

Year 2016

VCE

Mathematical Methods

Trial Examination 2



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**Victorian Certificate of Education
2016**

STUDENT NUMBER

Figures
Words

Letter

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**MATHEMATICAL METHODS
Trial Written Examination 2**

Reading time: 15 minutes

Total writing time: 2 hours

QUESTION AND ANSWER BOOK

Structure of book

Section	Number of questions	Number of questions to be answered	Number of marks
A	20	20	20
B	5	5	60
			Total 80

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners, rulers, a protractor, set squares, aids for curve sketching, one bound reference, one approved technology (calculator or software) and, if desired, one scientific calculator. Calculator memory DOES NOT need to be cleared. For approved computer-based CAS, their full functionality may be used.
- Students are NOT permitted to bring into the examination room: blank sheets of paper and/or white out liquid/tape.

Materials supplied

- Question and answer booklet of 31 pages.
- Detachable sheet of miscellaneous formulas at the end of this booklet.
- Answer sheet for multiple-choice questions.

Instructions

- Detach the formula sheet from the end of this book during reading time.
- Write your **student number** in the space provided above on this page.
- Write your **name** and **student number** on your answer sheet for multiple-choice questions, and sign your name in the space provided to verify this.
- Unless otherwise indicated, the diagrams in this booklet are **not** drawn to scale.
- All written responses must be in English.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.

SECTION A**Instructions for Section A**

Answer **all** questions in pencil on the answer sheet provided for multiple-choice questions. Choose the response that is **correct** for the question.

A correct answer scores 1 mark, an incorrect answer scores 0.

Marks will **not** be deducted for incorrect answers.

No mark will be given if more than one answer is completed for any question.

Unless otherwise indicated, the diagrams in this booklet are **not** drawn to scale.

Question 1

Consider the function $f : [2a, \infty) \rightarrow R$, $f(x) = x(a - x)$ where $a \in R \setminus \{0\}$.

The maximum value of the function is

- A. 0
- B. $-\frac{a^2}{4}$
- C. $\frac{a^2}{4}$
- D. $-2a^2$
- E. $2a^2$

Question 2

The point $P(3, -2)$ lies on the graph of the function f . The graph of f is translated one unit away from the y -axis, reflected in the x -axis then reflected in the line $y = x$. The coordinates of the image of P are

- A. $(4, 2)$
- B. $(2, 4)$
- C. $(2, 2)$
- D. $(-1, -3)$
- E. $(-2, -4)$

Question 3

Consider the functions $f(x) = a \sin\left(\frac{\pi x}{a}\right)$, $g(x) = a \tan\left(\frac{\pi x}{a}\right)$ and

$h(x) = a \cos\left(\frac{\pi}{a}\left(x - \frac{\pi}{2}\right)\right)$ where $a > 0$. Then which of the following is true?

- A. $h(x) = f(x)$
- B. $f(x)$ and $g(x)$ have the same period but different amplitudes.
- C. $f(x)$ and $g(x)$ have the same amplitude but different periods.
- D. $f(x)$ and $g(x)$ have the same amplitude and the same period.
- E. $f(x)$ and $h(x)$ have the same amplitude and the same period.

Question 4

Let f be a function with domain R . The function has the following properties

$f'(x) < 0$ for $x < a$ and $f'(x) < 0$ for $x > a$ and $f'(a) = 0$.

Then at $x = a$ the graph of f has a

- A. local minimum
- B. local maximum
- C. stationary point of inflection
- D. non-stationary point of inflection
- E. point of discontinuity

Question 5

Given that $f(x) = u(x) \log_e(2x)$ and $u(2) = 6$ and $u'(2) = 3$ then $f'(2)$ is equal to

- A. $3 \log_e(4e)$
- B. $3(\log_e(4) + 4)$
- C. $3 \log_e(4)$
- D. 6
- E. $\frac{3}{2} + 3 \log_e(4)$

Question 6

If $x + 2a$ is a factor of $x^3 + 6ka^2x + 4a^3$ where $a \in R \setminus \{0\}$, then

A. $k = -1$

B. $k = -\frac{1}{3}$

C. $k = \frac{1}{2}$

D. $k = \frac{1}{3}$

E. $k = 1$

Question 7

The function $f : (0, a] \rightarrow R$, $f(x) = \tan\left(\frac{\pi x}{3}\right)$ has an inverse function. The maximum possible value of a is

A. $\frac{3}{2}$

B. 3

C. 6

D. 9

E. 12

Question 8

Consider the functions $f(x) = \frac{1}{x^2}$ and $g(x) = \frac{1}{\sqrt{x}}$ defined on their maximal domains.

