse function $x = S^{-1}(y)$
so that $2y = e^x - e^{-x}$
 $2y - e^x + e^{-x} = 0$ multiplying by e^x
 $e^{2x} - 2ye^x - 1 = 0$ let $u = e^x$
 $u^2 - 2yu - 1 = 0$ solving for u using the quadratic formulae
 $\Delta = 4y^2 + 4 = 4(y^2 + 1)$ M1
 $u = e^x = \frac{2y \pm \sqrt{4(y^2 + 1)}}{2}$ take the positive since $u = e^x > 0$
 $e^x = y + \sqrt{y^2 + 1}$ so that $x = S^{-1}(y) = \log_e(y + \sqrt{y^2 + 1})$
so we have the result
 $y = S^{-1}(x) = \log_e(x + \sqrt{x^2 + 1})$ domain and range both R . A1
to check the result from **d.**
 $S(x) = \frac{3}{4}$
 $x = S^{-1}\left(\frac{3}{4}\right) = \log_e\left(\frac{3}{4} + \sqrt{\left(\frac{3}{4}\right)^2 + 1}\right) = \log_e\left(\frac{3}{4} + \sqrt{\frac{25}{16}}\right) = \log_e(2)$ M1

Question 4

a.i. by similar triangles $\frac{1.5}{y} = \frac{x}{2} \Rightarrow y = \frac{1.5 \times 2}{x} = \frac{3}{x}$ A1

ii. By Pythagoras's Theorem $AC = \sqrt{x^2 + 4}$ and $CB = \sqrt{y^2 + 1.5^2}$

$$L(x) = AC + CB = \sqrt{x^2 + 4} + \sqrt{y^2 + 1.5^2} \quad \text{substitute for } y,$$

$$L(x) = \sqrt{x^2 + 4} + \sqrt{\frac{9}{x^2} + \frac{9}{4}} \quad \text{M1}$$

$$= \sqrt{x^2 + 4} + \sqrt{\frac{9(x^2 + 4)}{4x^2}} \quad \text{since } x > 0$$

$$L(x) = \sqrt{x^2 + 4} + \frac{3}{2x} \sqrt{x^2 + 4} \quad \text{M1}$$

$$= \left(\frac{3}{2x} + 1 \right) \sqrt{x^2 + 4}$$

iii. $\frac{dL}{dx} = \frac{x^3 - 6}{x^2 \sqrt{x^2 + 4}}$ A1

$$\frac{dL}{dx} = 0 \Rightarrow x^3 = 6$$

$$x = \sqrt[3]{6} \quad \text{A1}$$

iv. $L(\sqrt[3]{6}) = 4.933\text{m}$ A1

Define $l(x) = \sqrt{x^2 + 4} + \sqrt{y^2 + \frac{9}{4}}$	Done
$l(x) _{y=\frac{3}{x}} \text{ and } x > 0$	$\frac{(2 \cdot x + 3) \cdot \sqrt{x^2 + 4}}{2 \cdot x}$
Define $l(x) = \frac{(2 \cdot x + 3) \cdot \sqrt{x^2 + 4}}{2 \cdot x}$	Done
$\frac{d}{dx}(l(x)) _{x > 0}$	$\frac{2 \cdot x + 3}{2 \cdot \sqrt{x^2 + 4}} - \frac{3 \cdot \sqrt{x^2 + 4}}{2 \cdot x^2}$
$\text{comDenom}\left(\frac{2 \cdot x + 3}{2 \cdot \sqrt{x^2 + 4}} - \frac{3 \cdot \sqrt{x^2 + 4}}{2 \cdot x^2}\right)$	$\frac{x^3 - 6}{x^2 \cdot \sqrt{x^2 + 4}}$
$\text{solve}\left(\frac{x^3 - 6}{x^2 \cdot \sqrt{x^2 + 4}} = 0, x\right)$	$\frac{1}{x = \sqrt[3]{6}}$
$l(x) _{x = \sqrt[3]{6}}$	4.93283

