

YEAR 12 *Trial Exam Paper*

2017

MATHEMATICAL METHODS

Written examination 1

Worked solutions

This book presents:

- worked solutions, giving you a series of points to show you how to work through the questions
- mark allocations
- tips on how to approach the exam

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Question 1a.**Worked solution**

Using the quotient rule gives

$$\frac{dy}{dx} = \frac{(x^3 - 3x)\cos(x) - (3x^2 - 3)\sin(x)}{(x^3 - 3x)^2}$$

Mark allocation: 2 marks

- 1 method mark for attempting to use the quotient rule
- 1 answer mark for correct solution

**Tips**

- You could also use the product rule to find this derivative.
- You do not need to simplify the answer further.

Question 1b.**Worked solution**

Using the product and chain rule gives

$$f'(x) = 3x^2e^{1-4x} - 4x^3e^{1-4x}$$

And then evaluate with $x = -1$, which gives $f'(-1) = 3e^5 + 4e^5 = 7e^5$.

Mark allocation: 2 marks

- 1 answer mark for correct derivative
- 1 answer mark for stating $f'(-1) = 7e^5$

**Tips**

- Don't forget to evaluate.
- Simplify your answer to get full marks.

Question 2a.**Worked solution**

Using the chain rule gives

$$\begin{aligned} f'(x) &= \frac{1}{2}(5-4x)^{-\frac{1}{2}} \times -4 \\ &= \frac{-2}{\sqrt{5-4x}} \end{aligned}$$

Mark allocation: 1 mark

- 1 answer mark for correct solution in simplified form

Question 2b.**Worked solution**

At $x = -1$, $\frac{dy}{dx} = \frac{-2}{3}$, and $y = 5$.

So the equation of the tangent line is

$$y - y_1 = m(x - x_1)$$

$$y - 5 = \frac{-2}{3}(x + 1)$$

$$y = \frac{-2x}{3} + \frac{13}{3}$$

$$3y + 2x = 13$$

$$2x + 3y = 13$$

Mark allocation: 3 marks

- 1 method mark for finding $m = \frac{-2}{3}$
- 1 method mark for finding $y = 5$ when $x = -1$
- 1 answer mark for correct equation, in the correct form

**Tips**

- This question could also be solved using the equation $y = mx + c$

Question 3a.**Worked solution**

To find the x -intercept, let $f(x) = 0$

$$\begin{aligned} 0 &= 3 - \frac{4}{2x+1} \\ 3 &= \frac{4}{2x+1} \\ 2x+1 &= \frac{4}{3} \\ 2x &= \frac{1}{3} \\ x &= \frac{1}{6} \end{aligned}$$

To find the y -intercept, let $x = 0$

$$\begin{aligned} y &= 3 - \frac{4}{0+1} = 3 - 4 \\ y &= -1 \end{aligned}$$

Find the asymptotes:

Method 1

To find the asymptotes, rewrite the equation as $f(x) = \frac{-4}{2\left(x + \frac{1}{2}\right)} + 3$.

In this form, by inspection, the equations of the asymptotes can be identified as