Then

A. $g(x) = \frac{1}{\sqrt{\sqrt{f(x)}}}$ for $x \in R$

B. $g^{-1}(x) = f(x)$ for $x \in R$

C. $g^{-1}(x) = f(x)$ for $x \in R \setminus \{0\}$

D. $f^{-1}(x) = g(x)$ for $x \in R$

E. $f^{-1}(x) = g(x)$ for $x > 0$

Question 9

If $\int_1^a f(x) dx = 4$ and $\int_a^1 g(x) dx = -3$ then $\int_1^a (2f(x) - 3g(x) + 1) dx$ is equal to

- A. 16
- B. $a - 1$
- C. $a - 2$
- D. $16 + a$
- E. $17 + a$

Question 10

The curve with the equation $y = e^x$ is transformed by a dilation from the y-axis by a scale factor of 2, a translation by one unit to the left in the x-direction and a translation of two units downwards in the y-direction. The equation of the transformed curve is

- A. $y = 0.5e^{x-1} - 2$
- B. $y = 2e^{x-1} - 2$
- C. $y = e^{0.5(x+1)} - 2$
- D. $y = e^{2(x+1)} - 2$
- E. $y = e^{2(x+1)} + 2$

Question 11

Consider the graph of the function $f : R \rightarrow R$, $f(x) = ax^3 + \sqrt{b}x^2 - x$ where $b > 0$. Then if

- A. $b + 4a < 0$ the graph crosses the x-axis three times.
- B. $b + 4a < 0$ the graph has no turning points.
- C. $b + 3a > 0$ the graph crosses the x-axis three times.
- D. $b + 3a < 0$ the graph has no turning points.
- E. $b + 3a = 0$ the graph has no turning points.

Question 12

A certain curve has its gradient given by $4e^{-\frac{x}{2}}$. If the curve crosses the y-axis at $y = 4$ then it crosses the x-axis at

A. the origin

B. $\log_e \left(\frac{\sqrt{3}}{3} \right)$

C. $\log_e \left(\frac{9}{4} \right)$

D. $\log_e \left(\frac{4}{9} \right)$

E. $-\log_e (9)$

Question 13

The transformation $T: R^2 \rightarrow R^2$, which maps the curve with equation $y = \frac{1}{x^2}$ to the curve with equation $y = 2 + \frac{8}{(x+4)^2}$, has the rule

A. $T \left(\begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -4 \\ 2 \end{bmatrix}$

B. $T \left(\begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 \\ 0 & 8 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 4 \\ -2 \end{bmatrix}$

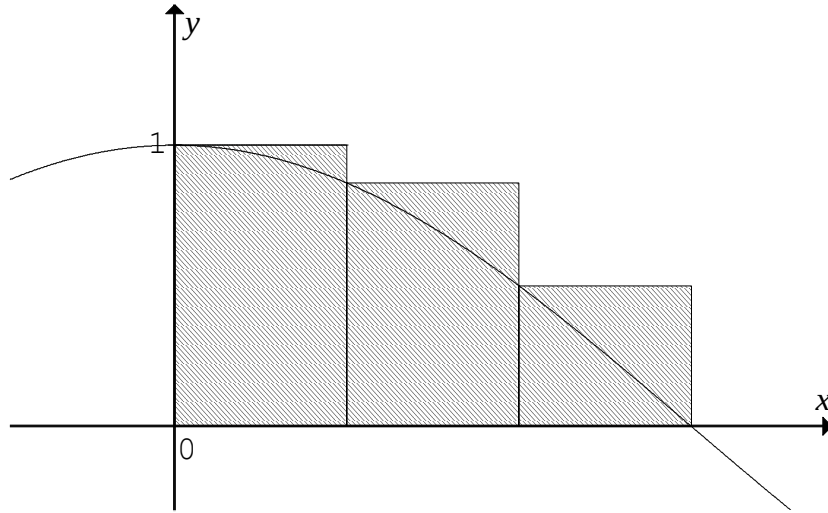
C. $T \left(\begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 4 \\ -2 \end{bmatrix}$