$$\text{b.i.} \quad \cos(\theta) = \frac{2}{AC} \Rightarrow AC = \frac{2}{\cos(\theta)}$$

$$\sin(\theta) = \frac{1.5}{CB} \Rightarrow CB = \frac{3}{2\sin(\theta)}$$

$$L(\theta) = AC + BC = \frac{2}{\cos(\theta)} + \frac{3}{2\sin(\theta)} \quad \text{for } 0 < \theta < 90^\circ \quad \text{A1}$$

$$\text{ii.} \quad \frac{dL}{d\theta} = \frac{\pi(4\sin^3(\theta) - 3\cos^3(\theta))}{360\sin^2(\theta)\cos^2(\theta)}$$

$$\frac{dL}{d\theta} = 0 \Rightarrow 4\sin^3(\theta) - 3\cos^3(\theta) = 0 \quad \text{A1}$$

$$\tan^3(\theta) = \frac{3}{4} \quad \tan(\theta) = \sqrt[3]{\frac{3}{4}}$$

$$\theta = \tan^{-1}\left(\sqrt[3]{\frac{3}{4}}\right) = 42.26^\circ \quad \text{A1}$$

Define $l(\theta) = \frac{2}{\cos(\theta)} + \frac{3}{2 \cdot \sin(\theta)}$		Done
	$\frac{d}{d\theta}(l(\theta))$	$\frac{-(3 \cdot (\cos(\theta))^3 - 4 \cdot (\sin(\theta))^3) \cdot \pi}{360 \cdot (\sin(\theta))^2 \cdot (\cos(\theta))^2}$
	solve($3 \cdot (\cos(\theta))^3 - 4 \cdot (\sin(\theta))^3 = 0, \theta$) $0 < \theta < 90$	$\theta = 42.257$
	$\tan^{-1}\left(\sqrt[3]{\frac{3}{4}}\right)$	42.257

c.i. Method 1

$$L(x) = AC + CB = \sqrt{x^2 + b^2} + \sqrt{y^2 + a^2} \quad \text{by similar triangles} \quad \frac{a}{y} = \frac{x}{b} \Rightarrow y = \frac{ab}{x}$$

$$L(x) = \sqrt{x^2 + b^2} + \sqrt{\frac{a^2 b^2}{x^2} + a^2} \quad \text{A1}$$

$$= \sqrt{x^2 + b^2} + \sqrt{\frac{a^2 (x^2 + b^2)}{x^2}} = \sqrt{x^2 + b^2} \left(\frac{a + x}{x} \right) \quad \text{since } x > 0, a > 0$$

ii. $\frac{dL}{dx} = \frac{x^3 - ab^2}{x^2 \sqrt{x^2 + b^2}}$

$$\frac{dL}{dx} = 0 \quad \text{solving for } x, \quad \text{M1}$$

$$\Rightarrow x^3 = ab^2 \Rightarrow x = \sqrt[3]{ab^2} = (ab^2)^{\frac{1}{3}}$$

$$L_{\min} = L(\sqrt[3]{ab^2}) = \sqrt{\left((ab^2)^{\frac{2}{3}} + b^2\right)} \left(\frac{a + \sqrt[3]{ab^2}}{\sqrt[3]{ab^2}}\right)$$

$$L_{\min} = \left(a^{\frac{2}{3}} + b^{\frac{2}{3}}\right)^{\frac{3}{2}} \quad m = \frac{2}{3}, n = \frac{3}{2} \quad \text{A2}$$

Define $l(x) = \sqrt{x^2 + b^2} + \sqrt{y^2 + a^2}$
Done

$l(x)|_{y=\frac{a \cdot b}{x}} \text{ and } a > 0 \text{ and } x > 0$
 $\frac{(x+a) \cdot \sqrt{x^2 + b^2}}{x}$

Define $l(x) = \frac{(x+a) \cdot \sqrt{x^2 + b^2}}{x}$
Done

$\frac{d}{dx}(l(x))$
 $\frac{x+a}{\sqrt{x^2 + b^2}} - \frac{a \cdot \sqrt{x^2 + b^2}}{x^2}$

