

Trial Examination 2016

VCE Mathematical Methods Units 3&4

Written Examination 2

Suggested Solutions

SECTION A

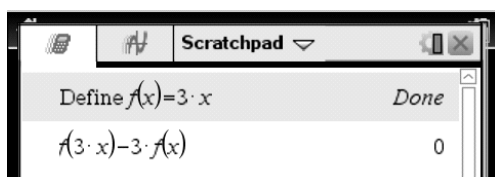
1	A	B	C	D	E
2	A	B	C	D	E
3	A	B	C	D	E
4	A	B	C	D	E
5	A	B	C	D	E
6	A	B	C	D	E
7	A	B	C	D	E
8	A	B	C	D	E
9	A	B	C	D	E
10	A	B	C	D	E

11	A	B	C	D	E
12	A	B	C	D	E
13	A	B	C	D	E
14	A	B	C	D	E
15	A	B	C	D	E
16	A	B	C	D	E
17	A	B	C	D	E
18	A	B	C	D	E
19	A	B	C	D	E
20	A	B	C	D	E

Question 1 E

$$\begin{aligned}
 g(x) &= 5 \cos\left(\frac{1}{3}x - \pi\right) - 3 \\
 &= 5 \cos\frac{1}{3}(x - 3\pi) - 3 \\
 &= f\left(\frac{1}{3}(x - 3\pi)\right) - 3
 \end{aligned}$$

This has a horizontal translation of 3π units in the positive direction of the x -axis.

Question 2 C**Question 3 E**

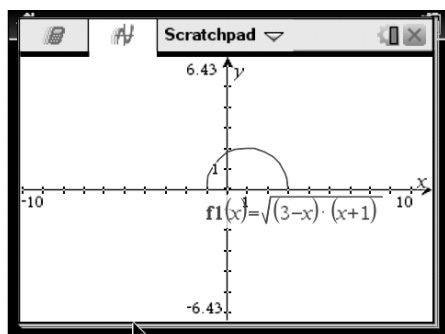
The population proportion, p , is $\frac{\text{number of population with attribute}}{\text{population size}}$.

The population proportion is a population parameter; its value is constant.

Since 73% of the entire population have the required attribute (at least two children), then 73% is a population parameter.

Question 4 B

$$\begin{aligned}
 h(x) &= f(x)g(x) \\
 &= \sqrt{(3-x)(x+1)} \text{ with a maximal domain of } [-1, 3]
 \end{aligned}$$



The function is not differentiable at the endpoints, therefore not at $x = -1$ or $x = 3$.

Question 5 D

$$E(X) = \sum xp(x)$$

$$\begin{aligned} &= -10(2p - p^2) + 0(p - 1)(3p - 1) + 10(2p - 2p^2) \\ &= -10p^2 \end{aligned}$$

Question 6 E

The 95% approximate margin of error is calculated using the formula $1.96 \sqrt{\frac{\hat{p}(1 - \hat{p})}{n}}$.

By increasing the sample size by a factor of 4, the new margin of error is calculated to be:

$$\begin{aligned} 1.96 \sqrt{\frac{\hat{p}(1 - \hat{p})}{4n}} &= \frac{1.96}{2} \sqrt{\frac{\hat{p}(1 - \hat{p})}{n}} \\ &= \frac{1}{2} \times 1.96 \sqrt{\frac{\hat{p}(1 - \hat{p})}{n}} \text{ as required.} \end{aligned}$$

(Halving the confidence level of 95% does not halve the related z-scores.)

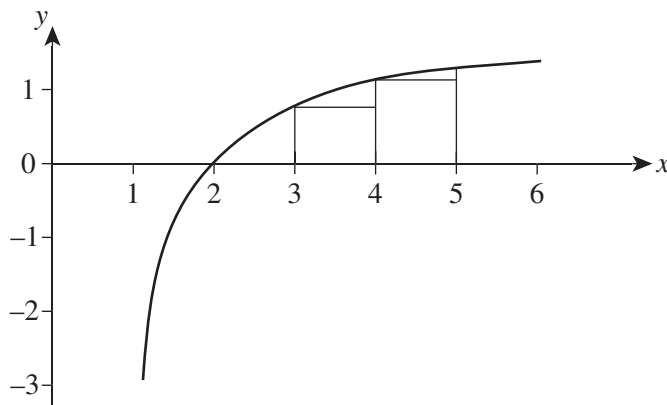
Question 7 E

There are 3 rectangles for the required area.

