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YEAR 12 *Trial Exam Paper*

2016
MATHEMATICAL METHODS

Written examination 1

Worked solutions

This book presents:

- worked solutions, giving you a series of points to show you how to work through the questions
- mark allocations
- tips on how to approach the exam

Question 1a.

Worked solution

$$\frac{dy}{dx} = 3x^2 \cos(x) - x^3 \sin(x)$$

Mark allocation: 2 marks

- 1 method mark for recognising that the product rule is to be used
- 1 answer mark for the correct answer

Tip

- *Look out for the product rule. There is always either a product rule or quotient rule question of this standard on Exam 1.*

Question 1b.

Worked solution

$$f(x) = \left(2 - x^2\right)^{\frac{1}{2}}$$

$$f'(x) = \frac{1}{2}(2 - x^2)^{-\frac{1}{2}} \times -2x$$

$$= \frac{-x}{\sqrt{2 - x^2}}$$

$$f'(1) = -\frac{1}{1} = -1$$

Alternatively:

Let $y = f(x) = \left(2 - x^2\right)^{\frac{1}{2}}$

and let $u = 2 - x^2 \Rightarrow \frac{du}{dx} = -2x$

$$y = u^{\frac{1}{2}} \Rightarrow \frac{dy}{du} = \frac{1}{2}u^{-\frac{1}{2}} = \frac{1}{2\sqrt{u}}$$

$$\therefore\ f'(x) = \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{1}{2\sqrt{u}} \times -2x$$

$$= \frac{-x}{\sqrt{2 - x^2}}$$

$$f'(1) = \frac{-1}{1} = -1$$

Mark allocation: 2 marks

- 1 answer mark for finding the correct derivative $\frac{-x}{\sqrt{2 - x^2}}$
- 1 answer mark for the correct answer $f'(1) = -1$

Tip

- *Don't forget to finish the question—the question asks for* $f'(1)$. *Many students forget to evaluate.*

Question 2

Worked solution

$$\int_4^7 \frac{3}{3x-2}\,dx$$

$$= 3\int_4^7 \frac{1}{3x-2}\,dx$$

$$= \log_e(3x-2)\Big|_4^7$$

$$= \log_e(19) - \log_e(10)$$

$$= \log_e\left(\frac{19}{10}\right)$$

So, $k = \frac{19}{10}$.

Mark allocation: 2 marks

- 1 method mark for getting $\log_e(3x-2)$ as the integral
- 1 answer mark for $\frac{19}{10}$ or its equivalent

Question 3

Worked solution

$$\sin(2\pi x + \frac{\pi}{4}) = \frac{1}{\sqrt{2}}$$

Reference angle is $\frac{\pi}{4}$, so:

$$2\pi x + \frac{\pi}{4} = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{9\pi}{4}$$

$$2\pi x = 0, \frac{\pi}{2}, 2\pi$$

$$x = 0, \frac{1}{4}, 1$$

Mark allocation: 2 marks

- 1 method mark for stating reference angle is $\frac{\pi}{4}$
- 1 answer mark for $x = 0, \frac{1}{4}, 1$

Tip

- *You can check your answers by substituting the values back into the original equation.*

Question 4

Worked solution

$$9^x - 9 = 8(3^x)$$

$$3^{2x} - 8(3^x) - 9 = 0$$

Let $k = 3^x$, giving:

$$k^2 - 8k - 9 = 0$$

$$(k-9)(k+1) = 0$$

$$3^x = 9, \quad 3^x = -1$$

$$\therefore x = 2$$

Substituting this into either equation gives $y = 9^2 - 9 = 72$.
So $A = (2, 72)$.

Mark allocation: 3 marks

- 1 method mark for substituting $k = 3^x$
- 1 method mark for factorising the quadratic
- 1 answer mark for $A = (2, 72)$

Tip

- *Again, check your answer by substituting back into the equations.*

Question 5

Worked solution

$$\log_e(x) - \log_e(x+6) = 4$$

$$\log_e\left(\frac{x}{x+6}\right) = 4$$

$$e^4 = \frac{x}{x+6}$$

$$e^4(x+6) = x$$

$$x(e^4 - 1) = -6e^4$$

$$x = \frac{-6e^4}{e^4 - 1} = \frac{6e^4}{1 - e^4}$$

Mark allocation: 2 marks

- 1 method mark for using a logarithmic law correctly
- 1 answer mark for $x = \frac{-6e^4}{e^4 - 1}$ or $\frac{6e^4}{1 - e^4}$

Question 6a.