$$y = 3 \text{ and } x = -\frac{1}{2}.$$

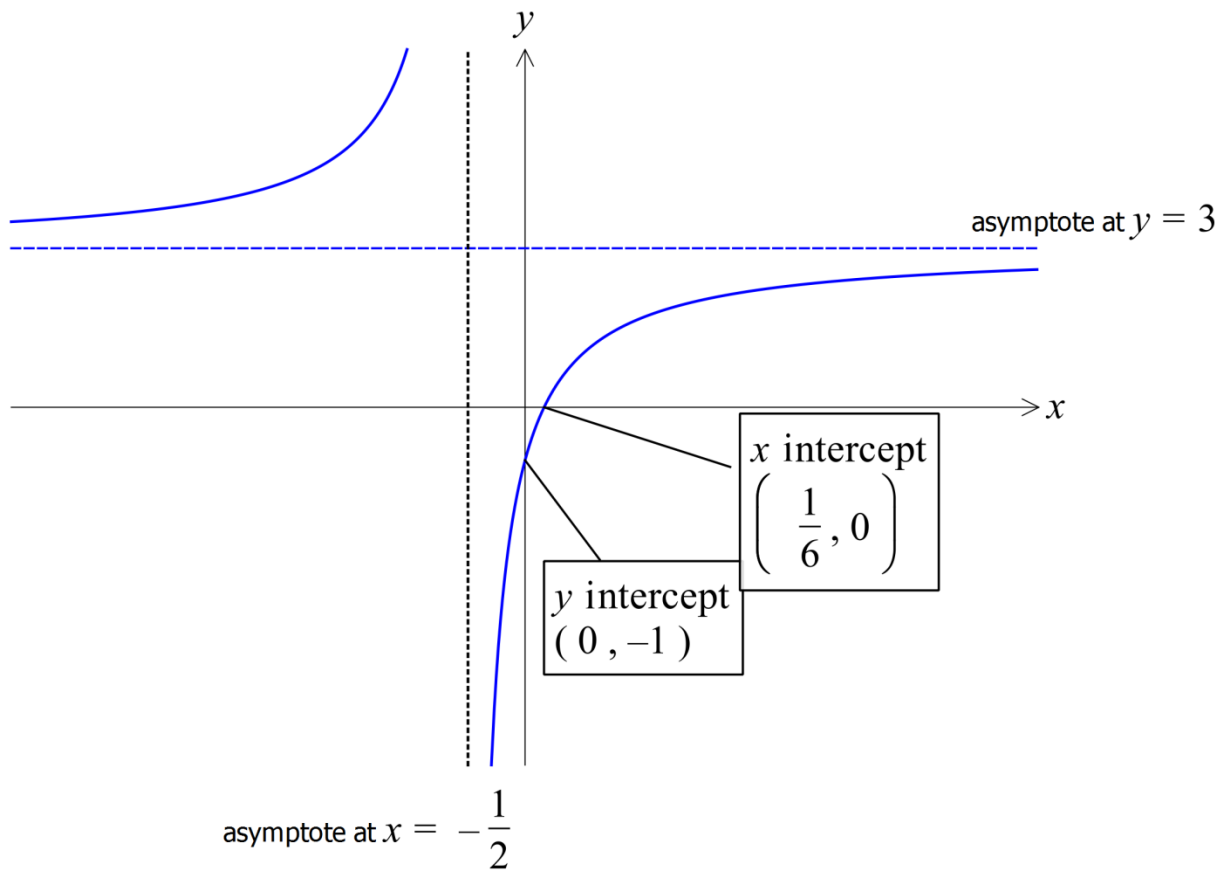
Alternatively,

Method 2

As $x \rightarrow +\infty$, $f(x) \rightarrow 3$ and as $x \rightarrow -\infty$, $f(x) \rightarrow 3$, therefore the horizontal asymptote is at $y = 3$.

Function is undefined when the denominator is equal to zero, so when $2x+1=0$, $x = -\frac{1}{2}$.

Therefore vertical asymptote is at $x = -\frac{1}{2}$.



Mark allocation: 3 marks

- 1 mark for shape; must be a negative hyperbola with asymptotic behaviour
- $\frac{1}{2}$ mark for each correctly labelled intercepts and equations of asymptotes (up to 2 marks)



Tip

- *Make sure asymptotes are marked in with dotted lines.*

Question 3b.**Worked solution**

$$\begin{aligned}
 \text{Area} &= \int_1^3 3 - \frac{4}{2x+1} dx \\
 &= [3x - 2\log_e(2x+1)]_1^3 \\
 &= (9 - 2\log_e(7)) - (3 - 2\log_e(3)) \\
 &= 6 - 2\log_e\left(\frac{7}{3}\right) \\
 &= 6 + 2\log_e\left(\frac{3}{7}\right) \\
 &= 6 + \log_e\left(\frac{9}{49}\right)
 \end{aligned}$$

So $a = 6$, $b = 7$ and $c = 49$.