Using left rectangles, we have:

$$\begin{aligned} \text{area} &= \log_e(2 - 1) \times 1 + \log_e(3 - 1) \times 1 + \log_e(4 - 1) \times 1 \\ &= \log_e(1) + \log_e(2) + \log_e(3) \end{aligned}$$

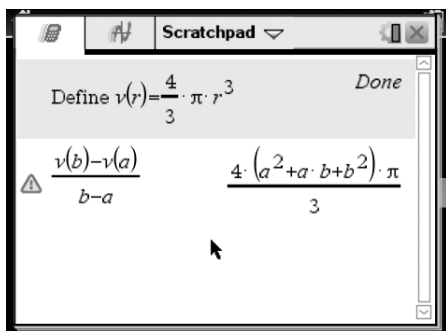
Since $\log_e(1) = 0$ the answer comes down to be $\log_e(2) + \log_e(3)$.



Question 8 **C**

$$V = \frac{4}{3}\pi r^3$$

$$\begin{aligned} \text{average rate of change of volume} &= \frac{V(b) - V(a)}{b - a} \\ &= \frac{\frac{4}{3}\pi b^3 - \frac{4}{3}\pi a^3}{b - a} \\ &= \frac{\frac{4}{3}\pi(b^3 - a^3)}{b - a} \\ &= \frac{\frac{4}{3}\pi(b - a)(b^2 + ba + a^2)}{b - a} \\ &= \frac{4}{3}\pi(a^2 + ab + b^2) \end{aligned}$$

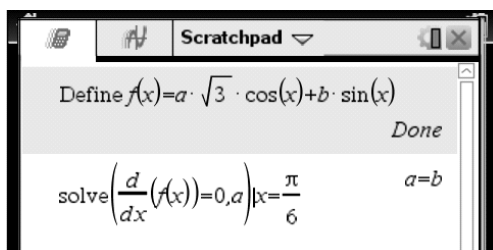
**Question 9** **D**

Doubling the (positive) mean would result in a translation of the normal graph in the positive direction of the x -axis by the value of the mean itself.

Halving the standard deviation would result in a dilation of the normal graph from the mean by factor of $\frac{1}{2}$. This would make the normal graph appear narrower.

Question 10 **D**

For turning point, derivative = 0, so $f'(x) = 0$ at $x = \frac{\pi}{6}$.



Question 11 C

The probability density function for X is

x	0	1	2	3	4
$\Pr(X = x)$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$

50% of the data lie below the median.

As $\Pr(X \leq 1) = 0.4$ the median is greater than 1.

As $\Pr(X \leq 2) = 0.6$, the median is less than or equal to 2.

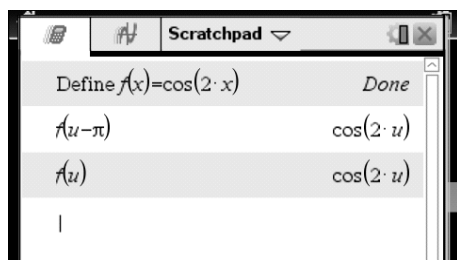
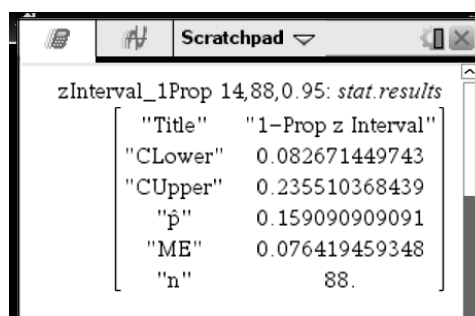
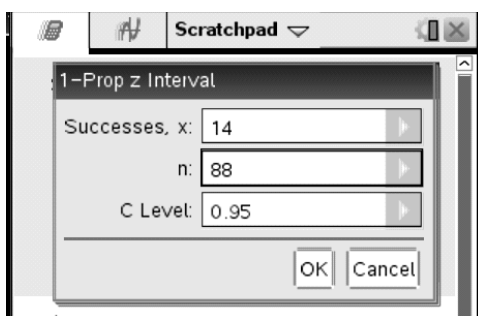
Hence the median is 2.

