The Mathematical Association of Victoria

Trial Examination 2016

MATHEMATICAL METHODS

Written Examination 2 - SOLUTIONS

SECTION A

1. A 2. C 3. E 4. C 5. A 6. D 7. E 8. B 9. B 10. A

11. E 12. D 13. A 14. E 15. B 16. D 17. D 18. C 19. C 20. B

SECTION A
Question 1

$f: R \to R, f(x) = -2 - 3\cos\left(\frac{x}{4} + 1\right)$

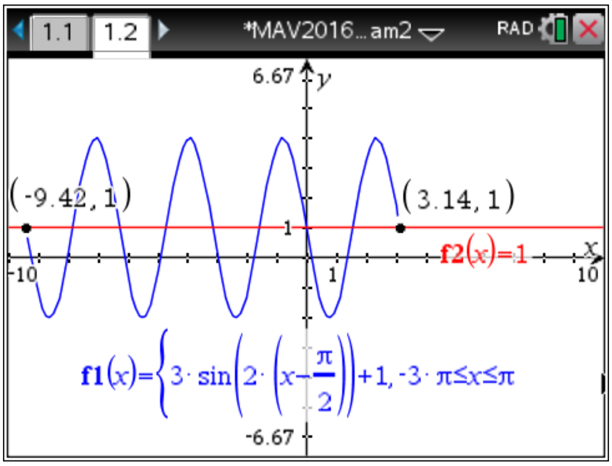
The range is $[-3-2, 3-2] = [-5, 1]$

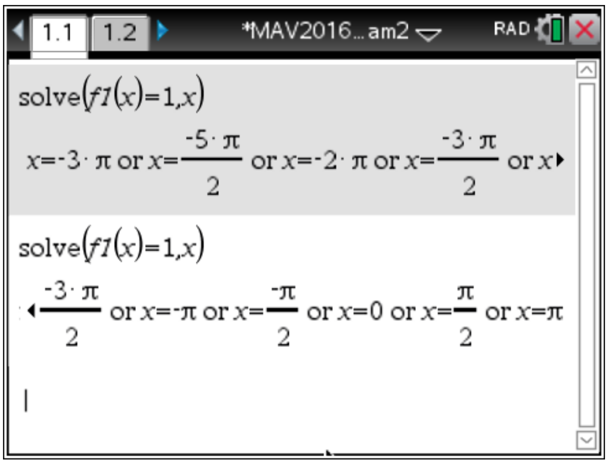
The Period is $\dfrac{2\pi}{\frac{1}{4}} = 8\pi$ **A**

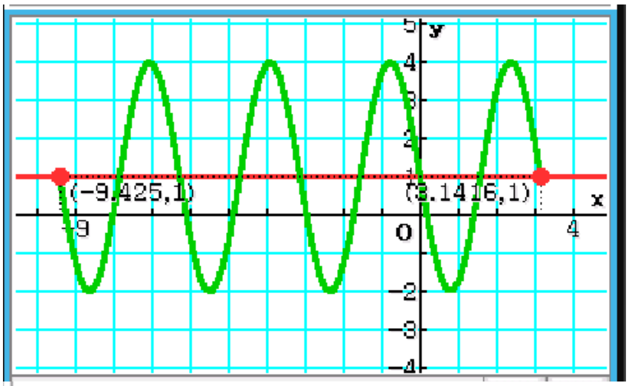
Question 2

$g: [-3\pi, \pi] \to R, g(x) = 3\sin\left(2\left(x - \frac{\pi}{2}\right)\right) + 1$

There are 9 intersections when $k = 1$. **C**







define $f(x)=3\sin\left(2\left(x-\frac{\pi}{2}\right)\right)+1$

done

solve$(f(x)=1 \mid -3\pi \le x \le \pi, x)$

$\left\{x=0, x=-3\cdot\pi, x=-2\cdot\pi, x=-\pi, x=\pi, x=\frac{-5\cdot\pi}{2}, x=\frac{-3\cdot\pi}{2}, x=\right.$

Question 3

The domain of $f\left(g(x)\right)$ is the same as the domain of g where $g(x)=\tan(\pi x)$.

The asymptotes of g have equations $y=\frac{1}{2}+k, k\in Z$.

Hence the domain of $f\left(g(x)\right)$ is $R\setminus\left\{\frac{1}{2}+k\right\}, k\in Z$ **E**

Question 4

$\sin\left(\frac{3\pi}{2}-x\right)=-\cos(x)$

$\sqrt{15^2-2^2}=\sqrt{221}$

$\sin\left(\frac{3\pi}{2}-x\right)=-\frac{\sqrt{221}}{15}$ **C**

Question 5

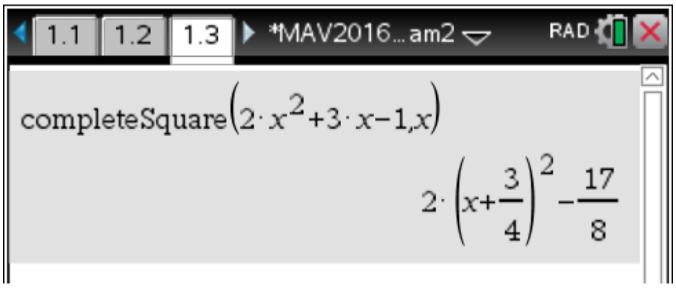
$y=2x^2+3x-1=2\left(x+\frac{3}{4}\right)^2-\frac{17}{8}$

The graph of $y=x^2$ has been dilated by a factor of 2 from the x-axis and

Then translated $\frac{3}{4}$ units to the left and $\frac{17}{8}$ units down.

A possible rule for the transformation $T: R^2 \to R^2$ is

$$T\left(\begin{bmatrix}x\\y\end{bmatrix}\right)=\begin{bmatrix}1&0\\0&2\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}-\begin{bmatrix}\frac{3}{4}\\\frac{17}{8}\end{bmatrix}$$ **A**

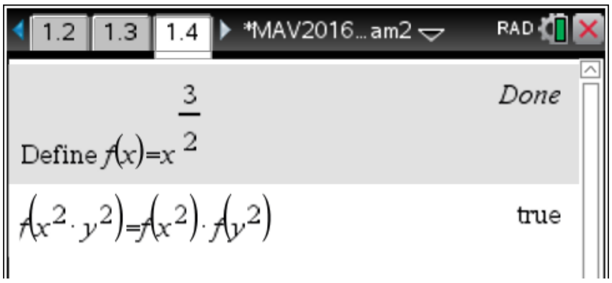


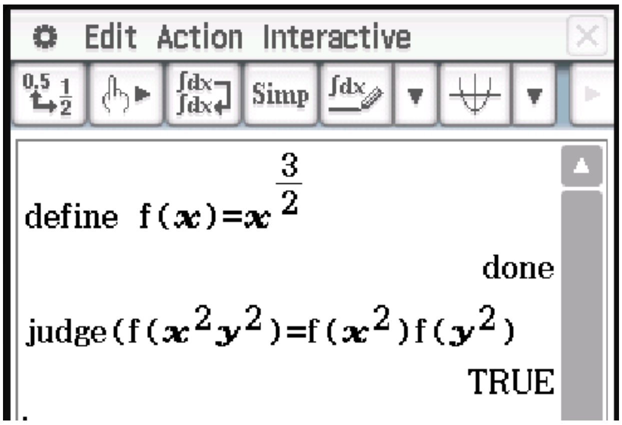
Question 6

$f(x)=x^{\frac{3}{2}}$

$f(x^2y^2)=\left(x^2y^2\right)^{\frac{3}{2}}=x^3y^3$

$f(x^2)f\left(y^2\right)=\left(x^2\right)^{\frac{3}{2}}\left(y^2\right)^{\frac{3}{2}}=x^3y^3$ **D**

