

MATHEMATICAL METHODS

Units 3 & 4 – Written examination 1



(TSSM's 2013 trial exam updated for the current study design)

SOLUTIONS

Question 1

a. $f(x) = 1$

$$ke^{2x} - 2e^x + k = 0$$

For two solutions $\Delta > 0$

$$4 - 4k^2 > 0 \text{ which gives } -1 < k < 1, k \neq 0$$

M1+A1

2 marks

b. $e^x = \frac{2 \pm \sqrt{4 - 4k^2}}{2k} = \frac{1 \pm \sqrt{1 - k^2}}{k}$

$$x = \left(\frac{(1 \pm \sqrt{1 - k^2})}{k} \right)$$

M1+A1

2 marks

Question 2

a. $f'(x) = 2x \times -\frac{1}{2\sqrt{1-x}} + 2\sqrt{1-x} \times 1$

$$f'(x) = \frac{-x}{\sqrt{1-x}} + 2\sqrt{1-x}$$

$$f'(x) = \frac{2-3x}{\sqrt{1-x}}$$

M1+A1

2 marks

b. $f'(x) = \frac{-x}{\sqrt{1-x}} + 2\sqrt{1-x}$

$$\frac{x}{\sqrt{1-x}} = 2\sqrt{1-x} - f'(x)$$

Integrating both sides with respect to x between the limits $x = -2$ and $x = 0$

$$\int_{-2}^0 \frac{x}{\sqrt{1-x}} dx = \int_{-2}^0 2\sqrt{1-x} - (2x\sqrt{1-x})_{-2}^0$$

$$\int_{-2}^0 \frac{x}{\sqrt{1-x}} dx = -\frac{4}{3} + 4\sqrt{3} - 4\sqrt{3}$$

$$\text{Area} = \frac{4}{3} \text{ square units}$$

M2+A1

3 marks

Question 3

a. $g(2x) = 0$

$$\sin(2x) + \cos(2x) = 0$$

$$\tan(2x) = -1$$

$$2x = n\pi + \tan^{-1}(-1), n \in \mathbb{Z}$$

$$2x = n\pi + \frac{3\pi}{4}, n \in \mathbb{Z}$$

$$x = \frac{n\pi}{2} + \frac{3\pi}{8}, n \in \mathbb{Z}$$

M1+A1

2 marks

b. $n = 0$ gives $x = \frac{3\pi}{8}$

$$n = -1 \text{ gives } x = \frac{-\pi}{8}$$

$$n = -2 \text{ gives } x = \frac{-5\pi}{8}$$

$$n = 1 \text{ gives } x = \frac{7\pi}{8}$$

$$\text{Solutions are } \frac{3\pi}{8}, \frac{-\pi}{8}, \frac{-5\pi}{8}, \frac{7\pi}{8}$$

M1+A1

2 marks

Question 4

a. $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

$$x' = 2x, y' = -3y$$

$$x = \frac{x'}{2}, y = \frac{-y'}{3}$$

$$\frac{-y'}{3} = \frac{1}{2 \times \frac{x'}{2}}$$

$$g(x) = -\frac{3}{x}$$

M2+A1

3 marks

- b.** *Reflection in the $x - axis$,
dilation of a factor of 3 units from the $x - axis$,
dilation by a factor of 2 units from the $y - axis$.*

A2
2 marks

c. $x = -\frac{3}{y}$ or $y = -\frac{3}{x}$
 $g^{-1}(x) = -\frac{3}{x}$

M2+A1
3 marks

- d.** *Domain of g^{-1} is $R \setminus \{0\}$
Range of g^{-1} is $R \setminus \{0\}$*

A2
2 marks

Question 5

a. $\int_0^4 0.3p \, dx + \int_4^5 xp \, dx = 1$

$$(0.3px)_0^4 + \left(p \frac{x^2}{2}\right)_4^5 = 1$$

$$1.2p + \frac{9}{2}p = 1 \text{ which gives } p = \frac{10}{57}$$

M2+A1
3 marks

b. $\int_1^{2.5} 0.3p \, dx = 0.3p(2.5 - 1)$

$$\int_1^{2.5} f(x) \, dx = \frac{3}{38}$$

M1+A1
2 marks

Question 6

- a.** $\hat{p}$ is the midpoint of the confidence interval

$$\hat{p} = \frac{s+r}{2}$$

M1+A1
2 marks

- b.** The margin of error is equal to half the width of the interval

$$M = \frac{s-r}{2}$$

M1+A1
2 marks

Question 7

$$(a, -a^2 - 8a)$$

$$\frac{dy}{dx} = -2x - 8$$

$$m_T = -2a - 8$$

$$\text{Also, } m_T = \frac{-a^2 - 8a - 3}{a}$$

$$\frac{-a^2 - 8a - 3}{a} = -2a - 8 \text{ which gives } a = \pm\sqrt{3}$$

The equations of tangents are:

$$y = (-8 - 2\sqrt{3})x + 3 \text{ and } y = (-8 + 2\sqrt{3})x + 3$$

M2+A2
4 marks

Question 8

$$\text{a. } \int_0^{\frac{\pi}{8}} \sin\left(\frac{4x}{3}\right) dx = \left(-\frac{\cos\left(\frac{4x}{3}\right)}{\frac{4}{3}}\right)_0^{\frac{\pi}{8}} = \frac{6-3\sqrt{3}}{8}$$

M1+A1
2 marks

$$\text{b. Average value} = \frac{1}{\frac{\pi}{4}-0} \times \left(-\frac{\cos\left(\frac{4x}{3}\right)}{\frac{4}{3}}\right)_0^{\frac{\pi}{4}} = \frac{3}{2\pi}$$

M1+A1
2 marks