THE HEFFERNAN GROUP
P.O. Box 1180
Surrey Hills North VIC 3127
Phone 03 9836 5021
Fax 03 9836 5025
info@theheffernangroup.com.au
www.theheffernangroup.com.au

MATHS METHODS 3 & 4 TRIAL EXAMINATION 1 SOLUTIONS 2017

Question 1 (4 marks)

a.

$$\frac{d}{dx}(2x\log_e(x))$$
$$= 2\log_e(x) + 2x \times \frac{1}{x} \qquad \text{(product rule)}$$
$$= 2\log_e(x) + 2 \qquad \text{(stop here!)}$$

(1 mark)

b.

$$f(x) = \frac{\tan(x)}{3x}$$
$$f'(x) = \frac{3x \times \sec^2(x) - 3\tan(x)}{9x^2} \qquad \text{(quotient rule)}$$

(1 mark)

$$f'\left(\frac{\pi}{3}\right) = \frac{3 \times \frac{\pi}{3} \times \sec^2\left(\frac{\pi}{3}\right) - 3\tan\left(\frac{\pi}{3}\right)}{9 \times \frac{\pi^2}{9}}$$
$$= \frac{\pi \times 4 - 3 \times \sqrt{3}}{\pi^2}$$
$$= \frac{4\pi - 3\sqrt{3}}{\pi^2}$$

(1 mark)

since $\sec^2\left(\frac{\pi}{3}\right) = \frac{1}{\cos^2\left(\frac{\pi}{3}\right)}$
$$= 1 \div \left(\frac{1}{2}\right)^2$$
$$= 4$$

(1 mark)

Question 2 (3 marks)

$$g'(x) = 3 - \frac{2}{x}, \quad x > 0$$
$$g(x) = \int\left(3 - \frac{2}{x}\right)dx$$

(1 mark)

$$= 3x - 2\log_e(x) + c, \quad \text{since } x > 0$$

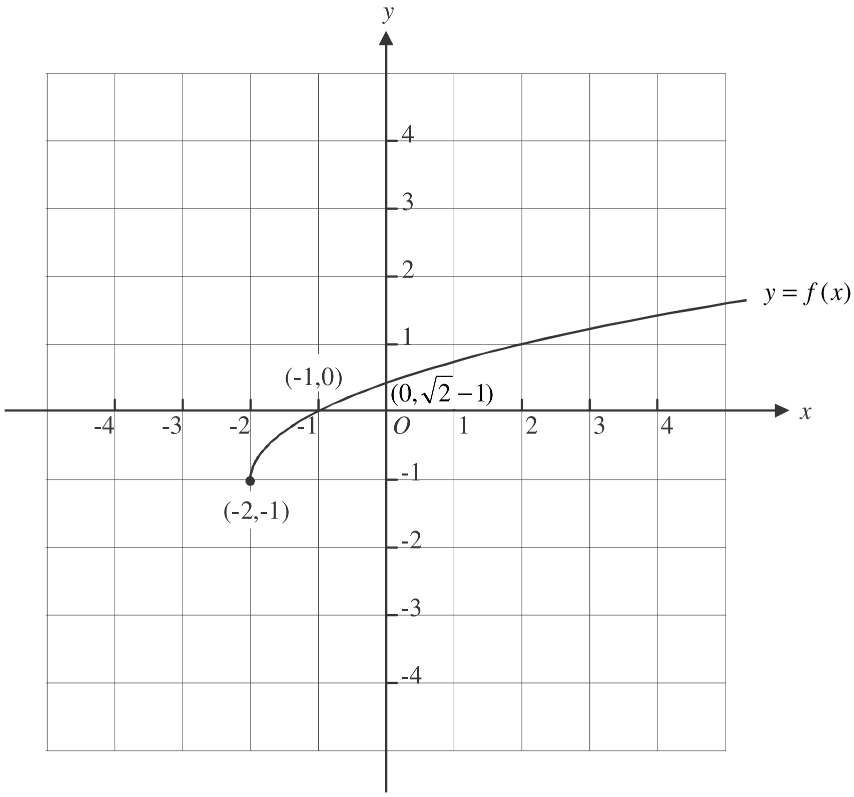
(1 mark)

Since $g(1) = 2$,
$$2 = 3 \times 1 - 2\log_e(1) + c$$
$$c = 2 - 3 \qquad (\text{since } \log_e(1) = 0)$$
$$c = -1$$
So $g(x) = 3x - 2\log_e(x) - 1$

(1 mark)

Question 3 (4 marks)

a.