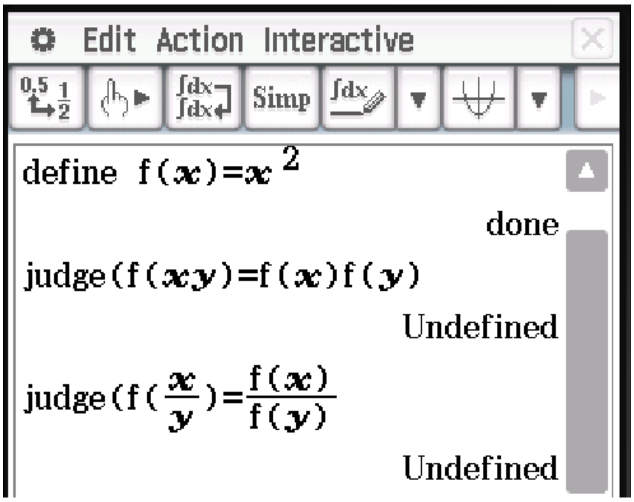




Using counter examples

$f(xy)=f(x)f(y),\ f(-1\times-1)=1, f(-1)\times f(-1)$ has no real solution

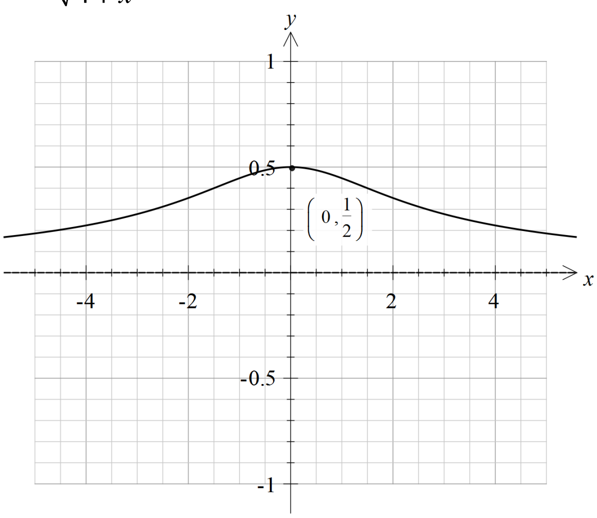
$f\left(\frac{x}{y}\right)=\frac{f(x)}{f(y)},\ f\left(\frac{-1}{-1}\right)=1,\frac{f(-1)}{f(-1)}$ has no real solution



$f(x+y)=f(x)+f(y)$, $f(1+1)=2^{\frac{3}{2}}, f(1)+f(1)=2$
$f(x-y)=f(x)-f(y)$, $f(2-1)=1, f(1)-f(1)=0$

Question 7
When $x=0$, $y=0.5$.
Either **C** or **E**
When $x=2$, $y\approx 0.35$

$y=\dfrac{1}{\sqrt{4+x^2}}$ **E**



Question 8

$f:(1,\infty)\to R, f(x)=\dfrac{3}{(x-1)^2}+2$

Range is $(2,\infty)$ which is the domain of the inverse function.

Let $y=\dfrac{3}{(x-1)^2}+2$

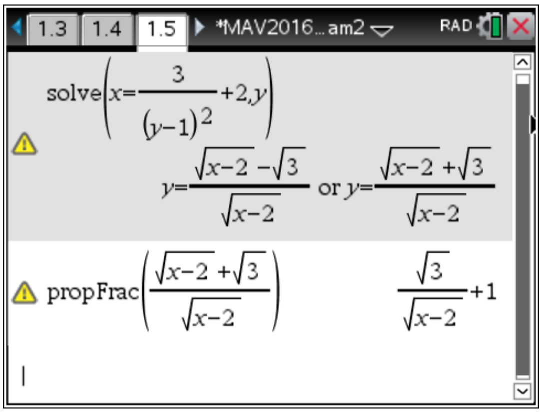
Inverse swap x and y.

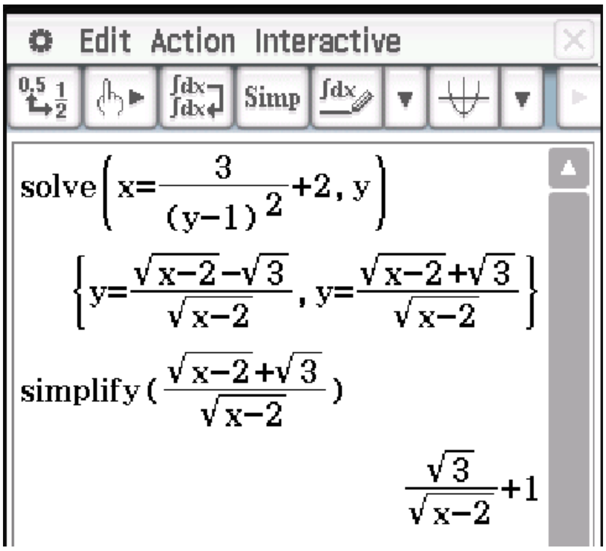
$x=\dfrac{3}{(y-1)^2}+2$

$y=\pm\sqrt{\dfrac{3}{x-2}}+1$

$y=\sqrt{\dfrac{3}{x-2}}+1$, as the domain of f is $(1,\infty)$.

$f^{-1}:(2,\infty)\to R, f^{-1}(x)=\sqrt{\dfrac{3}{x-2}}+1$ **B**

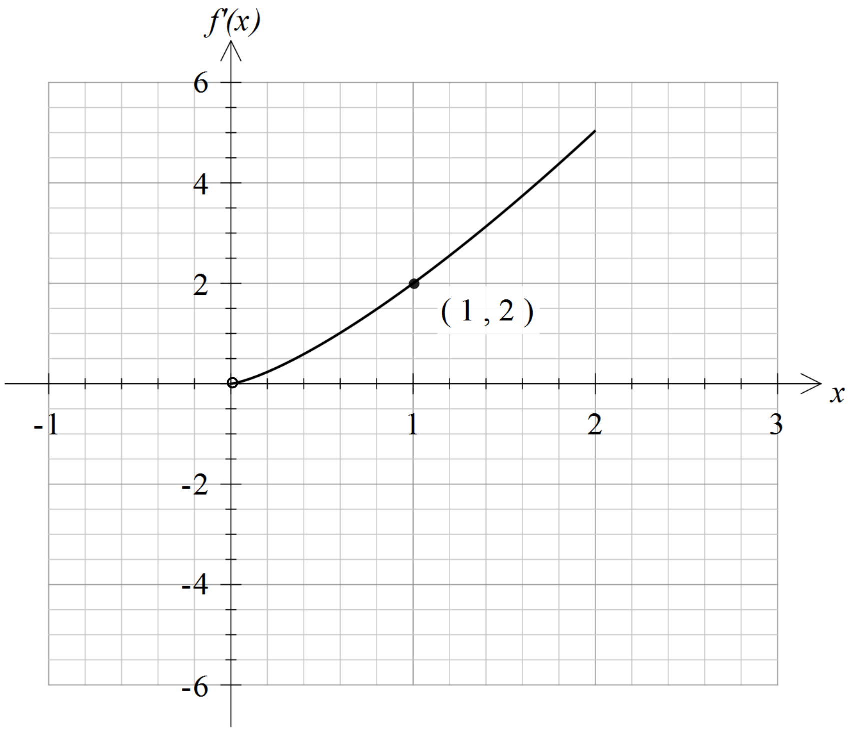




Question 9

The point (1, 2) is on the curve of f'.

Thus the gradient of the graph of f at $x = 1$ is 2. **B**



Question 10

If $f(x) = ax^5 + bx^4 + cx^3 + dx^2 + ex + h$ has a stationary point of inflection at $(-3, 0)$ and a turning point at (2, 0) then $f(x) = a(x+3)^3(x-2)^2$.