Question 12 E

To satisfy $f(u - \pi) = f(u)$, f must have period of π . $\cos(2x)$ has a period of $\frac{2\pi}{2} = \pi$ and is the only one of the given functions that does, hence it is the only one that satisfies the functional equation.

$$f(u) = \cos(2u)$$

$$\begin{aligned} f(u - \pi) &= \cos(2(u - \pi)) \\ &= \cos(2u - 2\pi) \\ &= \cos(2u) \\ &= f(u) \end{aligned}$$

**Question 13 A**

CLower = 0.083 and CUpper = 0.236, correct to 3 decimal places.

Question 14 **A**

The equation of the tangent can be found using $y = f'(e)(x - e) + f(e)$, where $f(x) = \log_e\left(\frac{a}{x}\right)$.

$$\therefore f(e) = \log_e\left(\frac{a}{e}\right)$$

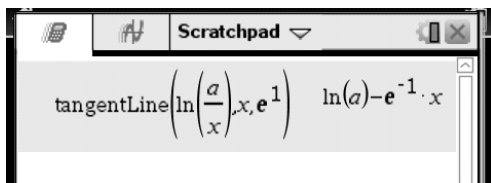
$$f'(x) = -\frac{1}{x}, f'(e) = -\frac{1}{e}$$

$$\therefore y = -\frac{1}{e}(x - e) + \log_e\left(\frac{a}{e}\right)$$

$$= -\frac{x}{e} + 1 + \log_e(a) - \log_e(e)$$

$$= -\frac{x}{e} + 1 + \log_e(a) - 1$$

$$= -\frac{x}{e} + \log_e(a)$$

**Question 15** **E**

The two equations are parallel when their gradients are the same,

or when $\text{Det} \begin{bmatrix} m-1 & 7 \\ 6 & 3m+2 \end{bmatrix} = 0$

$$\therefore (m-1)(3m+2) - 42 = 0$$

$$m = -\frac{11}{3}, 4$$

When $m = 4$, the equations become $3x + 7y = 12$ and $6x + 14y = -24n$, which are the same equation and have infinite solutions when $n = -1$.

Question 16 **E**

The population size cannot be determined by taking a single sample of 100 bats without knowing what proportion the sample is of the whole population.

A table of exact proportions is best suited to small samples ($n < 10$), so not appropriate.

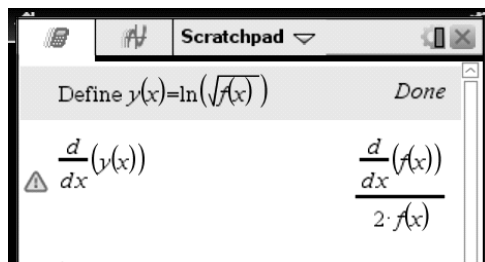
A point estimate (such as the sample proportion) is a statistic calculated on one sample and would be appropriate.

A confidence interval can also be calculated on one sample, giving us an interval where the population proportion is likely to lie, given the point estimate sample proportion.

Question 17 D

Using chain rule: $y = \log_e(u)$, where $u = \sqrt{f(x)}$.

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\ &= \frac{1}{u} \times \left(\frac{1}{2\sqrt{f(x)}} \times f'(x) \right) \\ &= \frac{1}{\sqrt{f(x)}} \times \frac{f'(x)}{2\sqrt{f(x)}} \\ &= \frac{f'(x)}{2f(x)}\end{aligned}$$

**Question 18 B**

$$ax^4 + 4x^3 + 2x^2 = 0$$

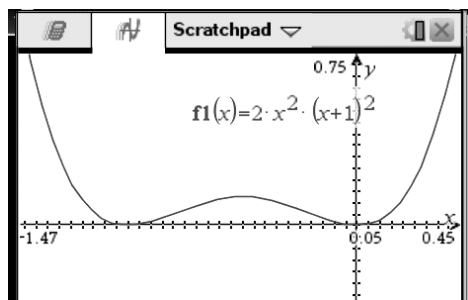
$$x^2(ax^2 + 4x + 2) = 0$$

$x = 0$ is one solution, therefore $ax^2 + 4x + 2$ must be a perfect square (or could use discriminant equals zero) for there to be the required two solutions only.