The graph of f is obtained by translating the graph of $y = \sqrt{x}$ two units left and one unit down. The endpoint is therefore located at $(-2,-1)$.

<u>x-intercept</u> occurs when $y = 0$

$$y = \sqrt{x+2} - 1$$

So, $0 = \sqrt{x+2} - 1$

$$1 = \sqrt{x+2}$$

$$1 = x + 2 \qquad \text{(ie square both sides)}$$

$$x = -1$$

x-intercept occurs at $(-1,0)$

<u>y-intercept</u> occurs when $x = 0$

$$y = \sqrt{0+2} - 1$$

$$y = \sqrt{2} - 1$$

y-intercept occurs at $\left(0, \sqrt{2} - 1\right)$

(1 mark) – correct endpoint **(1 mark)** – correct intercepts **(1 mark)** – correct shape

b.

$$\begin{aligned}\text{average rate of change} &= \frac{f(2) - f(-1)}{2 - {}^{-}1} \\ &= \frac{\left(\sqrt{4}-1\right) - \left(\sqrt{1}-1\right)}{3} \\ &= \frac{1-0}{3} \\ &= \frac{1}{3}\end{aligned}$$

(1 mark)

Question 4 (3 marks)

a. Let X represent the number of girls selected in a week.

$$X \sim Bi\left(3, \frac{2}{5}\right)$$

$$\begin{aligned}\Pr(X=0) &= {}^3C_0\left(\frac{2}{5}\right)^0\left(\frac{3}{5}\right)^3 \\ &= \frac{27}{125}\end{aligned}$$

(1 mark)

b.

$$\begin{aligned}\Pr(X \geq 2) &= \Pr(X=2)+\Pr(X=3) \\ &= {}^3C_2\left(\frac{2}{5}\right)^2\left(\frac{3}{5}\right)^1 + {}^3C_3\left(\frac{2}{5}\right)^3\left(\frac{3}{5}\right)^0 \\ &= 3\times\frac{4}{25}\times\frac{3}{5}+1\times\frac{8}{125} \\ &= \frac{36}{125}+\frac{8}{125} \\ &= \frac{44}{125}\end{aligned}$$

(1 mark)

c. We have a binomial distribution, so

$$\begin{aligned}\text{mean} &= E(X) \\ &= np \\ &= 3\times\frac{2}{5} \\ &= \frac{6}{5}\end{aligned}$$

(1 mark)

Question 5 (4 marks)

a. Draw a diagram.

$\Pr(X > 21)$ is shaded.

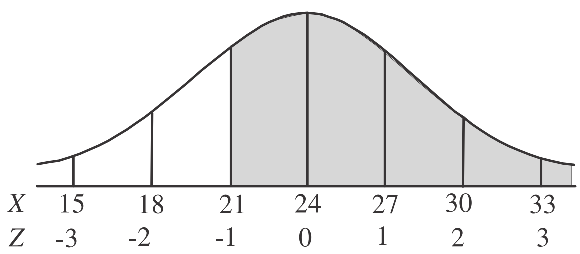
Since $\Pr(Z > 1) = 0.16$,

$\Pr(Z < -1) = 0.16$

because of the symmetry of the normal curve.

Now, $\Pr(Z < -1) = \Pr(X < 21)$

So
$$\begin{aligned}\Pr(X > 21) &= 1-\Pr(X < 21) \\ &= 1-0.16 \\ &= 0.84\end{aligned}$$



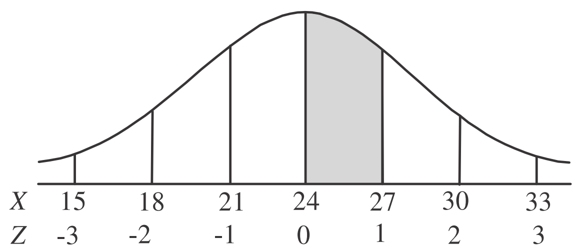
$\Pr(X > 21)$ is shaded

(1 mark)

b. Again, draw a diagram.

