

Trial Examination 2016

MATHEMATICAL METHODS

Written Examination 1 - SOLUTIONS

Question 1

a. $f(x) = \log_e(\cos(4x))$

$$f'(x) = \frac{-4\sin(4x)}{\cos(4x)} \quad \mathbf{1M}$$

$$= -4\tan(4x) \quad \mathbf{1A}$$

b.i. $x^3 - 3x^2 + 3x - 1$
 $= (x-1)^3 \quad \mathbf{1A}$

b.ii. $\int \left(\frac{1}{(1-x)(x^3 - 3x^2 + 3x - 1)} \right) dx$
 $= -\int \left(\frac{1}{(x-1)^4} \right) dx$
 $= -\int ((x-1)^{-4}) dx$
 $= \frac{1}{3(x-1)^3} + c \quad \mathbf{1A}$

Question 2

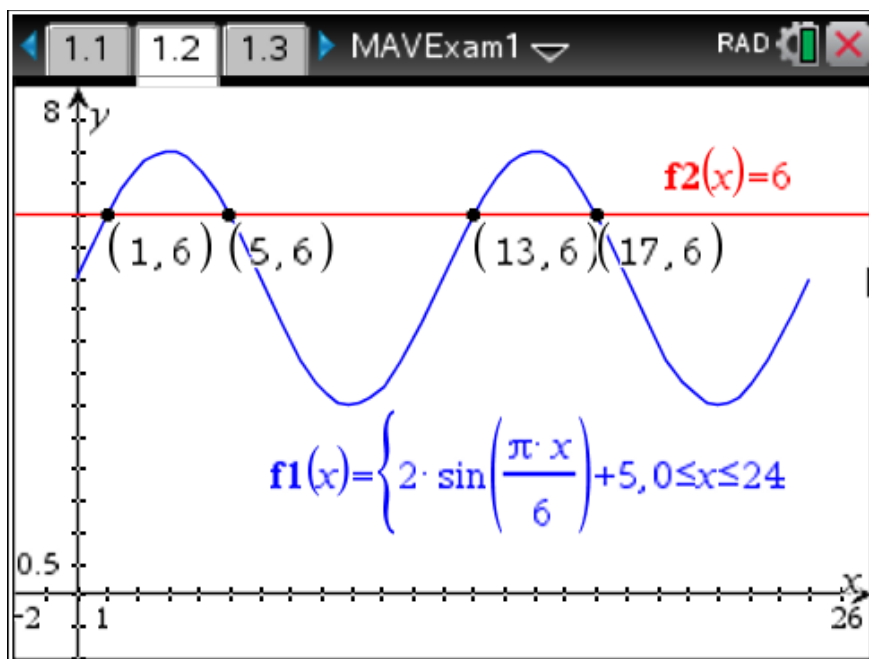
Let $2\sin\left(\frac{\pi t}{6}\right) + 5 = 6$