$2\left(\frac{a}{2}x^2 + 2x + 1\right)$ is a perfect square if $a = 2$.

$$2x^2(x+1)^2 = 0$$

Since a is positive, this is a positive quartic function and the graph of this function has x -intercepts at $x = -1$ and $x = 0$, which are both turning points and therefore local minimums.



Question 19 E

$$\begin{aligned}
 \int_0^8 f\left(\frac{1}{4}x\right) + 2 dx &= \int_0^8 f\left(\frac{1}{4}x\right) dx + \int_0^8 (2) dx \\
 &= 4 \times \int_0^2 f(x) dx + [2x]_0^8 \\
 &\quad \left(\text{as } \int_0^8 f\left(\frac{1}{4}x\right) dx \text{ is a dilation by a factor 4 of } \int_0^2 f(x) dx \text{ from the } y\text{-axis} \right) \\
 &= (4 \times 3) + (2 \times 8 - 2 \times 0) \\
 &= 28
 \end{aligned}$$

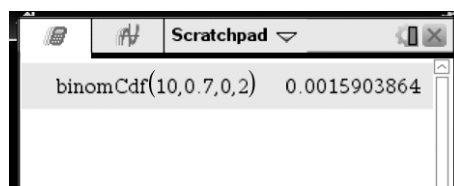
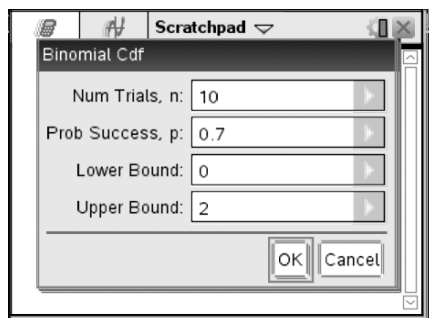
Question 20 B

This is a binomial distribution with $n = 10$. If we consider **not** becoming ill (as the question requires) as a success, then $p = 0.7$.

$$X \sim \text{Bi}(10, 0.7)$$

$$\Pr(\text{at most 2 successes}) = \Pr(X \leq 2)$$

$$= 0.0016 \text{ (correct to four decimal places)}$$



SECTION B**Question 1** (10 marks)

- a. $h = A \sin(nx) + B$, where A , n and B are real constants.

The amplitude of the graph is 125, therefore $A = 125$.

The graph has been translated 125 units up, therefore $B = 125$.

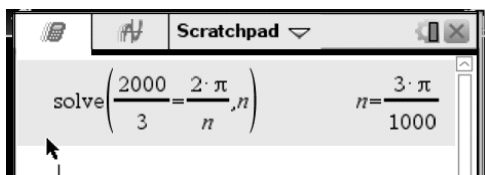
both correct A1

From the graph, $\frac{3}{4}$ of a period = 500 so period = $500 \times \frac{4}{3}$

$$= \frac{2000}{3}$$

Since period = $\frac{2\pi}{n}$, $n = \frac{3\pi}{1000}$.

A1



Scratchpad window showing the equation $\text{solve}\left(\frac{2000}{3} = \frac{2 \cdot \pi}{n}, n\right)$ and the result $n = \frac{3 \cdot \pi}{1000}$.

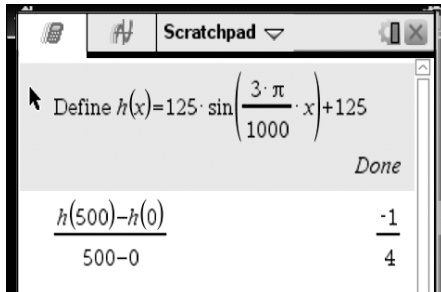
- b. First touches the ground at $x = 500$.

$$\text{average rate of change} = \frac{h(500) - h(0)}{500 - 0}$$

$$= -\frac{1}{4}$$

M1

A1



Scratchpad window showing the definition $h(x) = 125 \cdot \sin\left(\frac{3 \cdot \pi}{1000} \cdot x\right) + 125$ and the calculation $\frac{h(500) - h(0)}{500 - 0} = -\frac{1}{4}$.