D. $T \left(\begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{bmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -4 \\ -1 \end{bmatrix}$

E. $T \left(\begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

Question 14

The area of the three shaded rectangles, each of equal width, as shown in the diagram below can be used an approximation for the area between the curve $y = \cos(x)$ and the co-ordinate axes. The value of this area is



- A. 1
- B. $\frac{\pi}{12}(1 + \sqrt{3})$ and this under-estimates the exact area.
- C. $\frac{\pi}{12}(1 + \sqrt{3})$ and this over-estimates the exact area.
- D. $\frac{\pi}{12}(3 + \sqrt{3})$ and this under-estimates the exact area.
- E. $\frac{\pi}{12}(3 + \sqrt{3})$ and this over-estimates the exact area.

Question 15

Given a population where the probability of a success is 0.4, a random sample of 100 is taken. The probability that the proportion of successes in the sample will be greater than 0.5 is closest to

- A. 1
- B. 0.98
- C. 0.50
- D. 0.02
- E. 0

Question 16

The random variable Z has the standard normal distribution, with $\Pr(Z < a) = A$ and $\Pr(Z > b) = B$, where $b > a$. Then $\Pr(a < Z < b \mid Z < b)$ is equal to

- A. $\frac{1 - A - B}{1 - B}$
- B. $\frac{A - B}{B}$
- C. $\frac{B - A}{1 - B}$
- D. $\frac{0.5 - A - B}{B - 1}$
- E. $\frac{1 - A - B}{B - 1}$

Question 17

Consider the function $f : [0, \infty) \rightarrow \mathbb{R}$, $f(x) = xe^{-2x}$.

Several students were investigating properties of this function.

Albert stated that the average rate of change of the function over $0 \leq x \leq \frac{1}{2}$ is $\frac{1}{e}$.

Ben stated that the average value of the function over $0 \leq x \leq \frac{1}{2}$ is $\frac{e-2}{2e}$.

Colin stated that the gradient of the function is zero when $x = \frac{1}{2}$. Then

- A. Only Albert is correct.
- B. Only Ben is correct.
- C. Only Colin is correct.
- D. Only Albert and Colin are correct.
- E. Albert, Ben and Colin are all correct.

Question 18

All suitcases entering an airport are screened. Ninety-eight percent of the suitcases that contain prohibited substances activate an alarm. Ten percent of the suitcases that do not contain prohibited substances also activate the alarm. If one out of every 1,000 suitcases actually contain prohibited substances, then the probability that a suitcase that activates an alarm actually contains prohibited substances is closest to

- A. 0.0097
- B. 0.0098
- C. 0.1010
- D. 0.0200
- E. 0.0001

Question 19

A and B are two events with $\Pr(A) = p$ and $\Pr(B) = 2p$, where $0 < p \leq \frac{1}{3}$.

Which of the following is **false**?

- A. If A and B are independent then $\Pr(A \cap B) = 2p^2$
- B. If A and B are independent then $\Pr(A' \cap B') = 2p^2 - 2p + 1$
- C. If A and B are mutually exclusive then $\Pr(A \cap B) = 0$
- D. If A and B are mutually exclusive then $\Pr(A' \cap B') = 1 - 3p$
- E. If A and B are mutually exclusive then $\Pr(A' \cup B') = 1$

Question 20

The manager of a hospital is investigating the waiting time of patients in an emergency clinic. It has been found that the proportion of patients who wait more than one hour is 0.4. The manager chooses a random sample of size n of patients from the clinic. The manager requires the width of a 99% confidence interval, for patients who wait more than one hour to be at most 0.3. The value of the smallest integer n , is

- A. 100.
- B. 72.
- C. 71.
- D. 41.
- E. 18.

END OF SECTION A

SECTION B

Instructions for Section B

Answer **all** questions in the spaces provided.