$$\sin\left(\frac{\pi t}{6}\right) = \frac{1}{2}$$

$$\frac{\pi t}{6} = \frac{\pi}{6}, \frac{5\pi}{6}, \dots \quad \mathbf{1A}$$

$$t = 1, 5, 13, 17 \quad \mathbf{1A}$$

$$\{t : 1 < t < 5\} \cup \{t : 13 < t < 17\} \quad \mathbf{1A}$$

**Question 3**

a. $g(x) = 2x^5 - 10x^4 + 20x^3 - 20x^2 + 10x + 2$

$$g(x) = A(x+B)^5 + C = Ax^5 + 5Ax^4B + \dots + AB^5 + C$$

Equating coefficients

$$A = 2 \quad \quad \quad 1A$$

$$5AB = -10, \quad AB^5 + C = 2$$

$$10B = -10 \quad B = -1 \quad \quad \quad 1A$$

$$-2 + C = 2, \quad C = 4 \quad \quad \quad 1A$$

OR

$$g(x) = 2(x^5 - 5x^4 + 10x^3 - 10x^2 + 5x + 1)$$

$$= 2((x-1)^5 + 2)$$

$$= 2(x-1)^5 + 4$$

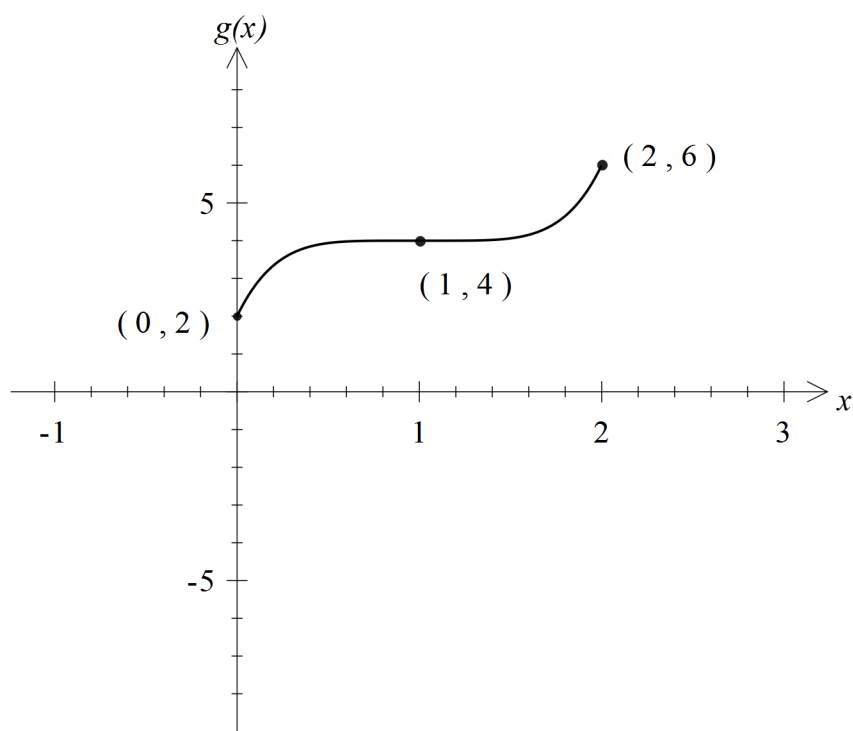
$$A = 2 \quad \quad \quad 1A$$

$$B = -1 \quad \quad \quad 1A$$

$$C = 4 \quad \quad \quad 1A$$

b. shape and stationary point of inflection 1A

coordinates of endpoints 1A

**Question 4**

a. $\frac{9}{e^{2x}} - \frac{6}{e^x} + 2e^x = 5$

$$\begin{aligned} \text{LHS} &= \frac{9}{e^{2\log_e(3)}} - \frac{6}{e^{\log_e(3)}} + 2e^{\log_e(3)} \\ &= \frac{9}{9} - \frac{6}{3} + 6 \\ &= 5 = \text{RHS} \end{aligned}$$

1M Show that

b. $\frac{9}{e^{2x}} - \frac{6}{e^x} + 2e^x = 5$

By inspection, let $x = 0$

$$\begin{aligned} \text{LHS} &= \frac{9}{e^0} - \frac{6}{e^0} + 2e^0 \\ &= 9 - 6 + 2 \end{aligned}$$

 $= 5$ as required**1M**

$$x = 0 \text{ or } x = \log_e(3)$$

1A**OR**

$$9 - 6e^x + 2e^{3x} = 5e^{2x}$$

$$2e^{3x} - 5e^{2x} - 6e^x + 9 = 0 \quad \mathbf{1M}$$

Let $a = e^x$

$$\text{Let } f(x) = 2a^3 - 5a^2 - 6a + 9 = 0$$

$$f(1) = 0, \quad a - 1 \text{ is a factor}$$

First term in the quadratic factor has to be $2a^2$.

