

2016 VCAA Mathematical Methods

Sample (v2 April) Exam 2 Solutions © 2016 itute.com

CAS should be used whenever possible to speed up the solution process.

SECTION A

1	2	3	4	5	6	7	8	9	10
A	D	C	E	A	A	C	E	E	C

11	12	13	14	15	16	17	18	19	20
B	B	A	B	D	E	D	B	D	E

Q1 A

Q2 D

Q3 $T = \frac{\pi}{2\pi} = \frac{1}{2}$ C

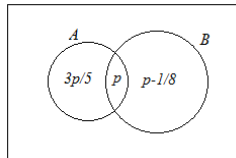
Q4 $P(-a) = 7(-a)^3 + 9(-a)^2 - 5a(-a) = 0$
 $-7a^3 + 9a^2 + 5a^2 = 0$, $a^2(-7a + 14) = 0$, $a = 2$ E

Q5 $-\frac{\pi}{2} \leq 2\left(x - \frac{\pi}{6}\right) \leq \frac{\pi}{2}$, $-\frac{\pi}{4} \leq x - \frac{\pi}{6} \leq \frac{\pi}{4}$, $-\frac{\pi}{12} \leq x \leq \frac{5\pi}{4}$
 $\therefore a = \frac{\pi}{12}$ A

Q6 $\int_1^4 (5 - 2f(x)) dx = \int_1^4 5 dx - 2 \int_1^4 f(x) dx = [5x]_1^4 - 2 \times 6 = 3$ A

Q7 $\Pr(A) = \frac{3p}{5} + p = \frac{8p}{5}$ C

$\Pr(B) = p + p - \frac{1}{8} = 2p - \frac{1}{8}$, $\therefore p \neq 0$
 $\Pr(A \cap B) = \Pr(A)\Pr(B)$, independent
 $\therefore p = \frac{8p}{5} \left(2p - \frac{1}{8}\right)$, $\therefore p = \frac{3}{8}$



Q8 $f(f(x)) = x$, $\therefore f^{-1}(x) = f(x)$ E

Q9 $\Pr(\text{different colours}) = \frac{{}^5C_1 \times {}^4C_1}{{}^9C_2} = \frac{5}{9}$ E

Q10 $x = 1 - x'$, $2y = 2 + y'$, $x - 2y = 3$
 $\therefore 1 - x' - (2 + y') = 3$, $\therefore -x' - y' = 4$ C

Q11 At $x = c$, $y = e^{ax} = e^{ac}$, $\frac{dy}{dx} = ae^{ax} = ae^{ac}$

Gradient of tangent $= \frac{e^{ac}}{c} = ae^{ac}$, $\therefore c = \frac{1}{a}$ B

Q12 $y = \frac{a}{3}x - \frac{5}{3}$ and $y = \frac{3}{a}x - \frac{8-a}{a}$

No solution: $\frac{a}{3} = \frac{3}{a}$ and $\frac{5}{3} \neq \frac{8-a}{a}$, $\therefore a = -3$ B



Q13 Binomial: $\Pr(X > 5) = 1 - \Pr(X \leq 5) \approx 0.0239$ A

Q14 $\Pr(-1.65 < Z < 1.65) \approx 0.90$ B

Q15 $f(x) = ax^3 - bx^2 + cx$, $f'(x) = 3ax^2 - 2bx + c$
 $f(x)$ has no stationary points when $3ax^2 - 2bx + c > 0$, i.e. when
its $\Delta < 0$, $(-2b)^2 - 4(3a)c < 0$, $3ac > b^2$, $c > \frac{b^2}{3a}$ D

Q16 $\int_0^a (x^2 - 4) dx = 0$, $\left[\frac{x^3}{3} - 4x\right]_0^a = 0$, $a = 2\sqrt{3}$ E

Q17 Consider $f(x) = x^3 - 9x^2 + 15x$.
 $f'(x) = 3x^2 - 18x + 15 = 3(x-1)(x-5)$
Local maximum (1, 7), local minimum (5, -25)
Translate $f(x) = x^3 - 9x^2 + 15x$ downwards more than 7 units, or
upwards more than 25 units. $\therefore w < -7$ or $w > 25$ D