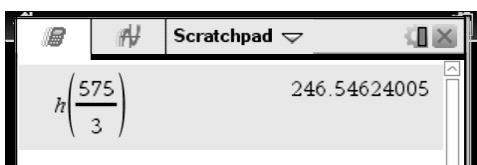
- c. $\frac{50}{2} = 25$ mm

$$\frac{500}{3} \pm 25 = \frac{575}{3} \text{ and } \frac{425}{3}$$

M1

So cuboid structure touches curve at $x = \frac{575}{3}$ and $x = \frac{425}{3}$.

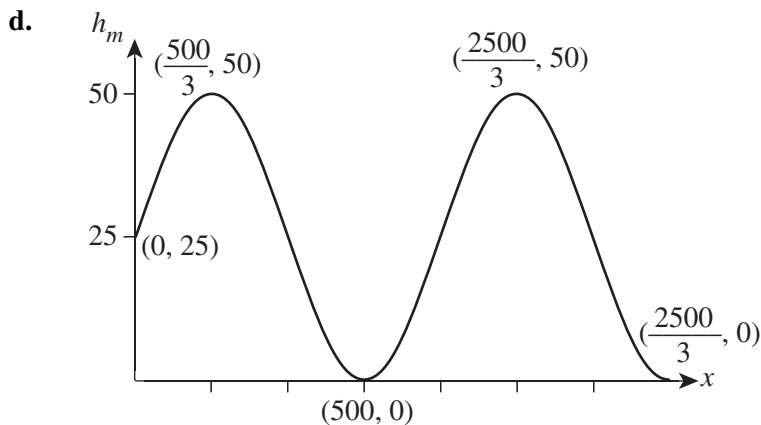
M1



Scratchpad window showing the calculation $h\left(\frac{575}{3}\right) = 246.54624005$.

So maximum height of cuboid is 247 mm.

A1



correct shape A1
all coordinates labelled A1

e. $h_m = 25 \sin\left(\frac{3\pi}{1000}x\right) + 25$

A1

Question 2 (16 marks)

- a. i. Let D be the number of children found to have tooth decay.

$E(D) = np$, $n = 100$, $p = 0.2$. Therefore, he would expect 20 children with tooth decay. A1

- ii. $D \sim \text{Bi}(100, 0.2)$

M1

$\Pr(D = 20) = 0.099$

A1

4.1	4.2	4.3	*NEAP 2016 ...ice
binomPdf(100,0.2,20)			0.0993
binomCdf(100,0.2,0,15)			0.128506

- b. $\Pr(D \leq 15) = 0.129$ (as above)

A1

- c. i. standard deviation $= \sqrt{np(1-p)}$ $n = 100$, $p = 0.2$
 $= 4$

mean $= np = 20$

$20 - 2 \times 4 = 12$

$20 + 2 \times 4 = 28$

2 standard deviation limit $= (12, 28)$

A1

29 children having tooth decay is more than 2 standard deviations away from the expected number of children with tooth decay. The chance of more than 28 children having tooth decay is less than approximately 2.5%. Xavier could have cause to doubt the accuracy of the information but should realise that one sample is not enough to be a reliable estimate.

feasible answer with justification A1

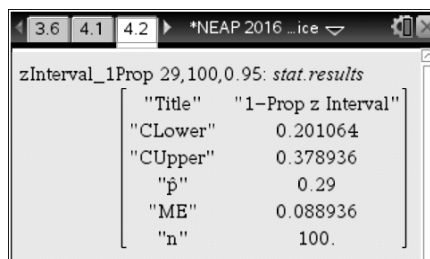
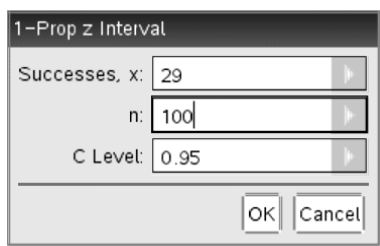
- ii. 0.29