Last term has to be -9 .

$$-2a^2 + ? = -5a^2, \quad ? = -3a^2$$

The middle term is $-3a$

$$(a-1)(2a^2-3a-9)=0$$

$$(a-1)(2a+3)(a-3)=0$$

$$a=1 \text{ or } a=-\frac{3}{2} \text{ or } a=3$$

$$e^x=1, \quad e^x \neq -\frac{3}{2}, \quad e^x=3$$

$$x=0 \text{ or } x=\log_e(3) \quad \mathbf{1A}$$

Question 5

a. $f(x) = xe^{2x}$

$$f'(x) = 2xe^{2x} + e^{2x} \quad \mathbf{1A}$$

b. $\int (2xe^{2x} + e^{2x}) dx = xe^{2x} + c \quad \mathbf{1M}$

$$\int (2xe^{2x}) dx = xe^{2x} - \int (e^{2x}) dx + c$$

$$\int (xe^{2x}) dx = \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + c_1$$

$$\text{Average Value} = \frac{1}{\frac{1}{2}-0} \int_0^{\frac{1}{2}} (xe^{2x}) dx \quad \mathbf{1M}$$

$$= 2 \int_0^{\frac{1}{2}} (xe^{2x}) dx$$

$$= \left[xe^{2x} - \frac{1}{2}e^{2x} \right]_0^{\frac{1}{2}}$$

$$= \left(\frac{1}{2}e - \frac{1}{2}e \right) - \left(0 - \frac{1}{2} \right)$$

$$= \frac{1}{2} \quad \mathbf{1A}$$

Question 6

$$10 = A \log_e(2 - B) \dots (1)$$

$$20 = A \log_e(8 - B) \dots (2) \quad \mathbf{1M}$$

Divide (2) by (1)

$$2 = \frac{\log_e(8 - B)}{\log_e(2 - B)}$$

$$2 \log_e(2 - B) = \log_e(8 - B)$$

$$\log_e((2 - B)^2) = \log_e(8 - B)$$

$$(2 - B)^2 = 8 - B$$

$$4 - 4B + B^2 = 8 - B$$

$$B^2 - 3B - 4 = 0$$

$$(B - 4)(B + 1) = 0$$

$$B = -1, \quad B \neq 4 \text{ as } B < 2 \quad \mathbf{1A}$$

Substitute $B = -1$ into (1)

$$A = \frac{10}{\log_e(3)} \quad \mathbf{1A}$$

Question 7**a.**

$$-2(m - 1)x + my = -m + 4$$

$$mx - 3y = 2m + 1$$

Using ratios for an infinite number of solutions or no solution gives

$$\frac{-2(m - 1)}{m} = \frac{m}{-3} \quad \mathbf{1M}$$

OR

The gradients will be the same for an infinite number of solutions or no solution

Rearranging the equations gives

$$y = \frac{2(m - 1)}{m}x - 1 + \frac{4}{m}$$

$$y = \frac{mx}{3} - \frac{2m}{3} - \frac{1}{3}$$

$$\frac{2(m - 1)}{m} = \frac{m}{3} \quad \mathbf{1M}$$

Solve for m

$$6(m - 1) = m^2$$

$$m^2 - 6m + 6 = 0$$

Using the quadratic formula gives

$$m = \frac{6 \pm \sqrt{36 - 24}}{2}$$

$$m = \frac{6 \pm \sqrt{12}}{2} = \frac{6 \pm 2\sqrt{3}}{2}$$

$$m = 3 \pm \sqrt{3}$$

So for unique solutions we need $m \in \mathbb{R} \setminus \{3 \pm \sqrt{3}\}$ **1A**

b. If $m = -1$, $y = 4x - 5$ and $y = -x^2 + 2x - 6$

Point of intersection $-x^2 + 2x - 6 = 4x - 5$ **1M**

$$x^2 + 2x + 1 = 0$$

$$(x + 1)^2 = 0$$

$x = -1$ only one unique solution, hence the line is a tangent to the parabola.

1A

(At $x = -1$, the gradient of the line and the parabola are both 4.)