Mark allocation: 3 marks

- 1 method mark for setting up the integral
- 1 method mark for correctly antidifferentiating
- 1 answer mark for the correct area in the correct form and stating the values of a , b and c

**Tip**

- *Make sure you write your answer for the area in the given form so that you can correctly state the values of a , b and c .*

Question 4a.**Worked solution**

$$\Pr(\text{germinating}) = \frac{1}{4}, \text{ therefore } \Pr(\text{not germinating}) = \frac{3}{4}$$

Assuming that each seed germinates or not independently of any other, then

$$\Pr(\text{the 6 seeds fail to germinate}) = \left(\frac{3}{4}\right)^6$$

Mark allocation: 1 mark

- 1 answer mark for correct solution

Question 4b.**Worked solution**

Let X be the random variable ‘Number of seeds that germinate’.

$$X \sim \text{Binomial}\left(p = \frac{1}{4}, n = 6\right)$$

$$\begin{aligned} \Pr(X = 2) &= \binom{6}{2} \left(\frac{1}{4}\right)^2 \left(\frac{3}{4}\right)^4 \\ &= \frac{15 \times 81}{4096} \\ &= \frac{1215}{4096} \end{aligned}$$

So $k = 1215$

Mark allocation: 2 marks

- 1 method mark for knowing to use $\binom{6}{2}$
- 1 answer mark for correct value of k

Question 4c.**Worked solution**

Four pots of six seeds, therefore 24 seeds in total.

We require each seed not to germinate and for this to happen on 24 occasions, so $\left(\frac{3}{4}\right)^{24}$.

Mark allocation: 1 mark

- 1 answer mark for correct solution

Question 5a.**Worked solution**

$$f'(x) = \frac{-1}{(x+2)^2} + \frac{1}{(4-x)^2}$$

Mark allocation: 1 mark

- 1 answer mark for correct derivative

**Tip**

- You must use the correct notation $f'(x)$.

Question 5b.**Worked solution**

Let $f'(x) = 0$

$$\Rightarrow \frac{1}{(x+2)^2} = \frac{1}{(4-x)^2}$$

$$x^2 + 4x + 4 = 16 - 8x + x^2$$

$$12x = 12$$

Therefore, $x = 1$, $f(1) = \frac{2}{3}$.

Check the endpoints: at $x = -1$, $f(-1) = \frac{6}{5}$

and at $x = 2$, $f(2) = \frac{3}{4}$.

So range is $\left[\frac{2}{3}, \frac{6}{5}\right]$

Mark allocation: 2 marks

- 1 method mark for finding the values of $f(-1)$ and $f(2)$
- 1 answer mark for correct range

**Tip**

- Always check endpoints and maximum and minimum points when finding the range of a restricted function.

Question 5c.**Worked solution**

$$\begin{aligned}
 f(x) &= \frac{1}{x+2} + \frac{1}{4-x} \\
 &= \frac{4-x}{(x+2)(4-x)} + \frac{x+2}{(x+2)(4-x)} \\
 &= \frac{6}{(x+2)(4-x)} \\
 &= \frac{-6}{(x+2)(x-4)} \\
 &= \frac{-6}{x^2 - 2x - 8}
 \end{aligned}$$

So $k = -6$.

Mark allocation: 2 marks

- 1 method mark
- 1 answer mark for $k = -6$

Question 5d.**Worked solution**

For the inverse function to exist, the graph needs to be one-to-one. For the graph to be one-to-one, restrict the domain of the graph at the turning point.

So $a = 1$.