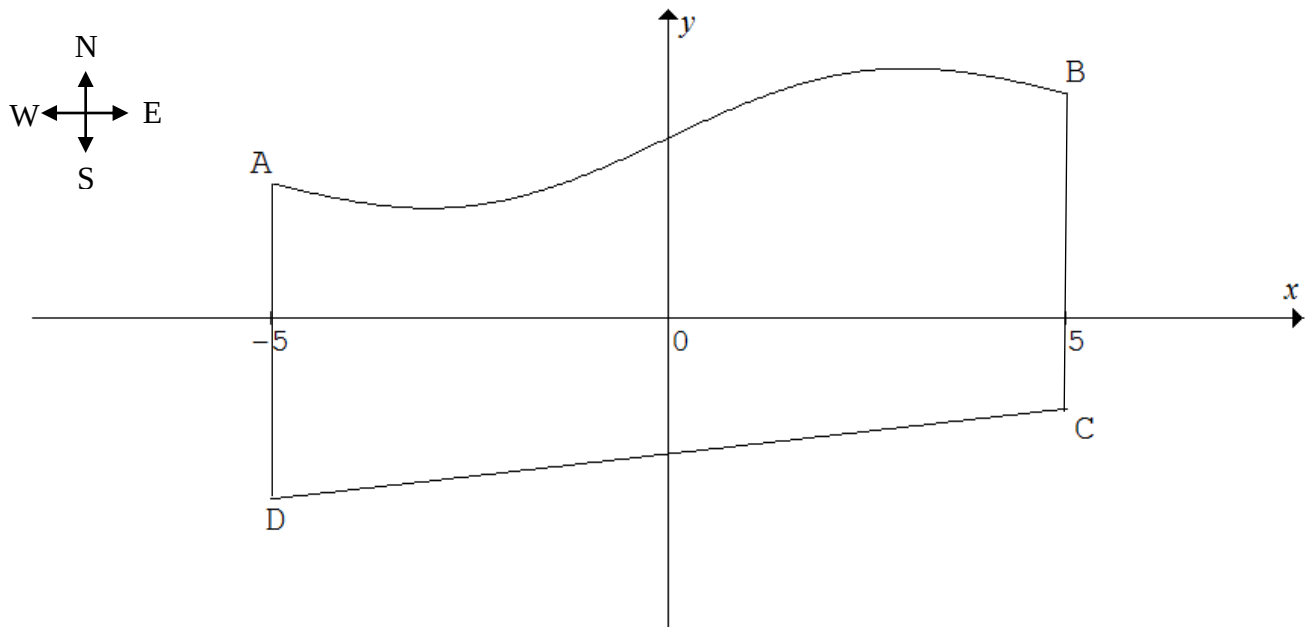
In all questions where a numerical answer is required an exact value must be given unless otherwise specified.

In questions where more than one mark is available, appropriate working **must** be shown.

Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Question 1 (10 marks)

The diagram shows a plain cross-sectional view of a swimming pool ABCD at a hotel resort. The x and y axes are shown, and are in the directions of east-west and north-south respectively. Dimensions are measured in metres. The length of the pool is 10 metres and the edges AD and BC are both 7 metres and run north-south.



The function connecting the points A and B is given by

$$f : [-5, 5] \rightarrow \mathbb{R}, \quad f(x) = 4 + \frac{x}{5} + \sin\left(\frac{\pi x}{5}\right)$$

- a. Write down the co-ordinates of the points A and B.

1 mark

- b. Find co-ordinates of the northern most point of the swimming pool, giving your answer correct to three decimal places.

1 mark

- c. The swimming pool has a straight line segment joining the points C and D.
Write down the function $g(x)$ which defines this straight line segment.

2 marks

- d. Find the maximum and minimum widths, measured north-south of the swimming pool.

3 marks

- e. Write down a definite integral which gives the total cross-sectional area of the swimming pool.