$\Pr(24 < X < 27)$ is shaded.

$$\begin{aligned}&\Pr(24 < X < 27) \\ &= \Pr(X < 27)-\Pr(X < 24) \\ &= \Pr(Z < 1)-0.5 \\ &= 1-0.16-0.5 \\ &= 0.34\end{aligned}$$



$\Pr(24 < X < 27)$ is shaded

(1 mark)

c. $\Pr(X<21 | X<24)$

$= \dfrac{\Pr(X<21 \cap X<24)}{\Pr(X<24)}$ (conditional probability formula) **(1 mark)**

$= \dfrac{\Pr(X<21)}{\Pr(X<24)}$

$= \dfrac{0.16}{0.5}$

$= \dfrac{16}{50}$

$= \dfrac{8}{25}$

Alternatively, $\dfrac{16}{50} = \dfrac{32}{100}$
$= 0.32$

(1 mark)

Question 6 (5 marks)

a. $L(x) = g(x) - f(x)$

$= 5 - x - \dfrac{4}{x}$

(1 mark)

b. $L(x) = 5 - x - 4x^{-1}$

$L'(x) = -1 + \dfrac{4}{x^2}$

$L'(x) = 0$ for max/min.

$-1 + \dfrac{4}{x^2} = 0$ **(1 mark)**

$\dfrac{4}{x^2} = 1$

$x^2 = 4$

$x = \pm 2$ reject $x = -2$ since $x > 0$

So $x = 2$

$L(2) = 5 - 2 - \dfrac{4}{2}$

$= 1$

Maximum length is 1 unit.

(1 mark)

c. gradient $= \tan(\theta)$

So $f'(2) = \tan(\theta)$ **(1 mark)**

Now $f'(x) = \dfrac{-4}{x^2}$

So $f'(2) = -1$

$\tan(\theta) = -1$

$\theta = \dfrac{3\pi}{4}^{c}$

or $\theta = 135°$

$(\theta = -\dfrac{\pi}{4}$ or $-45°$ also acceptable)

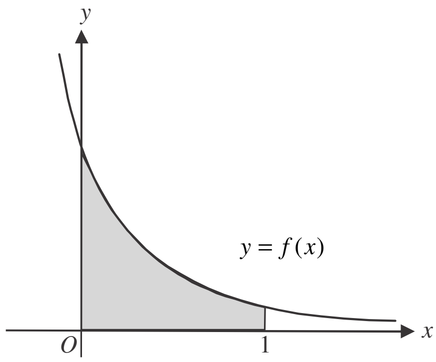
S	A
T	C

(1 mark)

Question 7 (5 marks)

a. The area required is shaded in the diagram below.

$$\begin{aligned}\text{area} &= \int_0^1 f(x)dx \\ &= \int_0^1 e^{-2x}dx \qquad \textbf{(1 mark)} \\ &= \left[-\frac{1}{2}e^{-2x}\right]_0^1 \\ &= -\frac{1}{2}(e^{-2}-e^0) \\ &= \frac{1}{2}\left(1-\frac{1}{e^2}\right)\end{aligned}$$



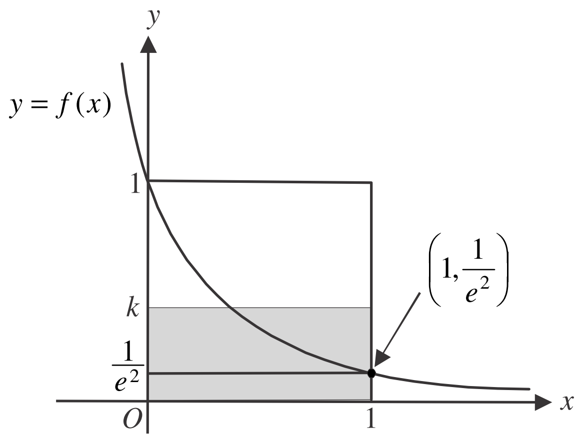
(1 mark)

b. $k=\frac{1}{1-0}\int_0^1 f(x)dx$ (using the average value formula which you must memorise because it isn't on the formula sheet)

$$\begin{aligned} &= \int_0^1 f(x)dx \\ &= \frac{1}{2}\left(1-\frac{1}{e^2}\right)\end{aligned}$$ using the answer from part **a.**

(1 mark)

c.