Mark allocation: 1 mark

- 1 answer mark for correct solution

Question 5e.**Worked solution**

Let $x = \frac{-6}{y^2 - 2y - 8}$, where $y = g^{-1}(x)$.

$$x = \frac{-6}{(y-1)^2 - 9}$$

$$(y-1)^2 - 9 = \frac{-6}{x}$$

$$(y-1)^2 = 9 - \frac{6}{x}$$

$$y-1 = \pm \sqrt{9 - \frac{6}{x}}$$

$$y = 1 \pm \sqrt{9 - \frac{6}{x}}$$

Since $\text{dom } g(x) = [-1, 1]$ and $\text{dom } g(x) = \text{ran } g^{-1}(x)$, only the negative square root is required.

$$g^{-1}(x) = 1 - \sqrt{9 - \frac{6}{x}}$$

Mark allocation: 2 marks

- 1 method mark for swapping x and y and for forming a completed square, $(y-1)^2 - 9$.
- 1 answer mark for correct inverse

**Tip**

- Ensure you use the correct notation, g^{-1} , and give consideration to the domain.

Question 6a.**Worked solution**

$$2 \cos(2x) - 1 = 0$$

$$2 \cos(2x) = 1$$

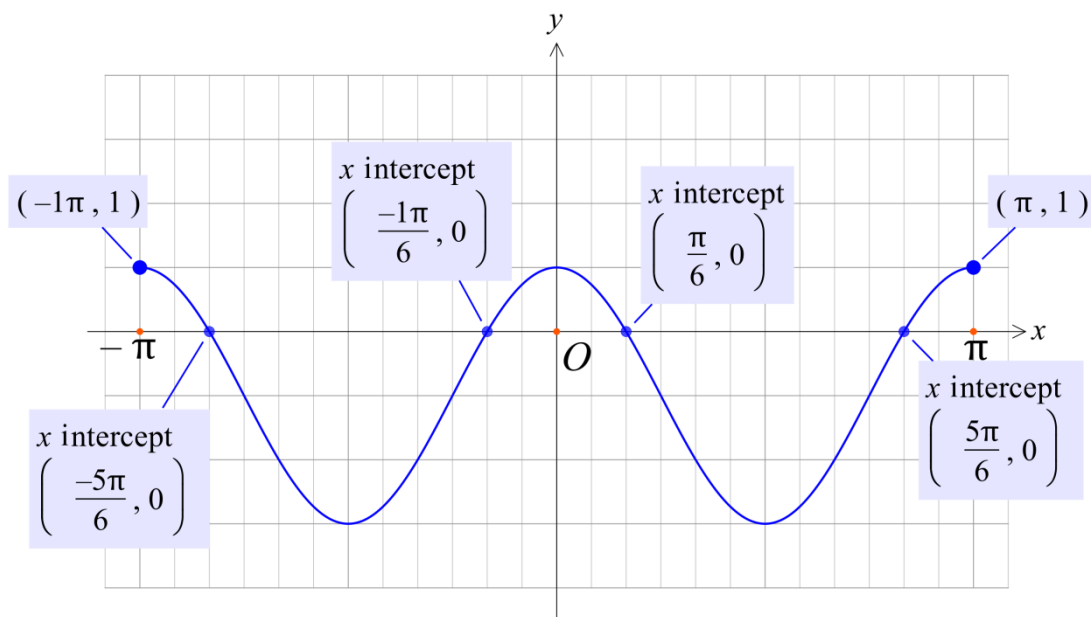
$$\cos(2x) = \frac{1}{2}, \text{ ref angle is } \frac{\pi}{3}$$

$$2x = \frac{-5\pi}{3}, \frac{-\pi}{3}, \frac{\pi}{3}, \frac{5\pi}{3}$$

$$x = \frac{-5\pi}{6}, \frac{-\pi}{6}, \frac{\pi}{6}, \frac{5\pi}{6}$$

Mark allocation: 2 marks

- 1 method mark for the ref angle of $\frac{\pi}{3}$
- 1 answer mark for all four angles correct

Question 6b.**Worked solution****Mark allocation: 3 marks**

- 1 method mark for a correct cos graph showing two cycles and an amplitude of 2
- 1 mark for endpoints labelled correctly
- 1 mark for intercepts labelled correctly

**Tip**

- Remember to label all points as coordinates.