$\text{comDenom}\left(\frac{x+a}{\sqrt{x^2 + b^2}} - \frac{a \cdot \sqrt{x^2 + b^2}}{x^2}\right)$
 $\frac{x^3 - a \cdot b^2}{x^2 \cdot \sqrt{x^2 + b^2}}$

$\text{solve}\left(\frac{d}{dx}(l(x)) = 0, x\right)$
 $x = a^{\frac{1}{3}} \cdot b^{\frac{2}{3}}$

$l(x)|_{x=a^{\frac{1}{3}} \cdot b^{\frac{2}{3}}}$
 $\left(\frac{2}{a^{\frac{2}{3}} + b^{\frac{2}{3}}}\right)^{\frac{3}{2}}$

c.i. Method 2

The angle θ is at A.

$$L(\theta) = \frac{a}{\sin(\theta)} + \frac{b}{\cos(\theta)} \quad \text{A1}$$

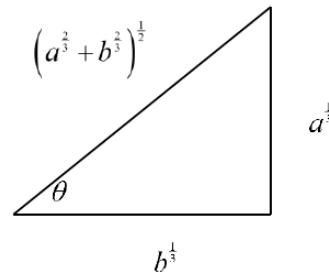
ii.
$$\frac{dL}{d\theta} = \frac{\pi(-a \cos^3(\theta) + b \sin^3(\theta))}{180 \sin^2(\theta) \cos^2(\theta)}$$

$$\frac{dL}{d\theta} = 0 \Rightarrow -a \cos^3(\theta) + b \sin^3(\theta) = 0 \quad \text{M1}$$

$$\tan^3(\theta) = \frac{a}{b} \Rightarrow \tan(\theta) = \left(\frac{a}{b}\right)^{\frac{1}{3}}$$

$$\Rightarrow \theta = \tan^{-1}\left(\frac{a}{b}\right)^{\frac{1}{3}}$$

so that



$$\frac{1}{\sin(\theta)} = \frac{(a^{\frac{2}{3}} + b^{\frac{2}{3}})^{\frac{1}{2}}}{a^{\frac{1}{3}}} \quad \text{and} \quad \frac{1}{\cos(\theta)} = \frac{(a^{\frac{2}{3}} + b^{\frac{2}{3}})^{\frac{1}{2}}}{b^{\frac{1}{3}}}$$

$$L_{\min} = L\left(\tan^{-1}\left(\frac{a}{b}\right)^{\frac{1}{3}}\right) = a \frac{(a^{\frac{2}{3}} + b^{\frac{2}{3}})^{\frac{1}{2}}}{a^{\frac{1}{3}}} + b \frac{(a^{\frac{2}{3}} + b^{\frac{2}{3}})^{\frac{1}{2}}}{b^{\frac{1}{3}}}$$

$$L_{\min} = (a^{\frac{2}{3}} + b^{\frac{2}{3}})^{\frac{1}{2}} [a^{\frac{2}{3}} + b^{\frac{2}{3}}]$$

$$L_{\min} = (a^{\frac{2}{3}} + b^{\frac{2}{3}})^{\frac{3}{2}} \quad m = \frac{2}{3}, n = \frac{3}{2} \quad \text{A2}$$

Define $l(\theta) = \frac{a}{\sin(\theta)} + \frac{b}{\cos(\theta)}$

Done

$\frac{d}{d\theta}(l(\theta))$

$$\frac{-(\cos(\theta))^3 \cdot a - (\sin(\theta))^3 \cdot b}{180 \cdot (\sin(\theta))^2 \cdot (\cos(\theta))^2} \cdot \pi$$

$l(\theta) | \theta = \tan^{-1}\left(\sqrt[3]{\frac{a}{b}}\right) \text{ and } b > 0$

$$\left(\frac{2}{a^{\frac{2}{3}} + b^{\frac{2}{3}}}\right)^{\frac{3}{2}}$$

END OF SUGGESTED SOLUTIONS