A1

$$\text{iii. } 0.29 \pm 1.96 \sqrt{\frac{0.29 \times (1 - 0.29)}{100}} = 0.29 \pm 0.0889 \dots$$

$$= (0.201, 0.379)$$

A1



d. i. $0.95 \times 200 = 190$

A1

ii. a normal distribution

A1

iii. mean $\mu = 0.2$, standard deviation $\sigma = \sqrt{\frac{0.2(1-0.2)}{200}} = 0.03$

both correct A1

iv. $\hat{p} \sim N(0.2, 0.03^2)$

$$\Pr(0.15 < \hat{p} < 0.2 | \hat{p} < 0.25) = \frac{\Pr(0.15 < (\hat{p} < 0.2 \cap \hat{p} < 0.25))}{\Pr(\hat{p} < 0.25)}$$

recognising conditional probability M1

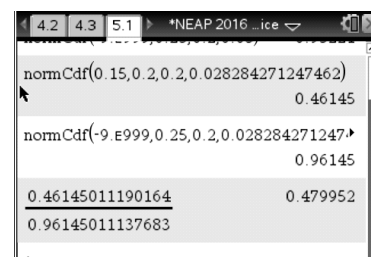
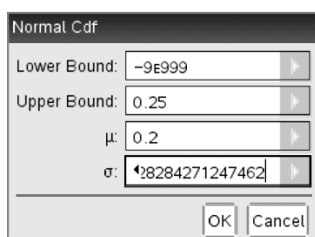
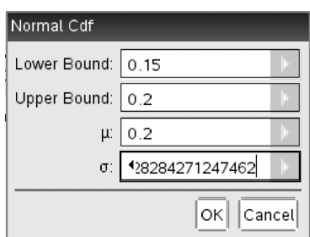
$$= \frac{\Pr(0.15 < \hat{p} < 0.2)}{\Pr(\hat{p} < 0.25)}$$

M1

$$= \frac{0.4615}{0.9615}$$

$$= 0.480 \text{ correct to three decimal places}$$

A1



e. $n = \left(\frac{1.96}{0.03}\right)^2 \times 0.2 \times 0.8$

M1

$$= 682.95$$

$$= 683$$

A1

He will need to sample 683 children.

Question 3 (8 marks)

a. domain: $f: [1000, 1200] \rightarrow R$

A1

rule: $f(x) = 400 - \frac{1}{100}(x - 1200)^2$

A1

b. A reflection in the y-axis maps $(x, y) \rightarrow (-x, y)$.

Hence $y = e^{-6}(x - 200)e^{\frac{x}{100}} \rightarrow y = e^{-6}(-x - 200)e^{\frac{-x}{100}}$.

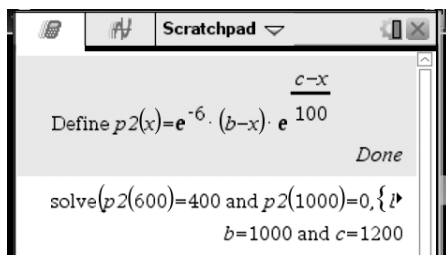
A1

Since the reflection is in the line $x = 600$, there is also a horizontal translation.

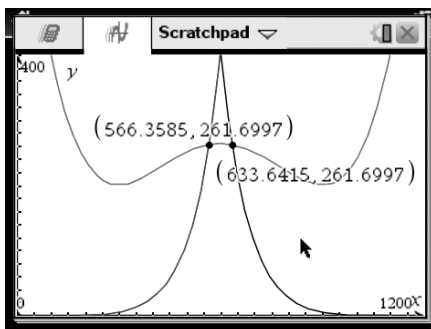
Using the points $(600, 400)$ and $(1000, 0)$ we can solve for b and c .

$b = 1000$ and $c = 1200$

A1



c. Method 1: graphically



Coordinates are $(566.4, 261.7)$ and $(633.6, 261.7)$.

A1 A1

Method 2: algebraically

Coordinates are $(566.4, 261.7)$ and $(633.6, 261.7)$.