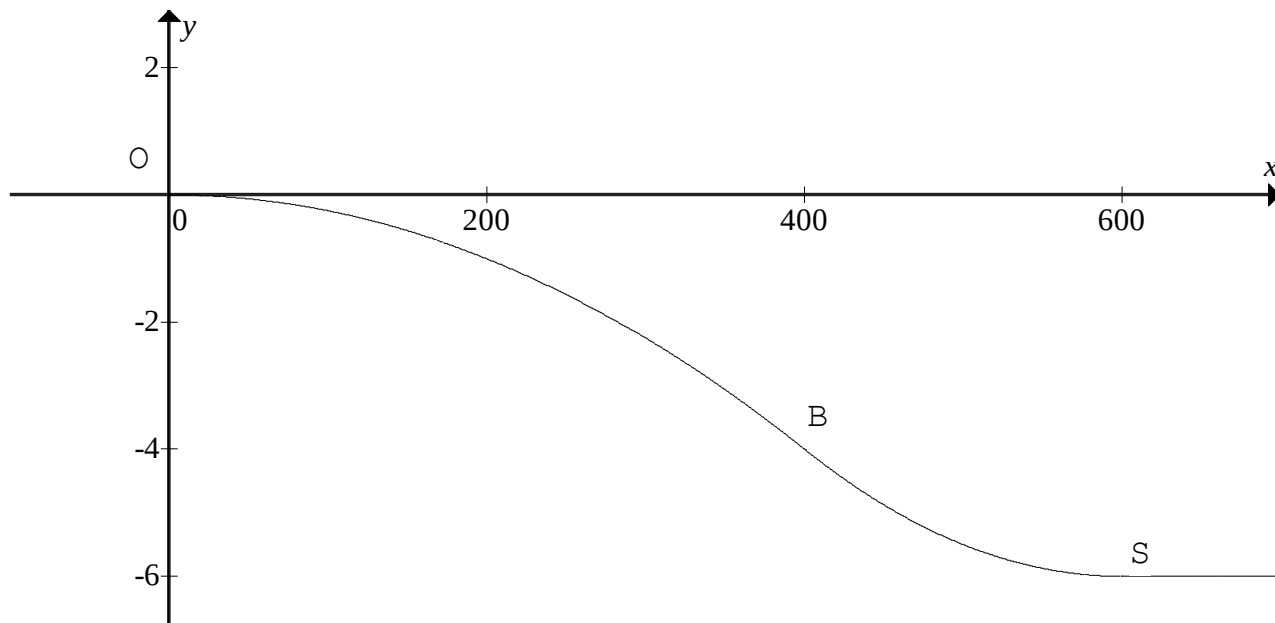
1 mark

- f. The swimming pool is filled with water to a constant depth of 1.5 metres.
Find the volume of water in the swimming pool in cubic metres.

2 marks

Question 2 (12 marks)

As part of the removal of railway level crossings, engineers are designing the track for a new railway station, to be built underground below an existing roadway. The diagram shows the track, together with the origin O at ground level.



The station is to be built at the point S , which is 600 metres horizontally forward and 6 metres below ground level. At O and S the track is horizontal. The track will consist of two sections one joining OB and another joining BS , where B is the point 400 metres horizontally forward and 4 metres below ground level. The two sections of the track must join smoothly at the point B . The rule for the function defining the section of the track BS is of the form

$$f(x) = a(x-h)^2 + c$$

- a.** State the values of h and c and show that $a = \frac{1}{20,000}$.

2 marks

- b.** Show that the gradient of the track at the point B is $-\frac{1}{50}$.

1 mark

- c.** Design 1 of the track OB is a function with the rule $g(x) = px^2 + qx + r$.
Write down linear simultaneous equations, involving p , q and r .

2 marks

- d.** Hence find the values of p , q and r .

1 mark

- e. Another design, design 2 of the track OB is a function with the rule $h(x) = mx^4 + nx^3$.
Write down two linear simultaneous equations, involving m and n .

2 marks

- f. Hence find the values of m and n .

1 mark

- g. The designers need to choose between either design 1 or 2. The decision depends on the one which has the smallest cross-sectional area to be excavated. Which design should they choose? Explain your findings.

3 marks

Question 3 (21 marks)

Car tyres come in many different brands. Even on a car, the tyres can wear out after travelling different distances, due to cornering, braking and wheel alignment. Manufacturers of car tyres are interested in the average distance in kilometres travelled by tyres, or the lifetime of tyres until they are considered un-usable and need to be replaced. At this time a tyre is considered un-roadworthy.

- a. For brand A of car tyres, records show that the lifetime is normally distributed with a mean of 45,000 km and a standard deviation of 8,000 km.

- i. Find the probability that a brand A tyre will last for at least 40,000 km.
Give your answer correct to four decimal places.