Worked solution

$$f(x) = \frac{1}{5}(x^4 - 4x^3)$$

$$f'(x) = \frac{1}{5}(4x^3 - 12x^2)$$

Let $f'(x) = 0$:

$$\Rightarrow 4x^3 - 12x^2 = 0$$

$$4x^2(x-3) = 0$$

$$x = 0, \;\; x = 3$$

$$y = 0, \;\; y = \frac{-27}{5}$$

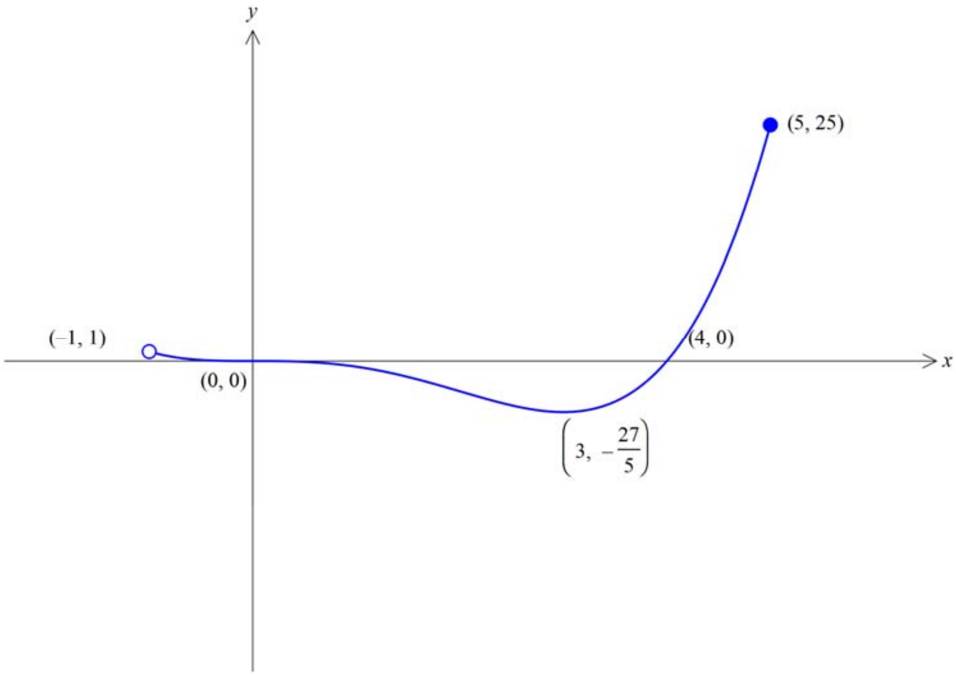
Hence, the coordinates of the stationary points are $(0, 0)$, $\left(3, \frac{-27}{5}\right)$.

Mark allocation: 2 marks

- 1 method mark for finding the derivative and setting it to zero
- 1 answer mark for giving both coordinates

Question 6b.

Worked solution



Mark allocation: 3 marks

- 1 method mark for the correct shape (i.e. a positive quartic)
- 1 answer mark for the correct end points
- 1 answer mark for correctly labelled intercepts and stationary points

Question 6c.

Worked solution

At $x = 1$, $f'(1) = \frac{1}{5}(4 - 12) = -\frac{8}{5}$

$$f(1) = -\frac{3}{5}$$

$$y - y_1 = m(x - x_1)$$

$$y + \frac{3}{5} = -\frac{8}{5}(x - 1)$$

$$y + \frac{3}{5} = -\frac{8x}{5} + \frac{8}{5}$$

$$y = -\frac{8x}{5} + 1$$

Mark allocation: 2 marks

- 1 method mark for finding the derivative at $x = 1$
- 1 answer mark for the correct equation of the tangent line

Question 7

Worked solution

$$f(x) = \int \left(3\sin(x) - \cos\left(\frac{x}{2}\right) \right) dx$$

$$= \left[-3\cos(x) - 2\sin\left(\frac{x}{2}\right) + c \right]$$

$$f\left(\frac{\pi}{3}\right) = -3\cos\left(\frac{\pi}{3}\right) - 2\sin\left(\frac{\pi}{6}\right) + c = 1$$

$$\Rightarrow \frac{-3}{2} - 1 + c = 1$$

$$c = \frac{7}{2}$$

$$f(x) = -3\cos(x) - 2\sin\left(\frac{x}{2}\right) + \frac{7}{2}$$

Mark allocation: 3 marks

- 1 method mark for attempting to antidifferentiate $f(x)$
- 1 method mark for attempting to find the value of c
- 1 answer mark for $f(x) = -3\cos(x) - 2\sin\left(\frac{x}{2}\right) + \frac{7}{2}$

Question 8a.