A1 A1

- d. average value of function over interval $[566.36, 633.64]$

$$\frac{1}{633.64-566.36} \int_{566.36}^{633.64} \left(\frac{1}{128000000} \right) dx$$

$$= 262.752790236$$

correct integral M1

$$= 262.8$$

A1

Question 4 (18 marks)

- a. i. driver A: $100 - 30 = 70$ km; driver B: $100 - 40 = 60$ km

both correct A1

ii. distance $= \sqrt{70^2 + 60^2}$
 $= 92.2$ km

A1

- b. i. $d_A = 100 - 30t$, $d_B = 100 - 40t$

A1

ii. $d(t) = \sqrt{(100 - 30t)^2 + (100 - 40t)^2}$

using Pythagoras M1 A1

iii. $d'(t) = 0$ for the minimum distance

M1

$$\text{Define } d(t) = \sqrt{(100 - 30t)^2 + (100 - 40t)^2}$$

$$\text{solve}\left(\frac{d}{dt}(d(t)) = 0, t\right) \quad t = \frac{14}{5}$$

$$d\left(\frac{14}{5}\right) \quad 20$$

$$t = 2.8 \text{ hours}$$

A1

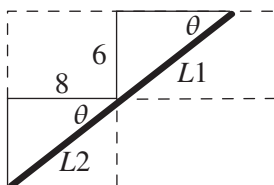
iv. $d(2.8) = 20$ km

A1

- c. i. Splitting the diagonal:

$L = L_1 + L_2$, where L_1 and L_2 are the diagonals which cross the respective roads.

splitting the diagonal A1



M1

From the diagram we see $\sin(\theta) = \frac{6}{L_1}$ and $\cos(\theta) = \frac{8}{L_2}$.

So $L_1 = \frac{6}{\sin(\theta)}$ and $L_2 = \frac{8}{\cos(\theta)}$.

Therefore $L(\theta) = \frac{6}{\sin(\theta)} + \frac{8}{\cos(\theta)}$ as required.

- ii. Maximum length occurs when $L'(\theta) = 0$.

M1

$$L'(\theta) = 0 \text{ gives } \theta = 0.7375.$$

A1

$$L(0.7375) = 19.73 \text{ m}$$

A1

- d. i. Following part c. i.:

$$L = \frac{b}{\sin(\theta)} + \frac{a}{\cos(\theta)}$$

$$\begin{aligned} L'(\theta) &= \frac{a \sin(\theta)}{\cos^2(\theta)} - \frac{b \cos(\theta)}{\sin^2(\theta)} \\ &= \frac{a \sin^3(\theta) - b \cos^3(\theta)}{\cos^2(\theta) \sin^2(\theta)} \end{aligned}$$

M1

$$L'(\theta) = 0$$

$$\Rightarrow \frac{a \sin^3(\theta) - b \cos^3(\theta)}{\cos^2(\theta) \sin^2(\theta)} = 0$$

$$\Rightarrow a \sin^3(\theta) - b \cos^3(\theta) = 0$$

A1

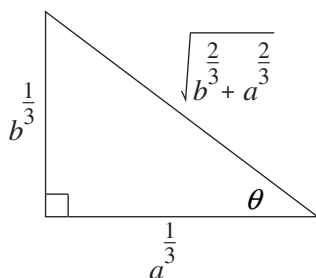
Note: Must have correct answer for both marks.

- ii. Maximum occurs when $a\sin^3(\theta) - b\cos^3(\theta) = 0$.

$$\Rightarrow \frac{\sin^3(\theta)}{\cos^3(\theta)} = \frac{b}{a}$$

$$\Rightarrow \tan(\theta) = \frac{b^{\frac{1}{3}}}{a^{\frac{1}{3}}}$$

A1



By considering the right-angled triangle where $\tan(\theta) = \frac{b^{\frac{1}{3}}}{a^{\frac{1}{3}}}$, we see $\sin(\theta) = \frac{b^{\frac{1}{3}}}{\sqrt{b^{\frac{2}{3}} + a^{\frac{2}{3}}}}$
 and $\cos(\theta) = \frac{a^{\frac{1}{3}}}{\sqrt{b^{\frac{2}{3}} + a^{\frac{2}{3}}}}$.