The area of the shaded rectangle (shown above) gives the average value of *f* between $x=0$ and $x=1$.
This rectangle has a height of *k* units and a width of 1 unit. **(1 mark)**

The larger rectangle with top left corner point at $(0,1)$ and bottom left corner at the origin has an area of 1 square unit.
The smaller rectangle with top left corner point at $\left(0,\frac{1}{e^2}\right)$ and bottom left corner at the origin has an area of $\frac{1}{e^2}$ square units.

Since all three rectangles have a width of one, then $\frac{1}{e^2}<k<1$. **(1 mark)**

Question 8 (4 marks)

a. $$\frac{d}{dx}\left(x\cos\left(\frac{x}{2}\right)\right)=1\times\cos\left(\frac{x}{2}\right)+x\times-\frac{1}{2}\sin\left(\frac{x}{2}\right)$$ (product rule)

$$=\cos\left(\frac{x}{2}\right)-\frac{x}{2}\sin\left(\frac{x}{2}\right)$$

(1 mark)

b. $$E(X)=\int_0^{\frac{2\pi}{3}} x\sin\left(\frac{x}{2}\right)dx.$$ **(1 mark)**

From part **a.**, we have

$$\frac{d}{dx}\left(x\cos\left(\frac{x}{2}\right)\right)=\cos\left(\frac{x}{2}\right)-\frac{x}{2}\sin\left(\frac{x}{2}\right)$$

Rearranging this gives,

$$\frac{x}{2}\sin\left(\frac{x}{2}\right)=\cos\left(\frac{x}{2}\right)-\frac{d}{dx}\left(x\cos\left(\frac{x}{2}\right)\right)$$

So $x\sin\left(\frac{x}{2}\right)=2\cos\left(\frac{x}{2}\right)-2\times\frac{d}{dx}\left(x\cos\left(\frac{x}{2}\right)\right)$ (multiply each and every term by 2)

So $$E(X)=\int_0^{\frac{2\pi}{3}} 2\cos\left(\frac{x}{2}\right)dx-\int_0^{\frac{2\pi}{3}} 2\times\frac{d}{dx}\left(x\cos\left(\frac{x}{2}\right)\right)dx$$

$$=2\left[2\sin\left(\frac{x}{2}\right)\right]_0^{\frac{2\pi}{3}}-2\left[x\cos\left(\frac{x}{2}\right)\right]_0^{\frac{2\pi}{3}}$$ (ie the antiderivative "undoes" the derivative) **(1 mark)**

$$=4\left(\sin\left(\frac{\pi}{3}\right)-\sin(0)\right)-2\left(\frac{2\pi}{3}\cos\left(\frac{\pi}{3}\right)-0\right)$$

$$=4\left(\frac{\sqrt{3}}{2}-0\right)-\frac{4\pi}{3}\times\frac{1}{2}$$

$$=2\sqrt{3}-\frac{2\pi}{3}$$

(1 mark)

Question 9 (8 marks)