Q18 Stationary points are at $x = -3, 2$.
 $f'(x)$ has a maximum, $\therefore y = f'(x)$ is an inverted parabola
 $\therefore y = f(x)$ has a local maximum at $x = 2$ B

Q19 Normal: $\Pr(T < 90) \approx \frac{150}{2000}$, $\Pr\left(Z < \frac{90-120}{\sigma}\right) \approx \frac{150}{2000}$
 $\frac{90-120}{\sigma} \approx -1.4395$, $\sigma \approx 20.8$ D

Q20 Let $g(x) = -|x|$ and $f(x) = x(x-1) = x^2 - x$.
 $f(g(x)) = x^2 + |x| = \begin{cases} x^2 + x & \text{for } x \geq 0 \\ x^2 - x & \text{for } x < 0 \end{cases}$ E

SECTION B

Q1a Period $= \frac{2\pi}{\frac{\pi}{3}} = 6$, amplitude $= 400$

Q1b
Max. pop. $= 1200 + 400 = 1600$, min. pop. $= 1200 - 400 = 800$

Q1c $n(10) = 1200 + 400 \cos \frac{10\pi}{3} = 1000$

Q1d $n(t) < n(10)$, when $2 < t < 4$ and $8 < t < 10$,
fraction of time $= \frac{4}{12} = \frac{1}{3}$



Q2a TSA = 6480, $2\left(hx + \frac{5hx}{2} + \frac{5x^2}{2}\right) = 6480$, $h = \frac{6480 - 5x^2}{7x}$

Q2b $V(x) = \frac{5x(6480 - 5x^2)}{14} > 0$, $\frac{25x(1296 - x^2)}{14} > 0$,
 $x(1296 - x^2) > 0$, $x(x - 36)(x + 36) > 0$ when $0 < x < 36$

Q2c $V(x) = \frac{16200x}{7} - \frac{25x^3}{14}$, $\frac{dV}{dx} = -\frac{75}{14}x^2 + \frac{16200}{7}$

Q2d Let $\frac{dV}{dx} = -\frac{75}{14}x^2 + \frac{16200}{7} = 0$, $x = \sqrt{432} = 12\sqrt{3}$ and
 $h = \frac{6480 - 5 \times 432}{7 \times 12\sqrt{3}} = \frac{360}{7\sqrt{3}} = \frac{120\sqrt{3}}{7}$ for maximum volume

Q3ai Binomial: $n = 20$, $p = \frac{5}{8}$, $\Pr(X \geq 10) \approx 0.9153$

Q3aii $\Pr(X \geq 15 | X \geq 10) = \frac{\Pr(X \geq 15)}{\Pr(X \geq 10)} \approx \frac{0.1788}{0.9153} \approx 0.195$

Q3aiii $E(\hat{p}) = p = \frac{5}{8}$, $\text{Var}(\hat{p}) = \frac{p(1-p)}{n} = \frac{\frac{5}{8} \times \frac{3}{8}}{20} = \frac{3}{256}$

Q3aiv $\sigma = \sqrt{\frac{3}{256}} = \frac{\sqrt{3}}{16}$

$\frac{5}{8} - 2 \times \frac{\sqrt{3}}{16} \approx 0.4085$, $\frac{5}{8} + 2 \times \frac{\sqrt{3}}{16} \approx 0.8415$
 $\Pr(0.4085 < \hat{p} < 0.8415) = \Pr(20 \times 0.4085 < X < 20 \times 0.8415)$
 $= \Pr(9 \leq X \leq 16) \approx 0.939$

Q3av $\Pr\left(\hat{p} \geq \frac{3}{4} | \hat{p} \geq \frac{5}{8}\right) = \Pr(X \geq 15 | X \geq 12.5) = \frac{\Pr(X \geq 15)}{\Pr(X \geq 13)}$
 $\approx \frac{0.1788}{0.5079} \approx 0.352$