Substituting into $L = \frac{b}{\sin(\theta)} + \frac{a}{\cos(\theta)}$ gives:

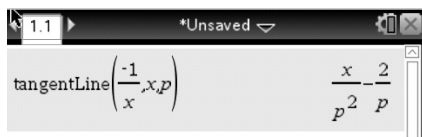
substitution M1

$$\begin{aligned} L &= \frac{b}{\frac{b^{\frac{1}{3}}}{\sqrt{b^{\frac{2}{3}} + a^{\frac{2}{3}}}}} + \frac{a}{\frac{a^{\frac{1}{3}}}{\sqrt{b^{\frac{2}{3}} + a^{\frac{2}{3}}}}} \\ &= \frac{b\sqrt{b^{\frac{2}{3}} + a^{\frac{2}{3}}}}{b^{\frac{1}{3}}} + \frac{a\sqrt{b^{\frac{2}{3}} + a^{\frac{2}{3}}}}{a^{\frac{1}{3}}} \\ &= b^{\frac{2}{3}}\sqrt{b^{\frac{2}{3}} + a^{\frac{2}{3}}} + a^{\frac{2}{3}}\sqrt{b^{\frac{2}{3}} + a^{\frac{2}{3}}} \\ &= \left(a^{\frac{2}{3}} + b^{\frac{2}{3}}\right)\left(b^{\frac{2}{3}} + a^{\frac{2}{3}}\right)^{\frac{1}{2}} \\ &= \left(a^{\frac{2}{3}} + b^{\frac{2}{3}}\right)^{\frac{3}{2}} \end{aligned}$$

A1

Question 5 (8 marks)

a. $f'(x) = \frac{1}{x^2}$



At the point $P\left(p, -\frac{1}{p}\right)$ the gradient of the tangent is $\frac{1}{p^2}$.

Substitution into $y - y_1 = m(x - x_1)$ gives:

$$y + \frac{1}{p} = \frac{1}{p^2}(x - p)$$

$$y = \frac{1}{p^2}(x - p) - \frac{1}{p}$$

$$y = \frac{1}{p^2}x - \frac{2}{p}$$

A1

b. i. Coordinates of midpoint of $T\left(t, -\frac{1}{t}\right)$ and $S\left(s, -\frac{1}{s}\right) = \left(\frac{t+s}{2}, \frac{\left(-\frac{1}{t} - \frac{1}{s}\right)}{2}\right)$

$$= \left(\frac{t+s}{2}, \frac{\left(\frac{-s-t}{st}\right)}{2}\right)$$

$$= \left(\frac{t+s}{2}, \frac{-(s+t)}{2st}\right)$$

A1

ii. Gradient of segment OM :

$$\begin{aligned}
 m &= \frac{y_2 - y_1}{x_2 - x_1} \\
 &= \frac{\left(\frac{-s-t}{2st}\right) - 0}{\left(\frac{t+s}{2}\right) - 0} \\
 &= \frac{-s-t}{2st} \times \frac{2}{t+s} \\
 &= -\frac{1}{st}
 \end{aligned}$$

A1

Gradient of segment OP :

$$\begin{aligned}
 m &= \frac{-\frac{1}{p} - 0}{p - 0} \\
 &= -\frac{1}{p^2}
 \end{aligned}$$

Equating gradient gives $\left(-\frac{1}{p^2}\right) = -\frac{1}{st}$

$$\Rightarrow p^2 = st$$

final solution with reasoning A1

c.
$$\begin{aligned}
 A &= \int_t^s -\frac{1}{x} dx \\
 &= \log_e\left(\frac{t}{s}\right)
 \end{aligned}$$

A1

d.
$$B = \int_s^t f(x) dx + \Delta OSS' - \Delta OTT'$$

M1

$$= -\int_s^t \frac{1}{x} dx + \frac{1}{2} \times s \times -\frac{1}{s} - \frac{1}{2} \times t \times -\frac{1}{t}$$

$$= -\int_s^t \frac{1}{x} dx - \frac{1}{2} + \frac{1}{2}$$

A1

$$= -\int_s^t \frac{1}{x} dx$$

$$= f(x)$$

Therefore $g(x) = f(x)$.

A1