Worked solution

Let X be the number of red fish in the sample. Hence:

$X = 0, 1, 2, 3$

$\hat{p} = 0, \frac{1}{3}, \frac{2}{3}, 1$

Mark allocation: 1 mark

- 1 answer mark for $\hat{p} = 0, \frac{1}{3}, \frac{2}{3}, 1$

Tip

- *Remember that the distribution of the sample proportions is linked directly to the distribution of the number of red fish in the sample.*

Question 8b.

Worked solution

$$\begin{aligned}\Pr(\hat{p} > 0.25) &= \Pr(X \geq 1) \\ &= 1 - \Pr(X = 0) \\ &= 1 - \frac{8}{16} \times \frac{7}{15} \times \frac{6}{14} \\ &= 1 - \frac{1}{10} = \frac{9}{10}\end{aligned}$$

Mark allocation: 2 marks

- 1 method mark for finding $\Pr(X = 0)$
- 1 answer mark for $\frac{9}{10}$

Tip

- *It is always easier to find* $\Pr(X \geq 1)$ *by finding* $1 - \Pr(X = 0)$.

Question 8c.

Worked solution

$$\Pr\left(\hat{p}=\frac{1}{3}\mid\hat{p}>\frac{1}{4}\right)=\frac{\Pr(\hat{p}=\frac{1}{3})}{\hat{p}>\frac{1}{4}}$$

$$=\frac{\Pr(\hat{p}=\frac{1}{3})}{0.9}$$

$$\Pr(\hat{p}=\frac{1}{3})=\Pr(X=1)$$

$$=\frac{\binom{8}{1}\binom{8}{2}}{\binom{16}{3}}=\frac{8\times\frac{8}{2}\times\frac{7}{1}}{\frac{16}{3}\times\frac{15}{2}\times\frac{14}{1}}$$

$$=\frac{2}{5}=0.4$$

$$\Pr\left(\hat{p}=\frac{1}{3}\mid\hat{p}>\frac{1}{4}\right)=\frac{\Pr\left(\hat{p}=\frac{1}{3}\right)}{0.9}$$

$$=\frac{0.4}{0.9}=\frac{4}{9}$$

Mark allocation: 3 marks

- 1 method mark for determining $\Pr\left(\hat{p}=\frac{1}{3}\mid\hat{p}>\frac{1}{4}\right)=\frac{\Pr\left(\hat{p}=\frac{1}{3}\right)}{\hat{p}>\frac{1}{4}}$
- 1 method mark for finding $\Pr\left(\hat{p}=\frac{1}{3}\right)=\Pr(X=1)=0.4$
- 1 answer mark for $\frac{4}{9}$

Question 9a.

Worked solution

$a < 2$ as $f(x) > 0$ in order to be a probability density function.

$\int_0^a \frac{\pi}{2}\sin\left(\frac{\pi x}{2}\right) dx = 1$ since X represents a probability density function.

$$\int_0^a \frac{\pi}{2}\sin\left(\frac{\pi x}{2}\right) dx = \left[-\cos\left(\frac{\pi x}{2}\right)\right]_0^a$$

$$= -\cos\left(\frac{\pi a}{2}\right) - -\cos(0)$$

$$= 1 - \cos\left(\frac{\pi a}{2}\right)$$

So, $1 - \cos\left(\frac{\pi a}{2}\right) = 1$

$$\cos\left(\frac{\pi a}{2}\right) = 0$$

$$a = 1$$

Mark allocation: 2 marks

- 1 method mark for setting integral equal to one
- 1 answer mark for *a* = 1

Question 9b.

Worked solution

$$E(X)=\int_0^1 \frac{\pi x}{2}\sin\left(\frac{\pi x}{2}\right)\,dx$$

$$\frac{d}{dx}\left(x\cos\left(\frac{\pi x}{2}\right)\right)=\frac{-\pi x}{2}\sin\left(\frac{\pi x}{2}\right)+\cos\left(\frac{\pi x}{2}\right)$$

$$\int \frac{d}{dx}\left(x\cos\left(\frac{\pi x}{2}\right)\right)dx=\int\left(\frac{-\pi x}{2}\sin\left(\frac{\pi x}{2}\right)+\cos\left(\frac{\pi x}{2}\right)\right)dx$$

$$\int\left(\frac{\pi x}{2}\sin\left(\frac{\pi x}{2}\right)\right)dx=\int\cos\left(\frac{\pi x}{2}\right)\,dx\;-\left(x\cos\left(\frac{\pi x}{2}\right)\right)$$

$$=\frac{2}{\pi}\sin\left(\frac{\pi x}{2}\right)-\left(x\cos\left(\frac{\pi x}{2}\right)\right)$$