1 mark

- ii. For a random sample of 5 tyres of brand A, find the probability more than half will last longer than 40,000 km. Give your answer correct to four decimal places.

1 mark

- b. Brand B of car tyres, have a lifetime which is normally distributed. Records show that 11% of these tyres have a lifetime of more than 61,000 km while 20% become un-usable before travelling 42,500 km. Determine the mean and standard deviation of the lifetime of brand B, giving your answers to the nearest thousand km.

3 marks

- c. Consider another brand C of car tyres. From a random sample of 36 car tyres, it has been found that 26 of these car tyres last longer than 40,000 km. Find a 95% confidence interval corresponding to the sample proportion of brand C car tyres which last longer than 40,000 km. Give values correct to three decimal places.

1 mark

- d. Consider another brand D of car tyres. It has been found that the probability that these car tyres which last longer than 40,000 km is 0.7. For samples of 36 tyres of brand D, $\hat{P}$ is the random variable of the sample proportion of these tyres which last longer than 40,000 km.

- i. Find the expected value and variance of $\hat{P}$

2 marks

- ii. Find the probability that the sample proportion of these car tyres lies within two standard deviations of the mean. Give your answer correct to three decimal places. Do not use a normal approximation.

2 marks

- e. The lifetime T in **thousands** of km of brand E of car tyres, satisfies a probability density function given by

$$f(t) = \begin{cases} at^2 & 0 \leq t \leq 50 \\ b(100-t) & 50 \leq t \leq 100 \\ 0 & \text{elsewhere} \end{cases}$$

- i. Show that $a = \frac{3}{312,500}$ and $b = \frac{3}{6,250}$.