$h = a \times 3^3 \times (-2)^2 = 108a$

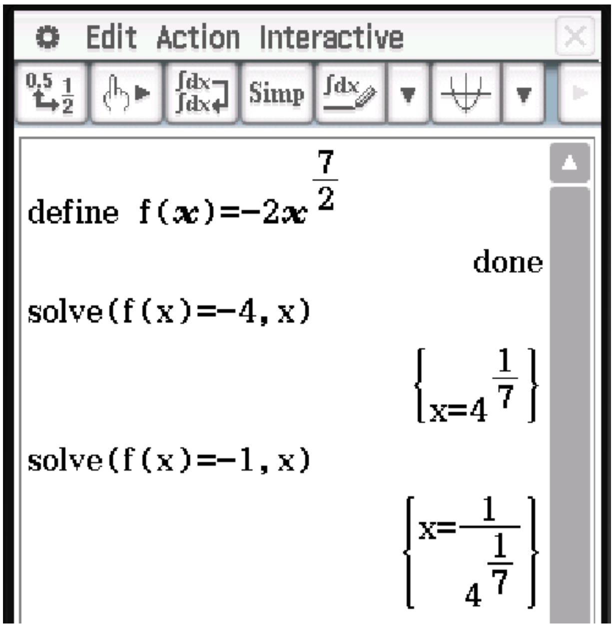
$\frac{h}{a} = 108$ **A**

Question 11

$f(x) = -2x^{\frac{7}{2}}$, $y = -4$ to $y = -1$

Solve $f(x) = -4, x = 2^{\frac{2}{7}}$

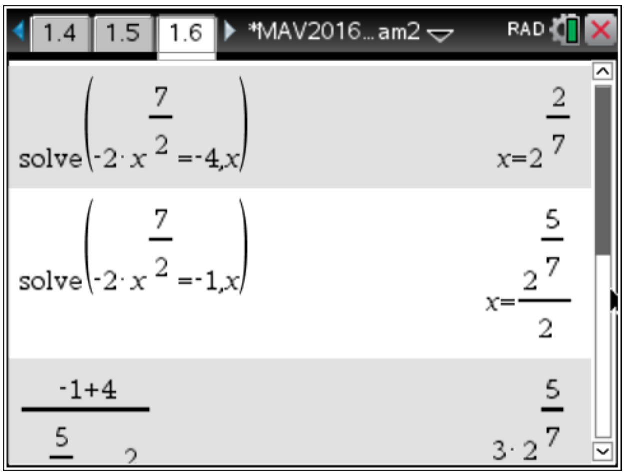
and $f(x) = -1, x = 2^{-\frac{2}{7}}$

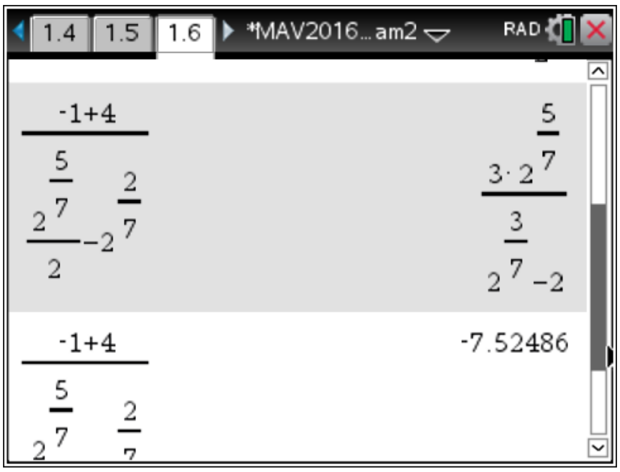


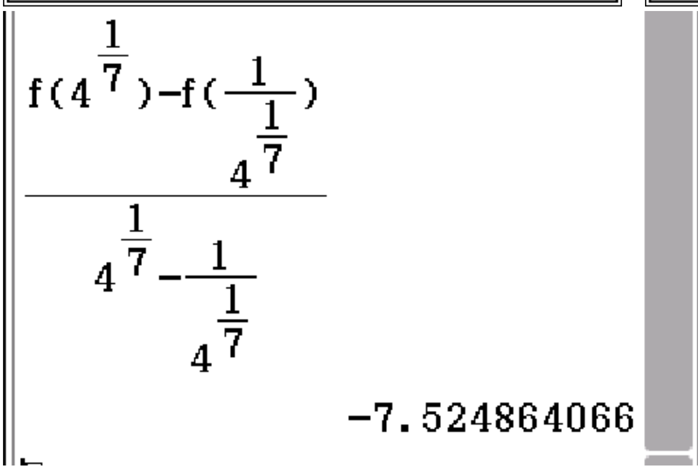
Average rate of change $= \dfrac{y_2 - y_1}{x_2 - x_1}$

$$= \frac{-1+4}{2^{-\frac{2}{7}} - 2^{\frac{2}{7}}}$$

≈ -7.5 **E**







Question 12

$\log_a(b) = c$ and $\log_c(a) = b$

$\log_a(b) + \log_c(a) = b + c$

$\log_a(b) + \dfrac{\log_a(a)}{\log_a(c)} = b + c$

$\log_a(b) + \dfrac{1}{\log_a(c)} = b + c$

$\log_a\left(\dfrac{b}{c}\right) \neq b + c$ **D**

Question 13

$f'(x) = 2x^3 - 2x$

$f(x) = \int\left(2x^3 - 2x\right)dx$

$= \dfrac{x^4}{2} - x^2 + c$

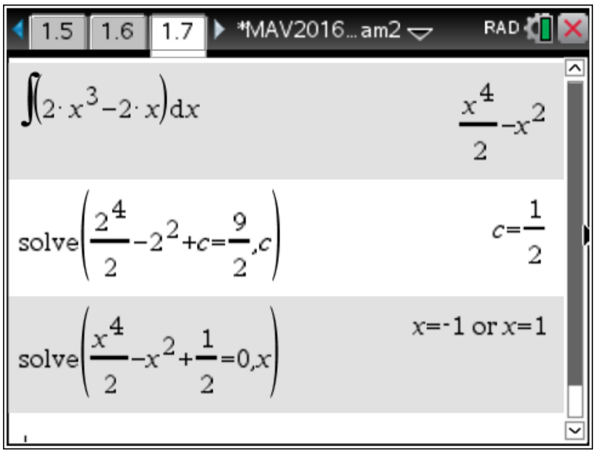
$f(2) = \dfrac{9}{2}$

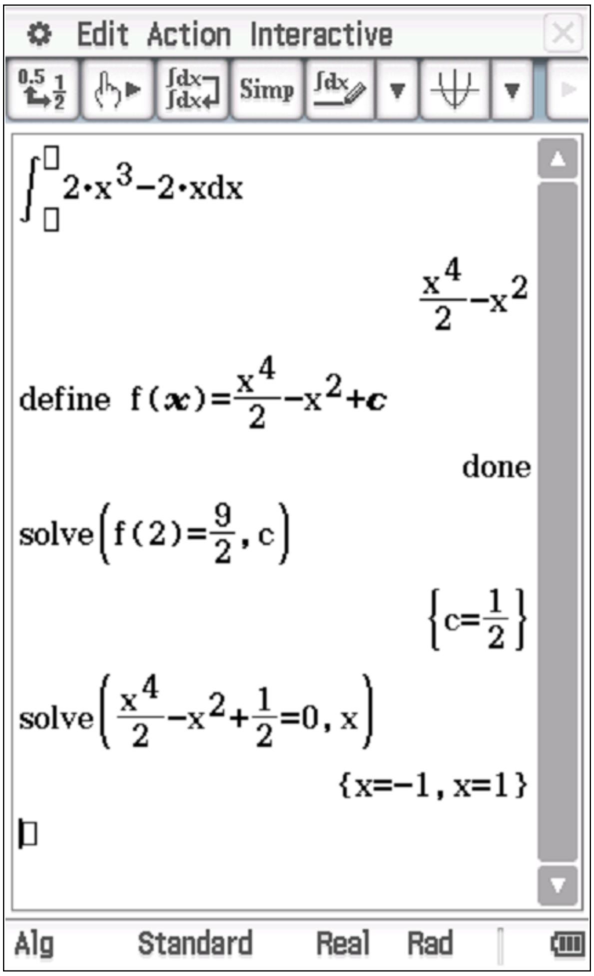
$\dfrac{9}{2} = \dfrac{2^4}{2} - 2^2 + c$

$c = \dfrac{1}{2}$

Solve $\dfrac{x^4}{2} - x^2 + \dfrac{1}{2} = 0$ for x.

$x = \pm 1$ **A**





Question 14

$$\int_2^3 \left(g(x)\right)dx = -3$$

$$= 2\int_3^2 \left(1-2g(x)\right)dx$$

$$= 2\int_3^2 (1)dx - 4\int_3^2 \left(g(x)\right)$$

$$= 2[x]_3^2 - 4\times 3$$

$$= -14$$ **E**

Question 15

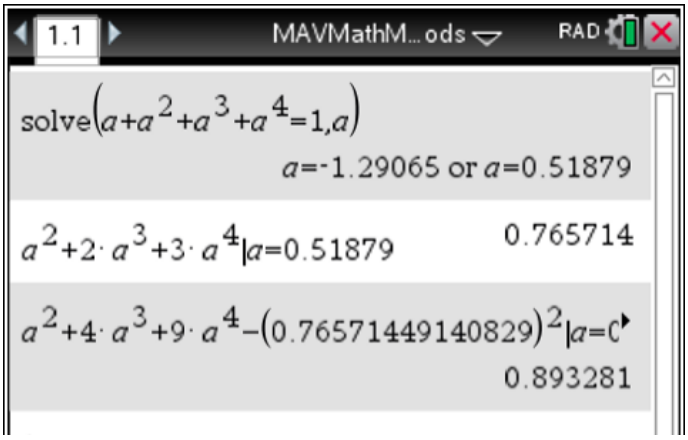
Area $= \frac{1}{2}\left(e^0 + e + e^2 + e^3\right)$