Q3b $\Pr(FFF') + \Pr(FF'F) + \Pr(F'FF)$
 $= \frac{1}{2} \times \frac{3}{4} \times \frac{1}{4} + \frac{1}{2} \times \frac{1}{4} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} \times \frac{3}{4} = \frac{11}{32}$

Q3ci $E(W) = \int_1^3 \frac{w((w-3)^3 + 64)}{256} dw + \int_3^5 \frac{w(w+29)}{128} dw$
 $\approx 0.978125 + 2.0677083 \approx 3.0458$ min

Q3cii $\Pr(W > 4) = \int_4^5 \frac{w+29}{128} dw \approx 0.261719$

Expected number of members $\approx 200 \times 0.261719 \approx 52$

Q3d $\left(0.6 - 1.96\sqrt{\frac{0.6 \times 0.4}{100}}, 0.6 + 1.96\sqrt{\frac{0.6 \times 0.4}{100}}\right) \approx (0.504, 0.696)$

Q4ai $\int_{-2}^0 e^x dx = [e^x]_{-2}^0 = 1 - e^{-2}$

Q4aii $1 - e^{-2}$

Q4aiii Area of shaded region $= \int_{-2}^1 e^x dx = [e^x]_{-2}^1 = e - e^{-2}$

Q4bi Let $\log_e(x) = -\log_e(a-x)$, $\log_e(x) + \log_e(a-x) = 0$
 $\log_e(x)(a-x) = 0$, $x(a-x) = 1$, $x^2 - ax + 1 = 0$, $x = \frac{a \pm \sqrt{a^2 - 4}}{2}$

Q4bii $a^2 - 4 > 0$, given $a > 0 \therefore a > 2$

Q4c $x = \frac{a}{2} = \sqrt{2}$, $a = 2\sqrt{2}$

Q5a

$x^4 - 8x = x(x^3 - 2^3) = x(x-2)(x^2 + 2x + 4) = x(x-2)((x+1)^2 + 3)$

Q5b $g(x) = f(x+1)$, i.e. translate $y = f(x)$ in the negative x direction by 1 unit.

Q5ci $1 \leq d < 3$

Q5cii $d < 1$

Q5d $y = g(x) = x^4 - 8x$, $g'(x) = 4x^3 - 8 = 0$, $x = \sqrt[3]{2}$
 $y = (\sqrt[3]{2})^4 - 8(\sqrt[3]{2})$, $\therefore n = (\sqrt[3]{2})^4 - 8(\sqrt[3]{2}) = -6(\sqrt[3]{2})$

Q5ei $y = g(x) = x^4 - 8x$, $g'(x) = 4x^3 - 8$
 $g'(u) = 4u^3 - 8 = m$, $g'(v) = 4v^3 - 8 = -m$
 $\therefore 4u^3 - 8 + 4v^3 - 8 = 0$, $u^3 + v^3 = 4$

Q5eii $u^3 + v^3 = (u+v)^3 - 3uv(u+v) = 4$ and given $u+v=1$
 $\therefore 1 - 3uv = 4$, $uv = -1$
 $\therefore u = \frac{1 \pm \sqrt{5}}{2}$ and $v = \frac{1 \mp \sqrt{5}}{2}$

Q5fi $y = g(x) = x^4 - 8x$, $g'(x) = 4x^3 - 8$

At $x = p$, $y = g(p) = p^4 - 8p$, $g'(p) = 4p^3 - 8$

Equation of the tangent: $y - (p^4 - 8p) = (4p^3 - 8)(x - p)$

$y = (4p^3 - 8)x - 4p^4 + 8p + p^4 - 8p$

$\therefore y = (4p^3 - 8)x - 3p^4$

Q5fii $y = (4p^3 - 8)x - 3p^4$ passes through $\left(\frac{3}{2}, -12\right)$

$\therefore -12 = (4p^3 - 8)\frac{3}{2} - 3p^4$, $-12 = 6p^3 - 12 - 3p^4$, $6p^3 - 3p^4 = 0$

$\therefore 3p^3(2-p) = 0$, $\therefore p = 0$ or 2

Equations of the tangents are:

$y = -8x$ and $y = 24x - 48$

Please inform mathline@itute.com re conceptual, mathematical and/or typing errors