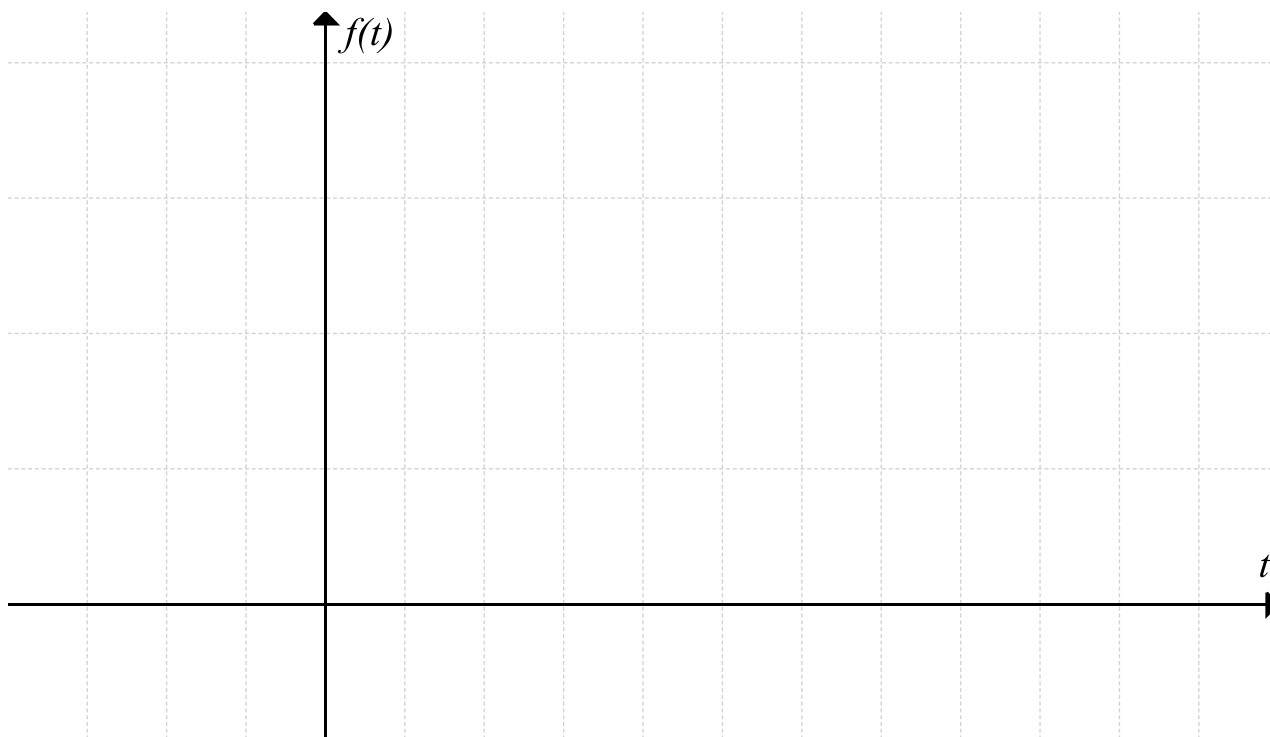
2 marks

- ii. For 36 tyres of brand E, find the expected number that will last longer than 40,000 km.

1 mark

iii. Sketch the graph of T on the axes below, clearly labelling the scale.

2 marks



iv. Find the expected value and variance of brand E car tyres.

2 marks

- v. Find the probability that the lifetime of brand E car tyres lies within two standard deviations of the mean. Give your answer correct to three decimal places.

2 marks

- vi. Find the median lifetime of brand E car tyres.
Give your answer correct to three decimal places.

2 marks

Question 4 (8 marks)

Let $f : \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = x(x+a)(x-a)$ where $a > 0$.

a.i. Find the equation of the tangent to the curve at the point $(k, f(k))$ where $k < 0$.

1 mark

ii. If the tangent crosses the x -axis at $x = a$, find k in terms of a .

1 mark

b. Find the values of x in terms of a , for which the function f is increasing.

1 mark

Consider the graph of $g(x) = f(x) + d$.

- c. Describe the transformation which maps the graph of $y = f(x)$ onto the graph of $y = g(x)$.

1 mark

- d. Find the values of d in terms of a , such that the graph of $y = g(x)$

- i. crosses the x -axis only once.

1 mark

- ii. crosses the x -axis three times.

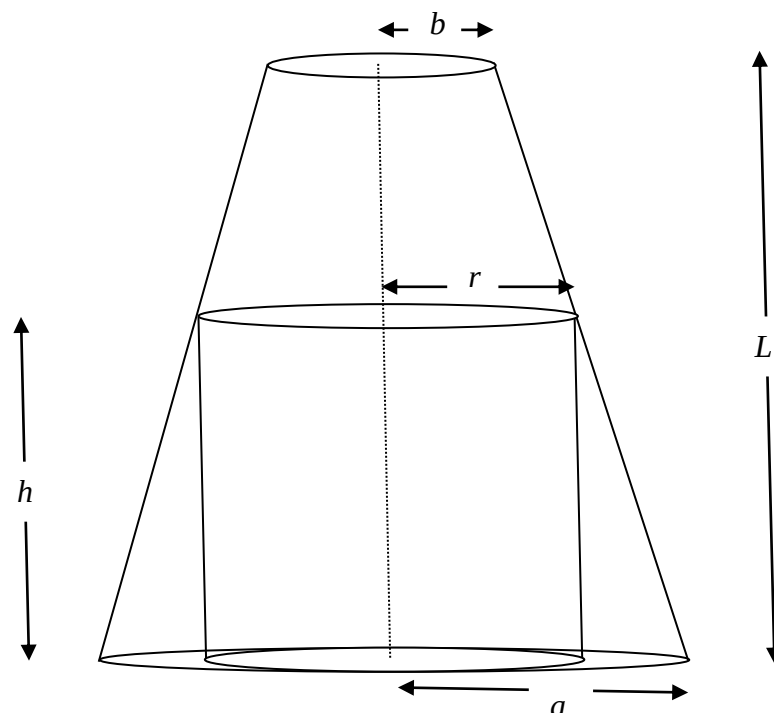
1 mark

- e. If the total area bounded by the graph of $y = f(x)$ is 128 square units, determine the value of a .

2 marks

Question 5 (9 marks)

A tree trunk is in the form of a truncated right circular cone. The radii of the end faces being a and b , where $a > b$ and the distance between these being L , where a , b and L are all positive constants. A log in the form of a right circular cylinder is cut from the trunk, the axis of the cylinder being perpendicular to the end faces. Let r and h be radius and height respectively of the log, as shown in the diagram below.



- a. Express h in terms of L , r , a and b .