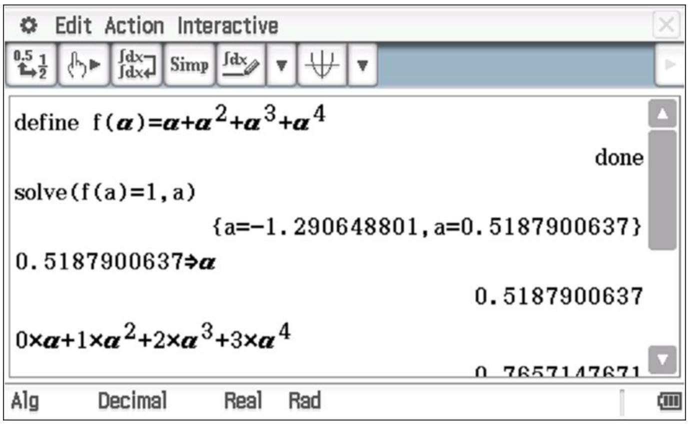
$= \frac{1}{2}\left(1 + e + e^2 + e^3\right)$

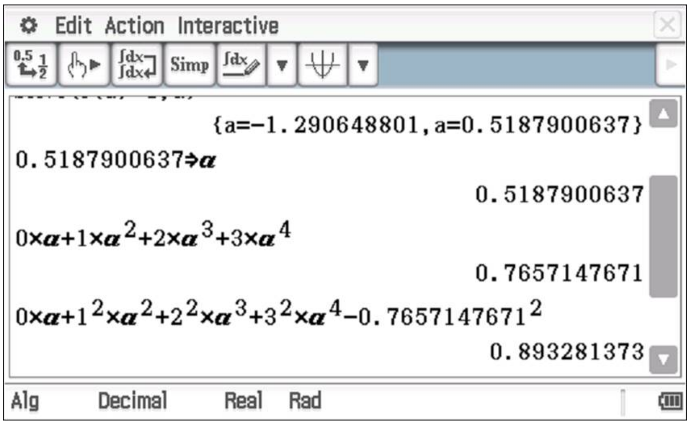
The rectangles are below the curve.
Underestimate **B**

Question 16

Solve $a + a^2 + a^3 + a^4 = 1$ for a
$a \approx 0.51879$
$E(X) = a^2 + 2\times a^3 + 3\times a^4 \approx 0.766$
$Var(X) = a^2 + 4a^3 + 9a^4 - (0.765714\ldots)^2 \approx 0.893$ **D**







Question 17

	J	J'	
R	$0.3p$	$0.7p$	p
R'	$0.3-0.3p$	p^2	$1-p$
	0.3	0.7	1

Independent events $\Pr(J \cap R) = \Pr(J) \times \Pr(R)$

Let $a = \Pr(J \cap R) = 0.3p$

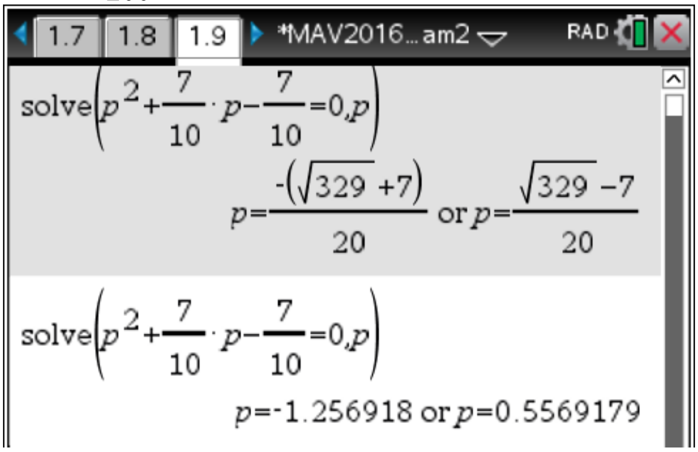
$0.3 + p - a + p^2 = 1$

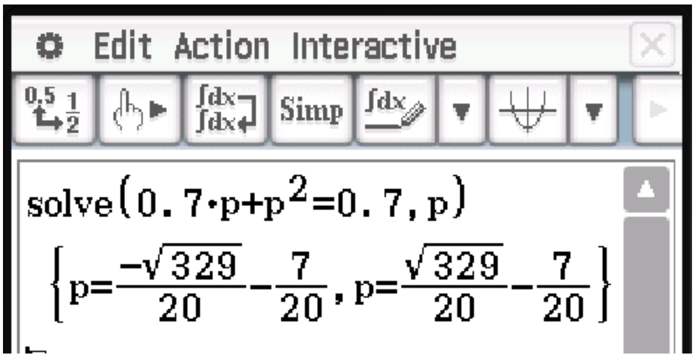
Solve $p^2 + 0.7p - 0.7 = 0$ for p.

$p = \frac{\sqrt{329} - 7}{20}$

$a = 0.3p$

$a = \frac{3\left(\sqrt{329} - 7\right)}{200}$ **D**





Question 18

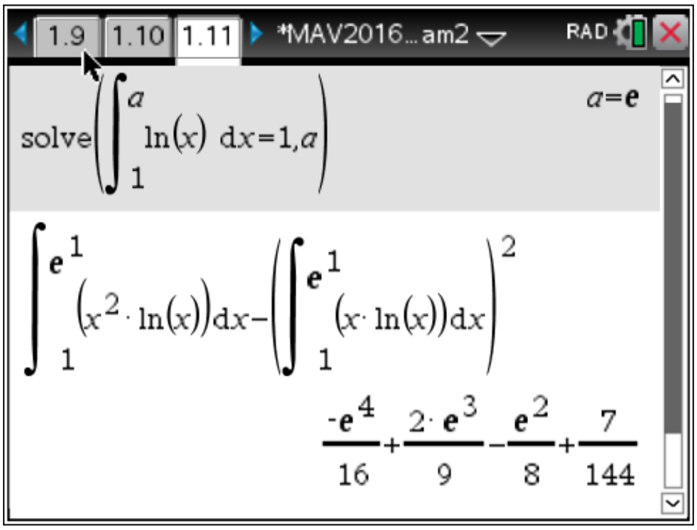
Solve $\int_1^a \left(\log_e(x)\right)dx = 1$ for a.

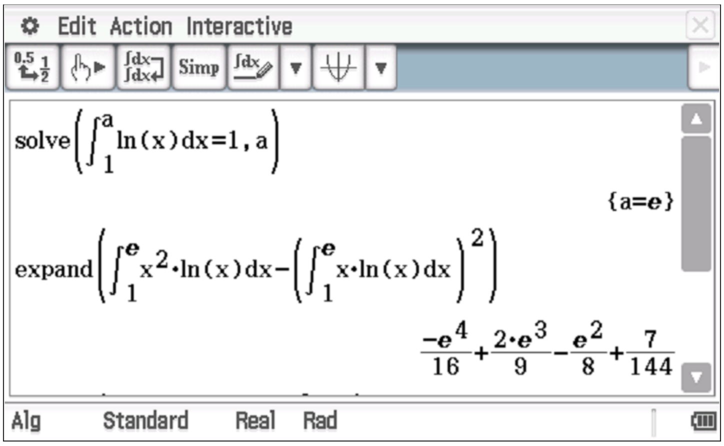
$a = e$

$\text{Var}(X) = \text{E}(X^2) - \left(\text{E}(X)\right)^2$

$$\text{Var}(X) = \int_1^e \left(x^2 \log_e(x)\right)dx - \left(\int_1^e \left(x \log_e(x)\right)dx\right)^2$$

