2017 Mathematical Methods Trial Exam 2 Solutions

SECTION A – Multiple-choice questions

1	2	3	4	5	6	7	8	9	10
E	D	C	E	A	D	B	A	A	B

11	12	13	14	15	16	17	18	19	20
D	C	E	D	E	A	A	B	A	B

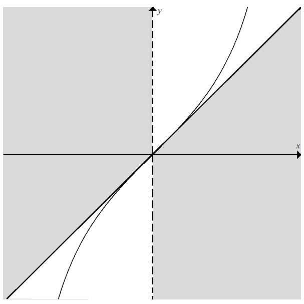
Q1 $y=\left(a\left(\frac{b+4}{a}\right)-b\right)^{1.5}=8$ E

Q2 $x-1\geq 0$ and $2+x>0$, .: $x\geq 1$ D

Q3 $g(x)=0.2f(0.1x+0.2)+0.1$
$g(10x-2)=0.2f(0.1(10x-2)+0.2)+0.1=0.2f(x)+0.1$
$h(x)=5g(10x-2)-0.5=f(x)$ C

Q4 The inverse is $x=e^{\left(\frac{y}{10}\right)}-\log_e\left(\frac{y}{10}\right)$ E

Q5 $y=mx$ lies in the shaded region, .: $m\leq 1$ A



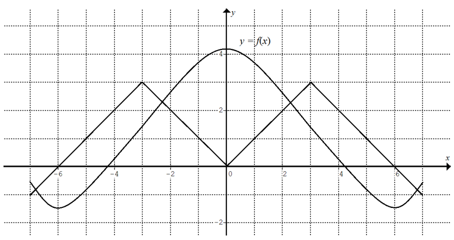
Q6 $\cos\left(\frac{\pi(x-1)}{2}\right)=\sin\left(\frac{\pi x}{2}\right)$, $\cos\left(\frac{\pi(x+1)}{2}\right)=-\sin\left(\frac{\pi x}{2}\right)$ D

Q7 $(x-p)^3=x^3-3px^2+3p^2x+p^3$, .: $a=-3p$, $b=3p^2$
.: $a^2=3b$ B

Q8 Try $x=0$, $by+z=1$, $dy+z=0$, eliminate z, $y=\frac{1}{b-d}$ and
$z=-\frac{d}{b-d}=\frac{d}{d-b}$ A

Q9 $\frac{15-6}{12}=0.75$ A

Q10 The graph shows that $y=f(x)$ is an even function.
.: $f(-x)=f(x)$, $f(x)-f(-x)=0$. B



Q11 $e^{2x}=e^x+c$, $\left(e^x\right)^2-e^x-c=0$, $e^x=\frac{1\pm\sqrt{1+4c}}{2}$

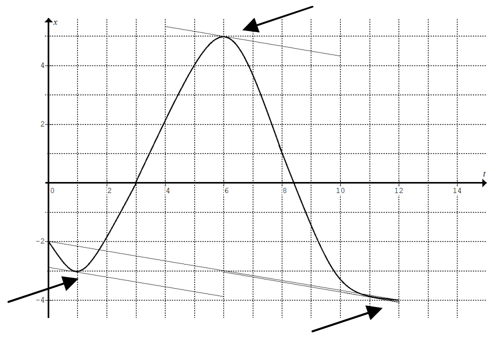
Two distinct roots if $1+4c>0$ and $\frac{1-\sqrt{1+4c}}{2}>0$

.: $-\frac{1}{4}<c<0$ D

Q12 The curve $y=a\sin(x)\cos\left(\frac{x}{2}\right)$ including the point at $x=\frac{1}{2}$ is dilated by the same factor in both *x* and *y* directions, .: same gradient at $x=\frac{1}{4}$. C

Q13 $\left[\frac{e^{bx}}{b}-\frac{2e^{\frac{bx}{2}}}{b}\right]_0^a=\frac{1}{b}$, $e^{\frac{ab}{2}}\left(e^{\frac{ab}{2}}-2\right)=0$, $ab=\log_e 4$ E

Q14 D



Q15 Number of squares bounded by the graph and the *t*-axis ≈ 32
Each square represents 1 metre distance.

Average speed $\approx\frac{32}{12}\approx 2.7$ E

Q16 $\frac{\Pr(5.5<X<11)}{\Pr(X>5.5)}=\frac{0.6}{0.65}\approx 0.92$ A

Q17 $\int_0^{2\pi}\frac{a}{2}(1-\cos x)dx=1$, .: $a=\frac{1}{\pi}$
$\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\frac{1}{2\pi}(1-\cos x)dx=\frac{\pi+2}{2\pi}$ A

Q18 Choice B is not a random variable, it is a subset of the sample space. B

Q19 $n=4$, $p=0.1$ and $q=0.9$
.: $\mu=np=0.4$ and $\sigma=\sqrt{npq}=0.6$ A

Q20 The mean of $\hat{P}=p=0.60$

The standard deviation of $\hat{P}=\sqrt{\frac{p(1-p)}{n}}=0.04$

$\Pr\left(\hat{P}<0.56\right)\approx 0.16$ (normal approx)
Number of samples $\approx 20\times 0.16\approx 3.2$ B

SECTION B

Q1a $y = x(2a - x)$, $p = \dfrac{2a+0}{2} = a$, $q = 2a^2 - a^2 = a^2$

Q1bi $y = mx + c$ is the equation of the tangent.