Question 6c.**Worked solution**

$$\begin{aligned}
\text{Average value} &= \frac{1}{b-a} \int_a^b f(x) \, dx \\
&= \frac{1}{\frac{\pi}{6} + \frac{\pi}{4}} \int_{-\frac{\pi}{4}}^{\frac{\pi}{6}} (2 \cos(2x) - 1) \, dx \\
&= \frac{24}{10\pi} \left[\sin(2x) - x \right]_{-\frac{\pi}{4}}^{\frac{\pi}{6}} \\
&= \frac{12}{5\pi} \left[\left(\sin\left(\frac{\pi}{3}\right) - \frac{\pi}{6} \right) - \left(\sin\left(-\frac{\pi}{2}\right) + \frac{\pi}{4} \right) \right] \\
&= \frac{12}{5\pi} \left[\frac{\sqrt{3}}{2} - \frac{\pi}{6} + 1 - \frac{\pi}{4} \right] \\
&= \frac{12}{5\pi} \left[\frac{\sqrt{3}}{2} + 1 - \frac{5\pi}{12} \right]
\end{aligned}$$

Mark allocation: 3 marks

- 1 method mark for setting up the integral for the average value
- 1 method mark for correctly antidifferentiating $2 \cos(2x)$
- 1 answer mark for correct value

Question 7a.**Worked solution**

$$\begin{aligned}
 y &= x \log_e(3x) - x \\
 \frac{dy}{dx} &= x \frac{1}{x} + \log_e(3x) - 1 \\
 &= 1 + \log_e(3x) - 1 \\
 &= \log_e(3x)
 \end{aligned}$$

Mark allocation: 1 mark

- 1 answer mark for correct solution

Question 7b.**Worked solution**

$$\begin{aligned}
 \text{Area} &= \int_{\frac{1}{3}}^{\frac{e}{3}} \log_e(3x) \, dx \\
 &= \left[x \log_e(3x) - x \right]_{\frac{1}{3}}^{\frac{e}{3}} \\
 &= \frac{e}{3} \log_e(e) - \frac{e}{3} - \log_e(1) + \frac{1}{3} \\
 &= \frac{1}{3}
 \end{aligned}$$

Mark allocation: 2 marks

- 1 method mark for correctly replacing $\int \log_e(3x) \, dx$ with $x \log_e(3x) - x$
- 1 answer mark for correct solution

Question 8**Worked solution**

As $f(x)$ is a probability density function, $\int_0^1 (ax^2 - bx^3) dx = 1$.

So

$$\left[\frac{ax^3}{3} - \frac{bx^4}{4} \right]_0^1 = \frac{a}{3} - \frac{b}{4} = 1$$

$$\Rightarrow 4a - 3b = 12 \quad [\text{equation 1}]$$

And with the turning point at $x = \frac{3}{4}$, then $f'(x) = 0$ at $x = \frac{3}{4}$

$$\text{So } f'(x) = 2ax - 3bx^2, \quad f'\left(\frac{3}{4}\right) = \frac{6a}{4} - \frac{27b}{16} = 0.$$

This gives

$$\frac{3a}{2} = \frac{27b}{8}$$

$$24a = 27b \quad [\text{equation 2}]$$

Therefore, multiplying equation 1 by 6 gives $24a - 18b = 72$.

$$27b - 18b = 72$$

$$9b = 72$$

$$b = 8, \quad a = 9$$

Mark allocation: 3 marks

- 1 method mark for setting up the probability density function equal to 1
- 1 method mark for finding the derivative and setting it equal to zero for $x = \frac{3}{4}$
- 1 answer mark for solving simultaneous equations to get $a = 9$ and $b = 8$