$$= -\frac{e^4}{16} + \frac{2e^3}{9} - \frac{e^2}{8} + \frac{7}{144}$$ **C**

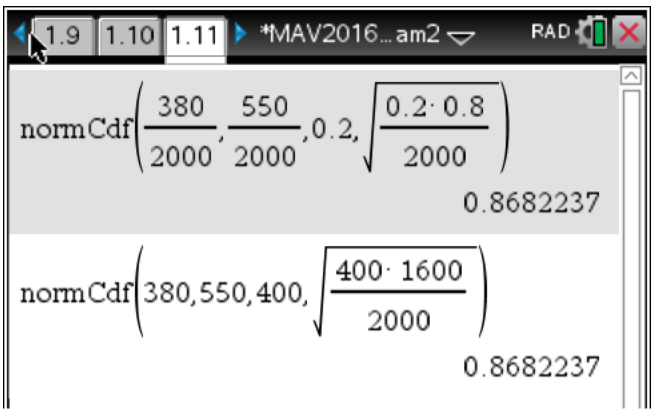


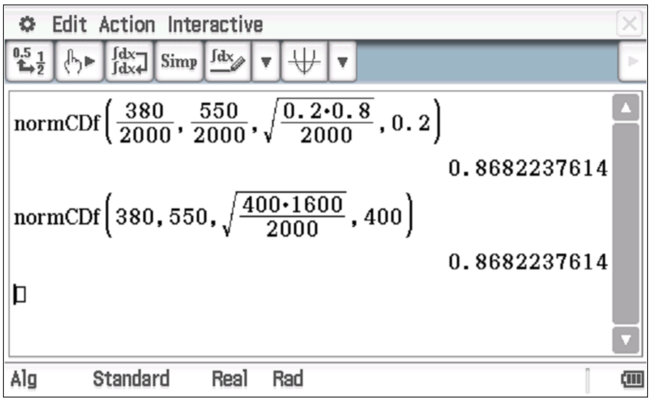


Question 19

$n = 2000$ and $p = 0.2$. Since n is large, the distribution of $\hat{P}$ is approximately normal with $\mu = 0.2$ and $\sigma=\sqrt{\frac{0.2\times 0.8}{2000}}$.

$\Pr\left(\frac{380}{2000} < X < \frac{550}{2000}\right) \approx 0.868$ **C**





Question 20

If 200 such samples were taken, $0.9 \times 200 = 180$ of the confidence intervals would be expected to contain *p*. **B**

SECTION B
EXTENDED RESPONSE QUESTIONS

Question 1 (14 marks)

a. $f(x) = ax^{\frac{2}{3}}$

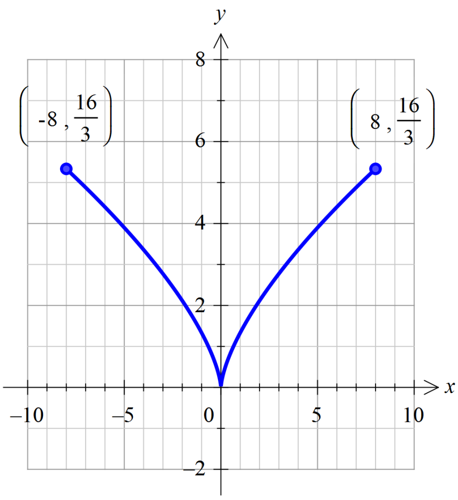
Substitute the point $\left(-8, \frac{16}{3}\right)$

$\frac{16}{3} = a(-8)^{\frac{2}{3}}$

$\frac{16}{3} = 4a$ **1M Show that**

$a = \frac{4}{3}$

b. Shape **1A**
Endpoints **1A**



c. $f_1(x) = f(-x)$ reflects the graph over the y-axis.

As the function $f(x) = \frac{4}{3}x^{\frac{2}{3}}$ is an even function the transformation described makes no difference. **1A**

d.

- a dilation of a factor of 2 units from the x-axis gives $y = \frac{8}{3}x^{\frac{2}{3}}$

  then

- a translation of 3 units up gives $f_2(x) = \frac{8}{3}x^{\frac{2}{3}} + 3$ **1A**

e.

- a reflection in the x-axis gives $y_1 = -\frac{4}{3}x^{\frac{2}{3}}$
- a reflection in the y-axis gives $y_2 = -\frac{4}{3}(-x)^{\frac{2}{3}} = -\frac{4}{3}x^{\frac{2}{3}}$
- a translation of 1 unit in the positive direction of the y-axis gives $y_3 = -\frac{4}{3}x^{\frac{2}{3}} + 1$
- a translation of 3 units in the positive direction of the x-axis gives $y_4 = -\frac{4}{3}(x-3)^{\frac{2}{3}} + 1$

The image equation is $f_3(x) = -\frac{4}{3}(x-3)^{\frac{2}{3}} + 1$ **2A**

f. $T: R^2 \to R^2, T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2 & 0 \\ 0 & -\frac{1}{3} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

gives $\begin{bmatrix} 2 & 0 \\ 0 & -\frac{1}{3} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2x+1 \\ -\frac{1}{3}y+2 \end{bmatrix} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$

Rearrange to get

$x = \frac{x_1 - 1}{2}$

$y = \frac{y_1 - 2}{-\frac{1}{3}} = -3(y_1 - 2)$ **1M**

Susbtitute to find the image equation for $f(x) = \frac{4}{3}x^{\frac{2}{3}}$.

$-3(y_1 - 2) = \frac{4}{3}\left(\frac{x_1 - 1}{2}\right)^{\frac{2}{3}}$ **1M**

Image equation is

$y_1 = -\frac{4}{9}\left(\frac{x_1 - 1}{2}\right)^{\frac{2}{3}} + 2$

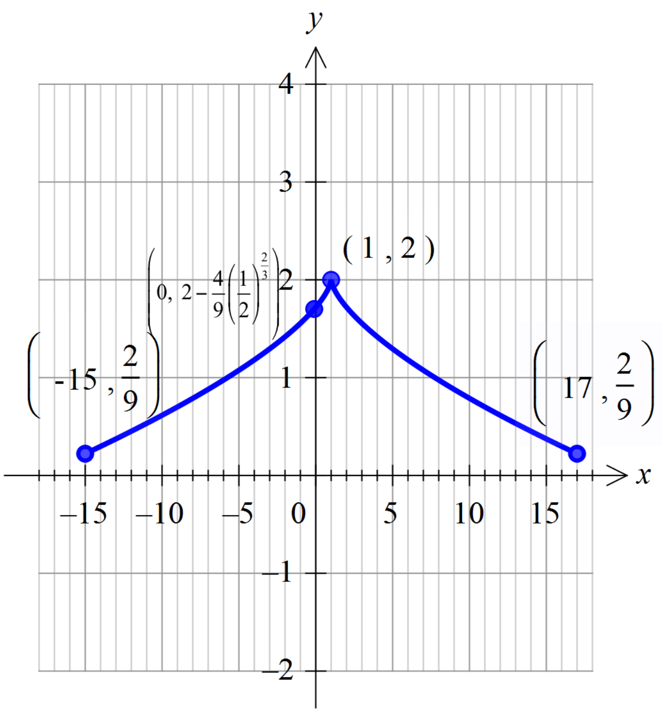
Giving $y = 2 - \frac{4}{9}\left(\frac{x-1}{2}\right)^{\frac{2}{3}}$ **1A**

g. $y = 2 - \frac{4}{9}\left(\frac{x-1}{2}\right)^{\frac{2}{3}}$ gives a dilation of a factor of 2 from y-axis and a translation of 1 unit in the positive direction of the x-axis from the original function $f(x) = \frac{4}{3}x^{\frac{2}{3}}$ where the domain was $[-8, 8]$.

A dilation of a factor of 2 from y-axis gives the domain $[-16, 16]$.

Then a translation of 1 unit in the positive direction of the x-axis gives a domain $[-15, 17]$. **1A**

h. Shape **1A**
Endpoints **1A**
Coordinates of Cusp and
y-intercept $\left(0,\ 2-\frac{4}{9}\left(\frac{1}{2}\right)^{\frac{2}{3}}\right)$ **1A**



Question 2 (15 marks)

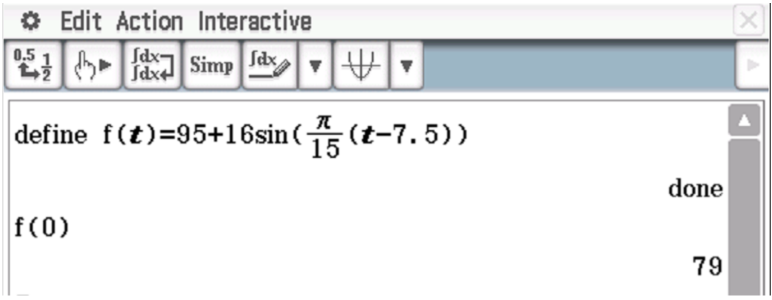
a. $P(0) = 95 + 16\sin\left(\frac{\pi}{15}(0 - 7.5)\right)$

$= 95 - 16\sin\left(\frac{\pi}{2}\right)$

$= 95 - 16$

$= 79$

79 beats per minute **1M Show that**



b. Period = $\frac{2\pi}{\pi/15} = 30$ **1A**

Amplitude = 16 **1A**

c. $P(t) = 95 + 16\sin\left(\frac{\pi}{15}(t - 7.5)\right)$

$P'(t) = 16 \times \frac{\pi}{15}\cos\left(\frac{\pi}{15}(t - 7.5)\right)$ **1A**