Question 8

a. length = $14 - 2x$

width = $11 - 2x$

Using Pythagoras

The diagonal

$$= \sqrt{(11 - 2x)^2 + (14 - 2x)^2}$$

$$= \sqrt{121 - 44x + 4x^2 + 4x^2 - 56x + 196}$$

$$= \sqrt{8x^2 - 100x + 317}$$

1A**1M Show that**

b. $\sqrt{8x^2 - 100x + 317} = 15$

$$8x^2 - 100x + 317 = 225$$

$$8x^2 - 100x + 92 = 0$$

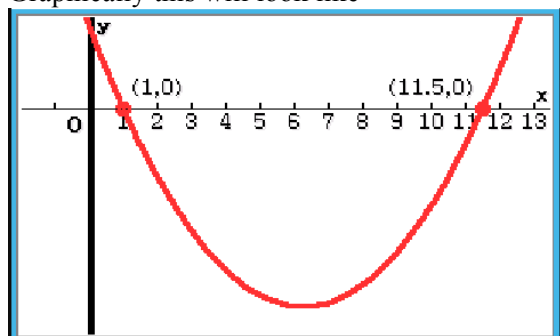
$$4(2x^2 - 25x + 23) = 0$$

$$4(2x - 23)(x - 1) = 0$$

Giving $x = \frac{23}{2}, x = 1$

1A**1A**

Graphically this will look like



So the inequation $4(2x - 23)(x - 1) \geq 0$

Has solutions $x \leq 1, x \geq \frac{23}{2}$

But implicit domain is $x \in \left(0, \frac{11}{2}\right)$

So solutions are $x \in (0, 1]$

1A

c. Area = $(11 - 2x)(14 - 2x), x \in \left(0, \frac{11}{2}\right)$

So area < 11×14

Area < 154 sq m.,

It is not possible to have an area of 155 square metres and above.

1M

Question 9

a. $\hat{P} = 0.8, n = 4$

$$\text{sd}(\hat{P}) = \sqrt{\frac{pq}{n}} = \sqrt{\frac{0.8 \times 0.2}{4}} = 0.2 \quad 1A$$

$$\begin{aligned} \Pr(0.6 \leq \hat{P} \leq 1) &= \Pr(2.4 \leq X \leq 4) \\ &= \Pr(3 \leq X \leq 4) \text{ as } X \text{ is an integer} \\ &= \binom{4}{3} (0.8)^3 (0.2)^1 + \binom{4}{4} (0.8)^4 (0.2)^0 \quad 1A \\ &= \binom{4}{1} (0.8)^3 (0.2) + 0.8^4 \\ &= 2 \times (0.8)^4 \\ &= 0.8192 \text{ or } \frac{512}{625} \quad 1A \end{aligned}$$

b. $Y : Bi(4, 0.2)$

$$\begin{aligned} \Pr(Y \leq 1) &= \Pr(Y = 0) + \Pr(Y = 1) \\ &= (0.8)^4 + \binom{4}{1} (0.2) (0.8)^3 \\ &= 0.8192 \text{ or } \frac{512}{625} \quad 1A \end{aligned}$$

$$\Pr(\text{first 2 shots were not goals}) = 2(0.8)^3 (0.2) + (0.8)^4, G'G'G'G + G'G'GG' + G'G'G'G'$$

$$\Pr(\text{first two shots were not goals} \mid \text{there was no more than 1 goal})$$

$$\begin{aligned} &= \frac{2(0.8)^3 (0.2) + (0.8)^4}{(0.8)^4 + 4(0.8)^3 (0.2)} \\ &= \frac{(0.8)^3 (1.2)}{(0.8)^3 (1.6)} \text{ or } \frac{0.6144}{0.8192} \text{ or } \frac{\frac{384}{625}}{\frac{512}{625}} \\ &= \frac{3}{4} \quad 1A \end{aligned}$$

c. $\Pr(W \geq 1) > 0.9$

$$1 - \Pr(W = 0) > 0.9 \quad 1M$$

$$1 - 0.8^n > 0.9$$

$$0.8^n < 0.1 \quad 1A$$

$$n > \frac{\log_{10}(0.1)}{\log_{10}(0.8)}$$

$$n > \frac{-1}{\log_{10}(8) - \log_{10}(10)}$$

$$n > \frac{-1}{-0.097} \text{ } (\approx 10.3 \text{ or is slightly greater than } 10)$$

$$n = 11 \quad 1A$$