a. Stationary points occur when $f'(x)=0$

$$f(x)=x^3-ax^2$$

$$f'(x)=3x^2-2ax=0$$ (remember a is a constant) **(1 mark)**

$$x(3x-2a)=0$$

$$x=0 \quad \text{or} \quad 3x-2a=0$$

$$3x=2a$$

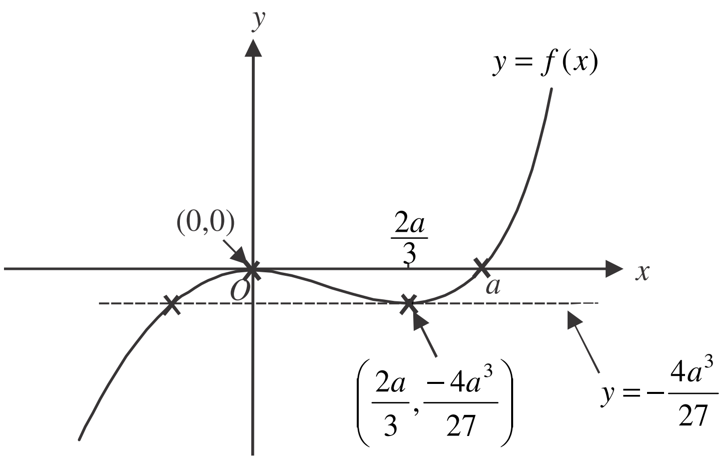
$$x=\frac{2a}{3}$$ **(1 mark)** correct x coordinates

Now, $f(0)=0$, so one stationary point occurs at $(0,0)$.

$$\begin{aligned} f\left(\frac{2a}{3}\right) &= \left(\frac{2a}{3}\right)^3-a\left(\frac{2a}{3}\right)^2 \\ &= \frac{8a^3}{27}-a\times\frac{4a^2}{9} \\ &= \frac{8a^3}{27}-\frac{4a^3}{9} \\ &= \frac{8a^3}{27}-\frac{12a^3}{27} \\ &= \frac{-4a^3}{27} \end{aligned}$$

The other stationary point occurs at $\left(\frac{2a}{3},-\frac{4a^3}{27}\right)$. **(1 mark)**

b. Look at the graph.



The equation $f(x)=n$ has two solutions when the graph of $y=f(x)$ intersects with the graph of $y=n$ twice. Note that the graph of $y=n$ is a horizontal straight line. The graph of $y=f(x)$ intersects with the graph of $y=0$ (which is of course is the x-axis) just twice. So one of the values of n is zero.

The graph of $y=f(x)$ also intersects with the graph of $y=n$ twice when

$n=-\frac{4a^3}{27}$ (using part **a.**).

So $n=0$ or $n=-\frac{4a^3}{27}$. **(1 mark)**

c. Method 1

Point U lies between the two stationary points and occurs when $f''(x)=0$. Point U is a point of inflection. **(1 mark)**

$$f''(x)=6x-2a=0$$
$$6x=2a$$
$$x=\frac{a}{3}$$

So $u=\frac{a}{3}$. **(1 mark)**

Method 2

Gradient of tangent given by $f'(x)=3x^2-2ax$ (from part **a.**).

Let the gradient of f at the point U be m.

So $3x^2-2ax=m$

i.e. $3x^2-2ax-m=0$

We want one solution to this equation. This is a quadratic equation in the variable x so one solution occurs when

$(-2a)^2-4\times3\times{}^-m=0$ (i.e. the discriminant b^2-4ac)

$$4a^2+12m=0$$
$$12m=-4a^2$$
$$m=\frac{-4a^2}{12}$$
$$=\frac{-a^2}{3}$$

So the gradient of the tangent is $\frac{-a^2}{3}$.

(1 mark)

Since $f'(x)=3x^2-2ax$

then $3x^2-2ax=\frac{-a^2}{3}$

$$3x^2-2ax+\frac{a^2}{3}=0$$
$$9x^2-6ax+a^2=0$$
$$(3x-a)(3x-a)=0$$
$$3x=a$$
$$x=\frac{a}{3}$$

So $u=\frac{a}{3}$.

(1 mark)

d. $f'(x)=3x^2-2ax$ from part **a**.

At $V(v,f(v))$, the gradient of the tangent is $3v^2-2av$.

At $W(w,f(w))$, the gradient of the tangent is $3w^2-2aw$.

We are told that the tangents at V and W have the same gradient so

$$3v^2-2av=3w^2-2aw$$ **(1 mark)**

$$3v^2-3w^2=2av-2aw$$
$$3(v^2-w^2)=2a(v-w)$$
$$3(v-w)(v+w)=2a(v-w)$$
$$3(v+w)=2a \quad \text{since } v-w\neq0 \text{ i.e. } v\neq w$$

So $v+w=\frac{2a}{3}$.

(1 mark)