$P'(t) = \frac{16\pi}{15}\cos\left(\frac{\pi t}{15} - \frac{\pi}{2}\right)$

$= \frac{16\pi}{15}\cos\left(\frac{\pi}{2} - \frac{\pi t}{15}\right)$

Giving $P'(t) = \frac{16\pi}{15}\sin\left(\frac{\pi t}{15}\right)$. **1M Show that**

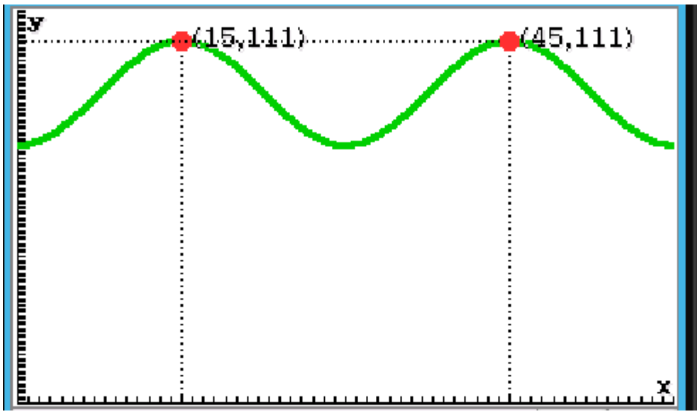
d. Solve $\frac{16\pi}{15}\sin\left(\frac{\pi t}{15}\right) = 0$ for $x \in [0, 60]$ 2 cycles **1M**

$\frac{16\pi}{15}\sin\left(\frac{\pi t}{15}\right) = 0$

$\sin\left(\frac{\pi t}{15}\right) = 0$

$\frac{\pi t}{15} = 0, \pi, 2\pi, 3\pi, 4\pi$

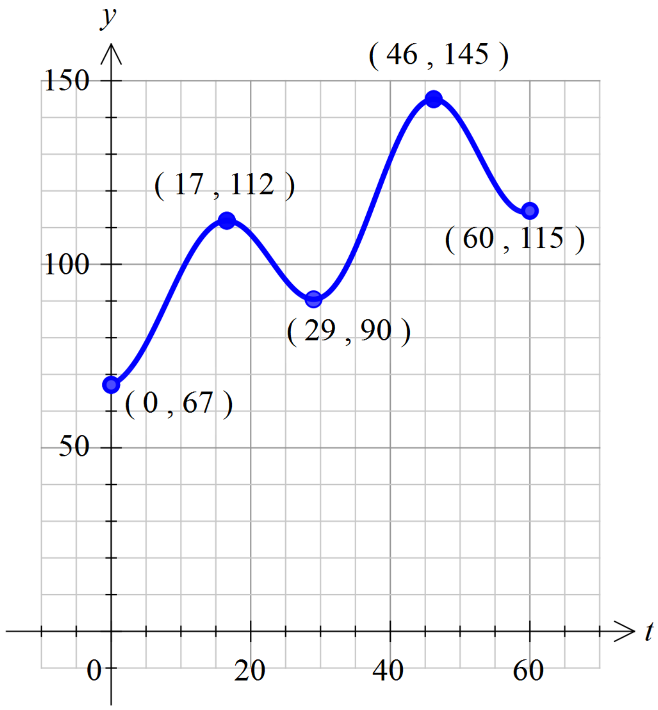
$t = 0, 15, 30, 45, 60$

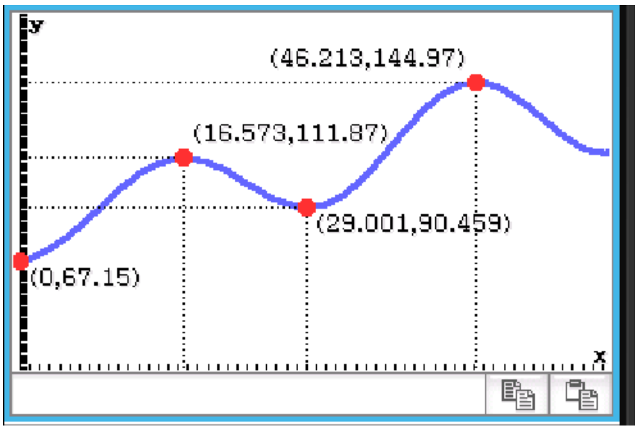


From the graph the local max occurs at $t = 15$ and $t = 45$ **1A**

e. $P_N : [0,60] \to R, P_N(t) = 0.01(t+85) \times P(t)$

Shape **1A**
Turning points **1A**
Endpoints **1A**

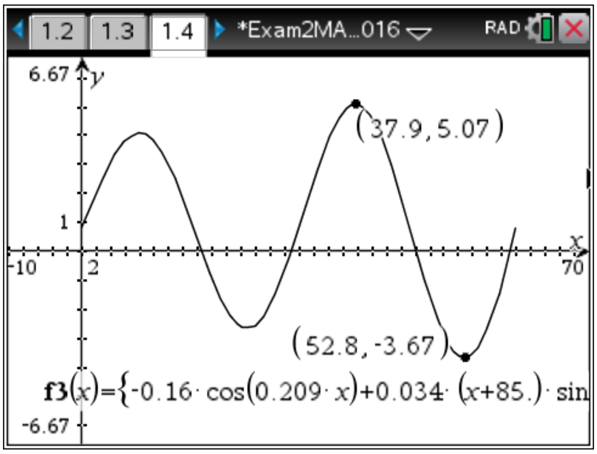




f. Maximum heart rate $\approx$ 144.97 beats /minute
Maximum heart rate rounded to the nearest whole number = 145 beats /minute **1A**

g. Find when $P_N'(t)$ has its greatest magnitude **1M**

$t = 37.9$ minutes **1A**



h. Solve the equation $0.01(t+85)=1$ gives $t = 15$ **1A**

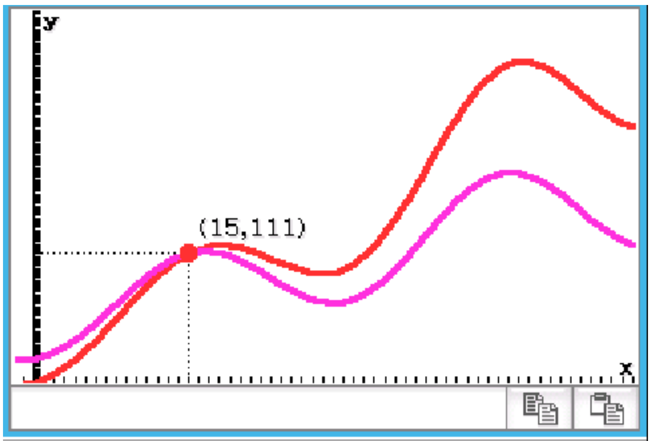
```
solve(0.01·(t+85)=1,t)
                          {t=15}
```

Solve $95+16\sin\left(\frac{\pi}{15}(t-7.5)\right)=0.01(t+85)\times P(t)$

$$95+16\sin\left(\frac{\pi}{15}(t-7.5)\right)=0.01(t+85)\times\left(95+16\sin\left(\frac{\pi}{15}(t-7.5)\right)\right)$$

Gives the equation $1=0.01(t+85)$ where previously found solution $t = 15$

So Sarah's pulse equals Stephen's pulse at $t = 15$.