So, $$\int_0^1\left(\frac{\pi x}{2}\sin\left(\frac{\pi x}{2}\right)\right)dx=\left[\frac{2}{\pi}\sin\left(\frac{\pi x}{2}\right)-\left(x\cos\left(\frac{\pi x}{2}\right)\right)\right]_0^1$$

$$=\frac{2}{\pi}\sin\frac{\pi}{2}-\frac{2}{\pi}\sin(0)-\cos\left(\frac{\pi}{2}\right)-0$$

$$=\frac{2}{\pi}$$

Mark allocation: 3 marks

- 1 method mark for setting up $E(X)=\int_0^1 \frac{\pi x}{2}\sin\left(\frac{\pi x}{2}\right)dx$
- 1 method mark for getting $\int\left(\frac{\pi x}{2}\sin\left(\frac{\pi x}{2}\right)\right)dx=\frac{2}{\pi}\sin\left(\frac{\pi x}{2}\right)-\left(x\cos\left(\frac{\pi x}{2}\right)\right)$
- 1 answer mark for $\frac{2}{\pi}$

Question 10a.

Worked solution

$$\frac{dy}{dx} = \frac{1}{\sqrt{a+2x}}$$

At $x = 4$, $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{a+8}}$

At $x = 4$, $y = \sqrt{a+8}$ and at $x = 0$, $y = 0$.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\sqrt{a+8}}{4}$$

So, $\dfrac{\sqrt{a+8}}{4} = \dfrac{1}{\sqrt{a+8}}$

$$\Rightarrow \; a + 8 = 4$$

$$a = -4$$

Mark allocation: 3 marks

- 1 method mark for finding $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{a+8}}$
- 1 method mark for setting gradient equal to the derivative at $x = a$
- 1 answer mark for $a = -4$

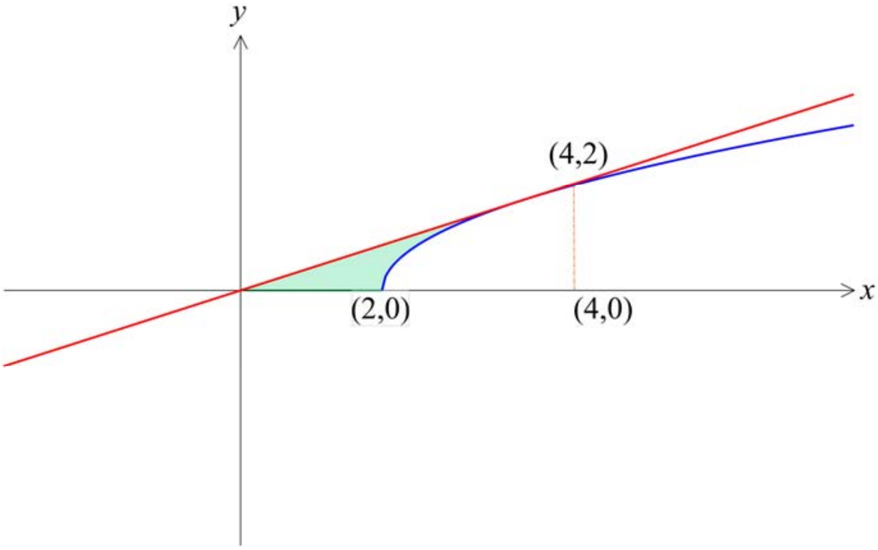
Tip

- *The solution can be checked by substituting* $a = -4$ *into the equation* $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{a+8}}$. *This gives* $\dfrac{dy}{dx} = \dfrac{1}{2}$, for $x = 4$, *so the equation of the tangent is* $y = \dfrac{x}{2}$. *The point on the curve* (4, 2) *lies on this tangent.*

Question 10b.

Worked solution

A quick sketch produces the following.



So the shaded area can be calculated by finding the area of triangle $-\int_{2}^{4}\sqrt{2x-4}\ dx$.

Shaded area:

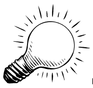
$$=\frac{1}{2}\times 4\times 2-\left[\frac{1}{3}(2x-4)^{\frac{3}{2}}\right]_{2}^{4}$$

$$=4-\frac{1}{3}(4)^{\frac{3}{2}}-\frac{1}{3}(0)$$

$$=4-\left(\frac{8}{3}\right)=\frac{4}{3}\text{ square units}$$

Mark allocation: 3 marks

- 1 method mark for determining the correct region
- 1 answer mark for finding $\int\sqrt{2x-4}\ dx=\frac{(2x-4)^{\frac{3}{2}}}{3}$
- 1 answer mark for $\frac{4}{3}$



Tip

- *Always draw a sketch to determine the region required. Use areas of simple shapes to help with calculating the area.*

END OF WORKED SOLUTIONS