Let $mx + c = 2ax - x^2$, .: $x^2 + (m - 2a)x + c = 0$

.: $(m-2a)^2 - 4c = 0$

Q1bii From part bi, $(m-2a)^2 = 4c$, $m - 2a = \pm 2\sqrt{c}$

.: $m = 2\left(a \pm \sqrt{c}\right)$

Given $c > q$, i.e. $c > a^2$, .: $\sqrt{c} > a$, $a - \sqrt{c} < 0$

.: $m = 2\left(a - \sqrt{c}\right)$ for negative m.

Q1ci When $a = 1$, $m = 2\left(1 - \sqrt{c}\right)$, $y = 2\left(1 - \sqrt{c}\right)x + c$

.: $2\left(\sqrt{c} - 1\right)x + y = c$

Q1cii Let $y = 0$ for x-intercept, .: $x = \dfrac{c}{2\left(\sqrt{c}-1\right)}$

.: $B = \left(\dfrac{c}{2\left(\sqrt{c}-1\right)}, 0\right)$

Q1d Area of $\Delta AOB = \dfrac{1}{2} \times \dfrac{c}{2\left(\sqrt{c}-1\right)} \times c = \dfrac{c^2}{4\left(\sqrt{c}-1\right)} = \dfrac{c^2\left(\sqrt{c}+1\right)}{4(c-1)}$

Q1e Let $\dfrac{d}{dc}\dfrac{c^2}{4\left(\sqrt{c}-1\right)} = 0$, $c = \left(\dfrac{4}{3}\right)^2 = \dfrac{16}{9}$

Min. area of $\Delta AOB = \dfrac{c^2}{4\left(\sqrt{c}-1\right)} = \dfrac{\left(\frac{4}{3}\right)^4}{\frac{4}{3}} = \left(\dfrac{4}{3}\right)^3$

Q1f Area bounded by the parabola and the x-axis:

$\int_0^2 \left(2x - x^2\right)dx = \dfrac{4}{3}$

.: min. shaded area $= \left(\dfrac{4}{3}\right)^3 - \dfrac{4}{3} = \dfrac{28}{27}$

Q2a $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

Q2b Same area as the region bounded by $y = e^x$, $y = -e^{-x}$ and $x = \pm 1$.

$\int_{-1}^{1} \left(e^x - \left(-e^{-x}\right)\right)dx = 2e$

Q2c $g'(x) = -\dfrac{1}{x}$ for $x < 0$

Q2d $Q(-\alpha, -\log_e \alpha)$

Q2e $\left(\overline{PQ}\right)^2 = (\alpha - (-\alpha))^2 + (\log_e \alpha - (-\log_e \alpha))^2 = 4\left(\alpha^2 + (\log_e \alpha)^2\right)$

.: $\overline{PQ} = 2\sqrt{\alpha^2 + (\log_e \alpha)^2}$

Q2f Let $\dfrac{d}{d\alpha} 2\sqrt{\alpha^2 + (\log_e \alpha)^2} = 0$

$\dfrac{1}{\sqrt{\alpha^2 + (\log_e \alpha)^2}} \times \left(2\alpha + \dfrac{2\log_e \alpha}{\alpha}\right) = 0$

.: $2\alpha + \dfrac{2\log_e \alpha}{\alpha} = 0$, .: $\alpha^2 + \log_e \alpha = 0$

Q2g $\alpha \approx 0.653$, $\overline{PQ} \approx 1.55954 \approx 1.560$

Q2h Since $\alpha^2 + \log_e \alpha = 0$, .: $\dfrac{\log_e \alpha}{\alpha^2} = -1$

Gradient of $\overline{PQ} = \dfrac{\log_e \alpha}{\alpha}$, gradient of both tangents $= \dfrac{1}{\alpha}$

.: gradient of $\overline{PQ} \times$ gradient of each tangent $= \dfrac{\log_e \alpha}{\alpha^2} = -1$

.: PQ is a common normal to both curves.