Looking at the graph

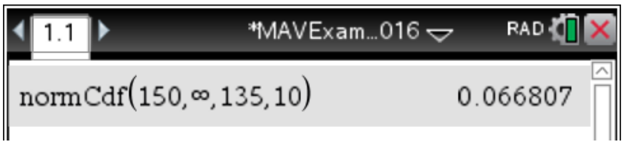
Sarah's pulse can be measured to be lower than her twin brother Stephen's

for $t \in (15,60]$ **1A**

Question 3 (17 marks)

a. $X \sim \mathrm{N}(135,100)$

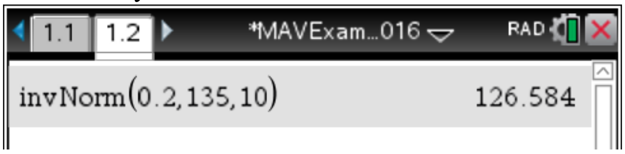
$\Pr(X > 150) = 0.0668... = 7\%$ to the nearest whole percent. **1A**



b. $\Pr(X > x) = 0.8$

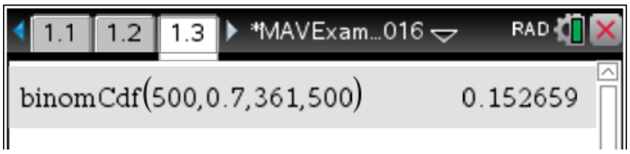
$\Pr(X < x) = 0.2$

$x = 127$ days to the nearest whole number **1A**



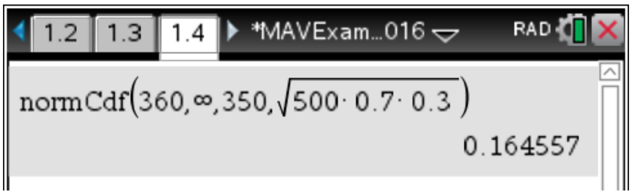
c. $Y \sim \mathrm{Bi}(500,0.7)$ **1A**

$\Pr(Y > 360) = \Pr(Y \geq 361) = 0.1527$ correct to 4 decimal places. **1A**



d. $W \sim \mathrm{N}(350, 500\times 0.7\times 0.3)$ **1M**

$\Pr(W > 360) = 0.1646$ correct to 4 decimal places. **1A**



e. $\mu = 0.7$

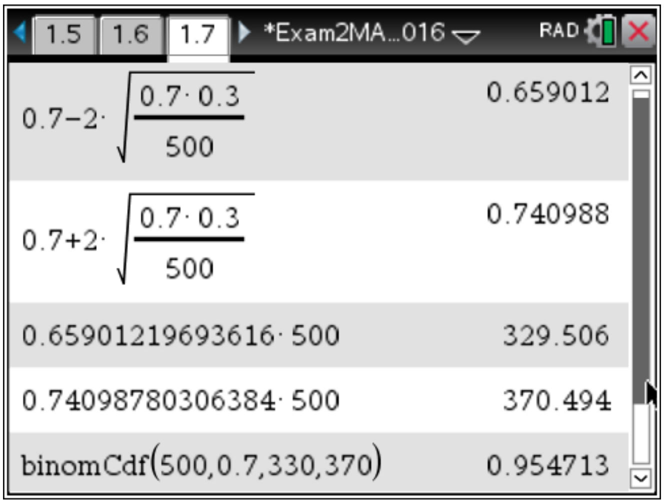
$$\sigma = \sqrt{\frac{0.7\times 0.3}{500}} = \frac{\sqrt{105}}{500}$$

$$\left(0.7 - 2\times\frac{\sqrt{105}}{500}, 0.7 + 2\times\frac{\sqrt{105}}{500}\right)$$ **1M**

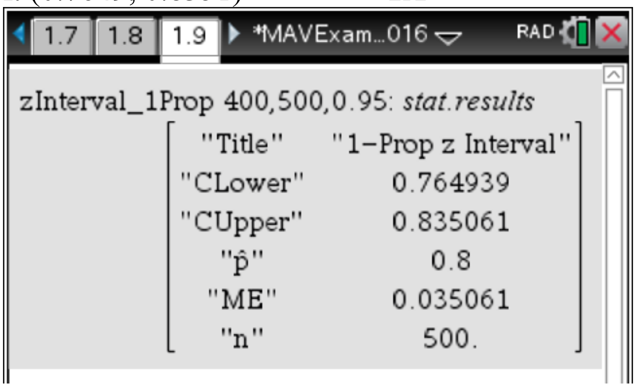
$\Pr\left(0.6590... \leq \hat{P} \leq 0.7409...\right) = \Pr(329.5... \leq X \leq 370.4...)$ **1M**

$= \Pr(330 \leq X \leq 370)$

$= 0.9547$ correct to four decimal places. **1A**



f. (0.7649, 0.8351) **1A**



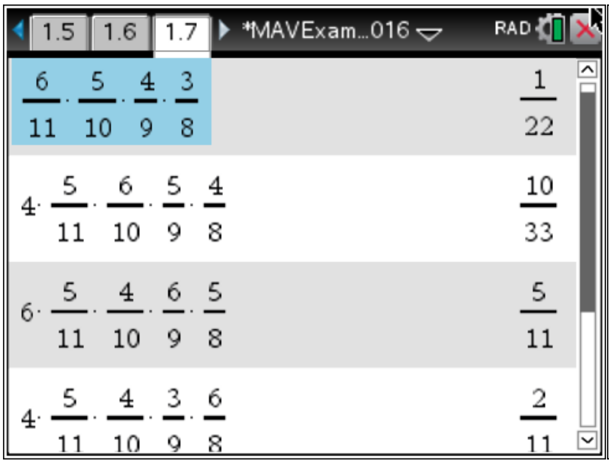
g.

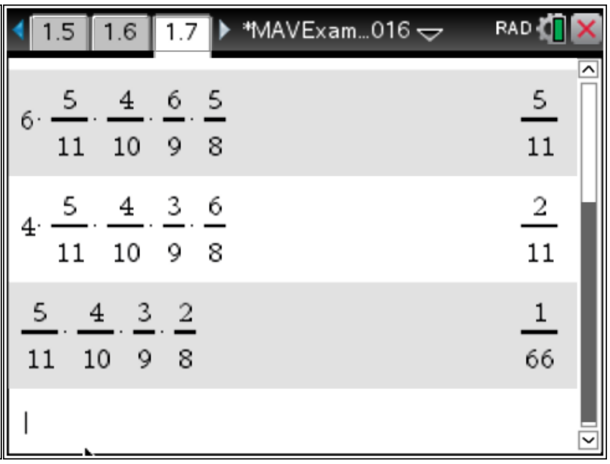
Proportion of yellow balls $\hat{p}$	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1
$\Pr\left(\hat{P}=\hat{p}\right)$	$\frac{1}{22}$	$\frac{10}{33}$	$\frac{5}{11}$	$\frac{2}{11}$	$\frac{1}{66}$

Without replacement

$\frac{6}{11}\times\frac{5}{10}\times\frac{4}{9}\times\frac{3}{8}=\frac{1}{22}$ **1A**

$4\times\frac{5}{11}\times\frac{6}{10}\times\frac{5}{9}\times\frac{4}{8}=\frac{10}{33}$ **1A**





h. $\Pr\left(\hat{P}=1 \mid \hat{P}>\frac{1}{2}\right)=\dfrac{\Pr\left(\hat{P}=1 \cap \hat{P}>\frac{1}{2}\right)}{\Pr\left(\hat{P}>\frac{1}{2}\right)}$ **1M**

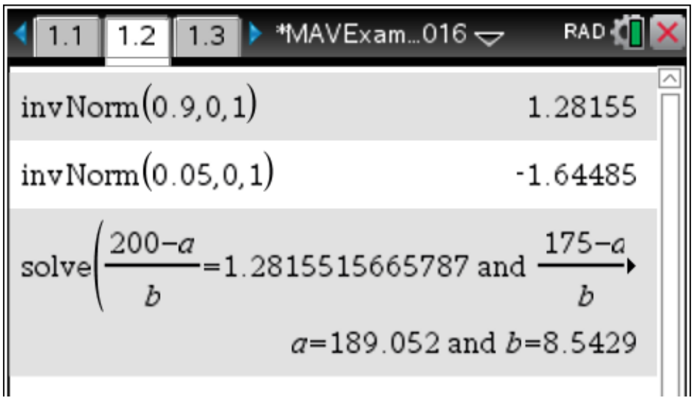
$=\dfrac{\frac{1}{66}}{\frac{13}{66}}=\dfrac{1}{13}$ **1A**

i. $\Pr(X>200)=0.1, \Pr(X<185)=0.05$ **1M**

$\Pr\left(Z>\dfrac{200-\mu}{\sigma}\right)=0.1$, $\Pr\left(Z<\dfrac{175-\mu}{\sigma}\right)=0.05$

$\dfrac{200-\mu}{\sigma}=1.2815\ldots, \dfrac{175-\mu}{\sigma}=-1.6448\ldots$ **1M**