3 marks

- b.** Show that the volume V of the log is given by $V(r) = \frac{\pi L}{a-b}(ar^2 - r^3)$

and find $\frac{dV}{dr}$.

2 marks

- c.** If the log is to have a maximum volume, express r in terms of a , stating conditions on a and b .

2 marks

- d.** Hence find in terms of a , b and L the volume of the largest log that can be cut from the cone.

2 marks

END OF EXAMINATION

MATHEMATICAL METHODS

Written examination 2

FORMULA SHEET

Directions to students

Detach this formula sheet during reading time.

This formula sheet is provided for your reference.

Mathematical Methods formulas

Mensuration

area of a trapezium	$\frac{1}{2}(a+b)h$	volume of a pyramid	$\frac{1}{3}Ah$
curved surface area of a cylinder	$2\pi rh$	volume of a sphere	$\frac{4}{3}\pi r^3$
volume of a cylinder	$\pi r^2 h$	area of triangle	$\frac{1}{2}bc \sin(A)$
volume of a cone	$\frac{1}{3}\pi r^2 h$		

Calculus

$\frac{d}{dx}(x^n) = nx^{n-1}$	$\int x^n dx = \frac{1}{n+1}x^{n+1} + c, n \neq -1$
$\frac{d}{dx}((ax+b)^n) = na(ax+b)^{n-1}$	$\int (ax+b)^n dx = \frac{1}{a(n+1)}(ax+b)^{n+1} + c, n \neq -1$
$\frac{d}{dx}(e^{ax}) = ae^{ax}$	$\int e^{ax} dx = \frac{1}{a}e^{ax} + c$
$\frac{d}{dx}(\log_e(x)) = \frac{1}{x}$	$\int \frac{1}{x} dx = \log_e(x) + c, x > 0$
$\frac{d}{dx}(\sin(ax)) = a \cos(ax)$	$\int \sin(ax) dx = -\frac{1}{a} \cos(ax) + c$
$\frac{d}{dx}(\cos(ax)) = -a \sin(ax)$	$\int \cos(ax) dx = \frac{1}{a} \sin(ax) + c$
$\frac{d}{dx}(\tan(ax)) = \frac{a}{\cos^2(ax)} = a \sec^2(ax)$	
product rule $\frac{d}{dx}(uv) = u \frac{dv}{dx} + v \frac{du}{dx}$	quotient rule $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$
chain rule $\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$	

Probability

$\Pr(A) = 1 - \Pr(A')$		$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$	
$\Pr(A B) = \frac{\Pr(A \cap B)}{\Pr(B)}$			
mean	$\mu = E(X)$	variance	$\text{var}(X) = \sigma^2 = E((X - \mu)^2) = E(X^2) - \mu^2$

Probability distribution		Mean	Variance
discrete	$\Pr(X = x) = p(x)$	$\mu = \sum x p(x)$	$\sigma^2 = \sum (x - \mu)^2 p(x)$
continuous	$\Pr(a < X < b) = \int_a^b f(x) dx$	$\mu = \int_{-\infty}^{\infty} x f(x) dx$	$\sigma^2 = \int_{-\infty}^{\infty} (x - \mu)^2 f(x) dx$

Sample proportions

$\hat{p} = \frac{X}{n}$		mean	$E(\hat{p}) = p$
standard deviation	$\text{sd}(\hat{p}) = \sqrt{\frac{p(1-p)}{n}}$	approximate confidence interval	$\left(\hat{p} - z \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}, \hat{p} + z \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \right)$

END OF FORMULA SHEET

ANSWER SHEET

STUDENT NUMBER

Figures
Words

Letter

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SIGNATURE _____

SECTION A

1	A	B	C	D	E
2	A	B	C	D	E
3	A	B	C	D	E
4	A	B	C	D	E
5	A	B	C	D	E
6	A	B	C	D	E
7	A	B	C	D	E
8	A	B	C	D	E
9	A	B	C	D	E
10	A	B	C	D	E
11	A	B	C	D	E
12	A	B	C	D	E
13	A	B	C	D	E
14	A	B	C	D	E
15	A	B	C	D	E
16	A	B	C	D	E
17	A	B	C	D	E
18	A	B	C	D	E
19	A	B	C	D	E
20	A	B	C	D	E