Q2i Area of the largest square $\approx 1.55954^2 \approx 2.432$

Q3a

$f(x)=0.0012833\,x^2\left(x^2-8\pi^2\right)=0.0012833\,x^2\left(x^2-\left(2\sqrt{2}\pi\right)^2\right)$

$=0.0012833\,x^2\left(x+2\sqrt{2}\pi\right)\left(x-2\sqrt{2}\pi\right)$

.: $a=0.0012833$ and $b=c=2\sqrt{2}\pi$

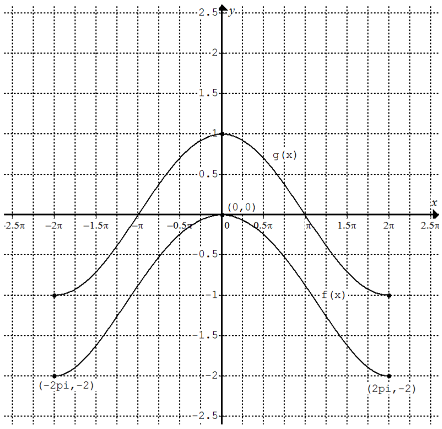
Q3b $f'(x)=0.0051332\,x^3-0.0205328\,\pi^2 x$

Let $f'(x)=0$, $0.0051332\,x^3-0.0205328\,\pi^2 x=0$

$0.0051332\,x\left(x^2-4\,\pi^2\right)=0.0051332\,x(x+2\pi)(x-2\pi)=0$

.: stationary points are at $x=-2\pi$, 0 and 2π

Q3c and d



Q3e Area $=\int_{-2\pi}^{2\pi}\left(f(x)+1-g(x)\right)dx\approx 0.837$ by CAS

Q3f $x\approx -3.400352\approx -3.400$, or $x\approx 3.400$ by CAS

x-intercepts: $(-3.400, 0)$ and $(3.400, 0)$

Q3g $3.400352\to\pi$, .: $m\approx\dfrac{\pi}{3.400352}\approx 0.923902\approx 0.924$

Q3h $x\to\dfrac{x}{m}$, .: $p=m^4\approx 0.729$ and $q=m^2\approx 0.854$

Q4a $\int_0^{1.5} a\,x^2\,(x-1.5)^2 e^x\,dx=1$,

$$a=\frac{1}{\int_0^{1.5} x^2\,(x-1.5)^2 e^x\,dx}\approx\frac{1}{0.55773524}\approx 1.793 \text{ by CAS}$$

Q4b $\overline{X}=\int_0^{1.5} x\,f(x)dx\approx 0.82965\approx 0.830$ by CAS

Q4c $\int_0^{m}\left(\dfrac{1}{0.55773524}\right)x^2\,(x-1.5)^2 e^x\,dx=0.5$, $m\approx 0.843$ by CAS

Q4di Binomial: $n=10$, $p=0.137$

$\Pr(X<3)=\Pr(X\le 2)\approx 0.8527664\approx 0.853$ by CAS

Q4dii $\Pr(X<3\mid X\ge 1)=\dfrac{\Pr(1\le X\le 2)}{1-\Pr(X=0)}\approx 0.8089999\approx 0.809$

Q4e The mean of sample proportion $\approx p=0.137$,

the standard deviation of sample proportion

$\approx\sqrt{\dfrac{p(1-p)}{n}}\approx 0.03438\approx 0.034$

Q4f $\Pr\left(\hat{P}<0.1\mid \hat{P}<0.2\right)=\dfrac{\Pr\left(\hat{P}<0.1\right)}{\Pr\left(\hat{P}<0.2\right)}\approx 0.1428\approx 0.143$ by CAS

Q4g For sample size of 10, assuming the mean of sample proportion $\approx p=0.137$, the standard deviation of sample proportion

$\approx\sqrt{\dfrac{p(1-p)}{n}}\approx 0.109$

$\Pr\left(\hat{P}<0.1\mid \hat{P}<0.2\right)=\dfrac{\Pr\left(\hat{P}<0.1\right)}{\Pr\left(\hat{P}<0.2\right)}\approx 0.511$

For sample size of 200, the mean of sample proportion $\approx p=0.137$,

the standard deviation of sample proportion $\approx\sqrt{\dfrac{p(1-p)}{n}}\approx 0.024$

$\Pr\left(\hat{P}<0.1\mid \hat{P}<0.2\right)=\dfrac{\Pr\left(\hat{P}<0.1\right)}{\Pr\left(\hat{P}<0.2\right)}\approx 0.062$

$\Pr\left(\hat{P}<0.1\mid \hat{P}<0.2\right)\approx 0.062$ is more reliable than

$\Pr\left(\hat{P}<0.1\mid \hat{P}<0.2\right)\approx 0.511$ because the latter has a small sample size and normal approximation is not valid.

Q4h An approximate 95% confidence interval

$$\approx\left(0.137-1.96\sqrt{\frac{0.137(1-0.137)}{100}},\ 0.137+1.96\sqrt{\frac{0.137(1-0.137)}{100}}\right)$$

$\approx(0.070, 0.204)$

Please inform mathline@itute.com re conceptual and/or mathematical errors