$\mu=189$ days, $\sigma=9$ days **1A**

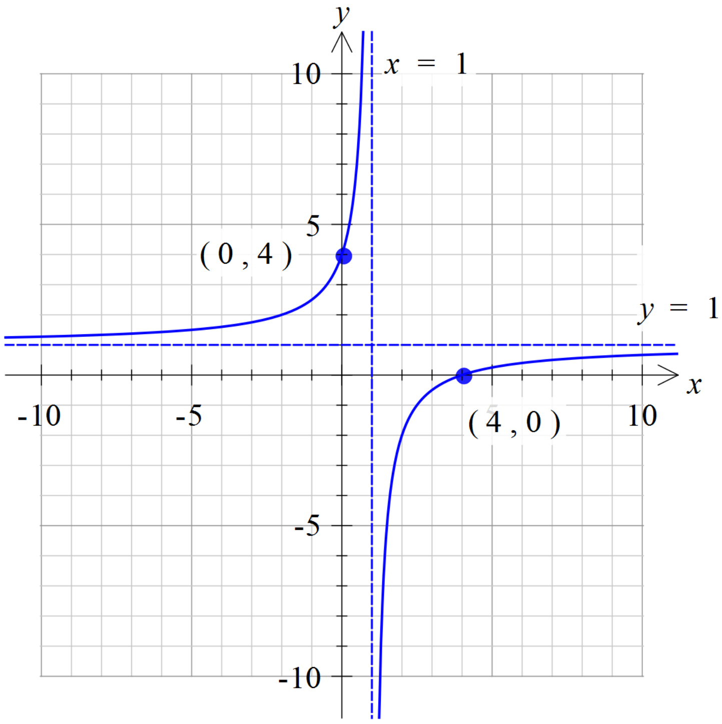


Question 4 (14 marks)

a. shape **1A**

asymptotes with equations **1A**

coordinates of axial-intercepts **1A**



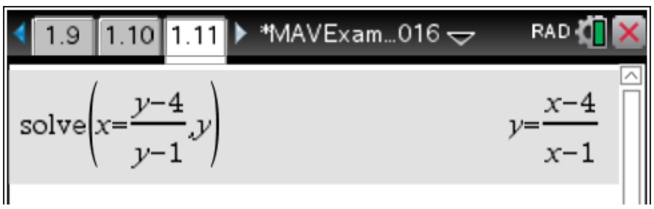
b. Let $y = \dfrac{x-4}{x-1}$

Inverse swap x and y.

Solve $x = \dfrac{y-4}{y-1}$ for y.

$f^{-1}(x) = \dfrac{x-4}{x-1}$

Hence $f(x) = f^{-1}(x)$ **1A**



c. $g: R\backslash\{-n\} \to R, g(x) = \dfrac{x+m}{x+n}$

$g^{-1}(x) = \dfrac{-nx+m}{x-1}$

$n = -1$ **1A**

$m \in R\backslash\{-1,0\}$ **1A**

OR

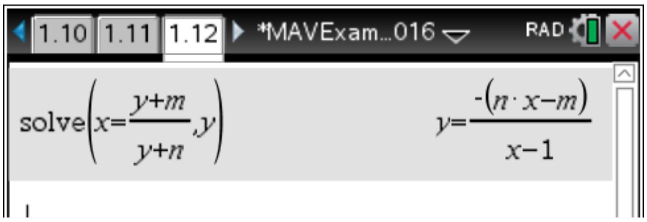
$g(x) = 1 + \dfrac{m+n}{x+n}$

Horizontal asymptote at $y = 1$

Vertical asymptote at $x = 1$

$n = -1$ **1A**

$m \in R \setminus \{-1, 0\}$ **1A**

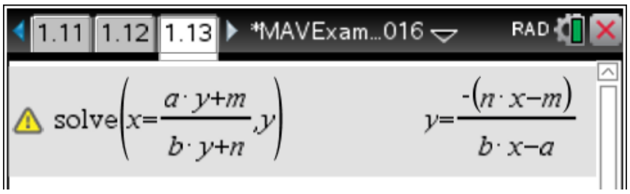


d. $h: R \setminus \left\{-\dfrac{b}{n}\right\} \to R, h(x) = \dfrac{ax+m}{bx+n}$

$h^{-1}(x) = \dfrac{-nx+m}{bx-a}$

$ax + m = -nx + m$, $bx - a = bx + n$

$a = -n$ **1M Show that**



e. $h_1: R \setminus \left\{-\dfrac{n}{4}\right\} \to R, h_1(x) = \dfrac{3x+2}{4x+n}$

$n = -3$ **1A**

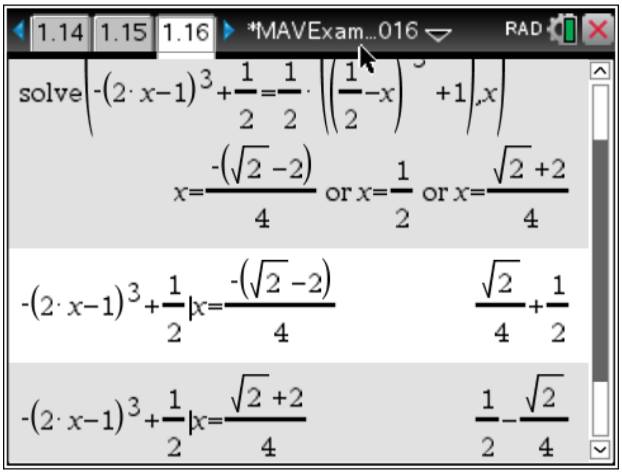
f. $w: R \to R, w(x) = -(2x-1)^3 + \dfrac{1}{2}$

$w^{-1}(x) = \dfrac{1}{2}\left(\sqrt[3]{\left(\dfrac{1}{2} - x\right)} + 1\right)$

Solve $w(x) = w^{-1}(x)$ for x. **1M**

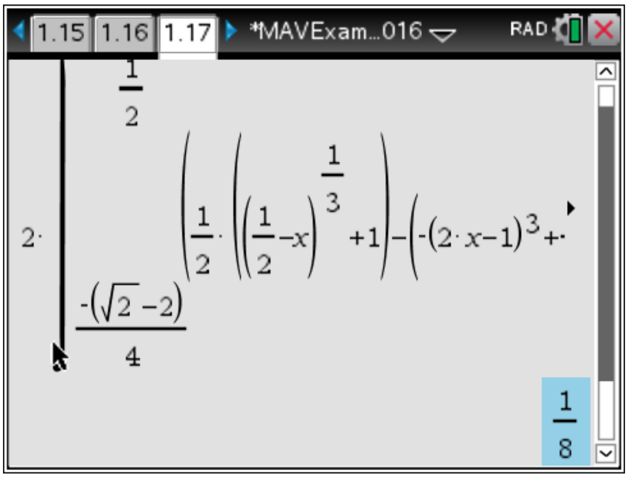
$x = \dfrac{2-\sqrt{2}}{4}, x = \dfrac{1}{2}$ or $x = \dfrac{2+\sqrt{2}}{4}$ **1A**

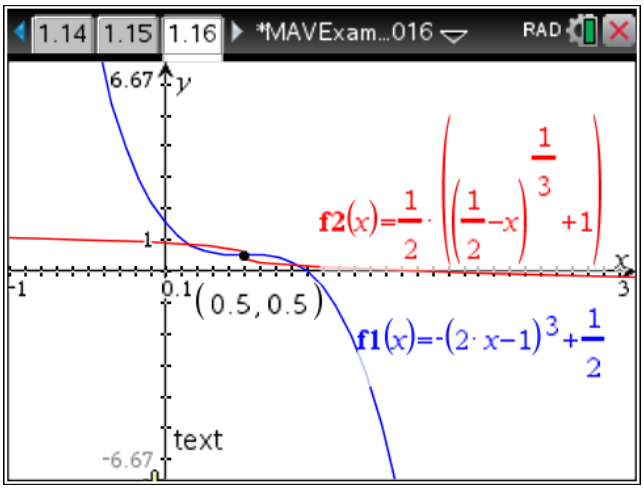
$\left(\dfrac{2-\sqrt{2}}{4}, \dfrac{2+\sqrt{2}}{4}\right), \left(\dfrac{1}{2}, \dfrac{1}{2}\right), \left(\dfrac{2+\sqrt{2}}{4}, \dfrac{2-\sqrt{2}}{4}\right)$ **1H**



g. Area $= 2\int_{\frac{2-\sqrt{2}}{4}}^{\frac{1}{2}} \left(\frac{1}{2}\left(\sqrt[3]{\left(\frac{1}{2}-x\right)}+1\right)-\left(-(2x-1)^3+\frac{1}{2}\right)\right)dx$ **1M**

$= \frac{1}{8}$ **1A**





h. $w_k : R \to R, w_k(x) = -(rx-1)^3 + s$

$s = \frac{1}{r}$ **1A**