

Trial Examination 2015

VCE Mathematical Methods (CAS) Units 3&4

Written Examination 2

Suggested Solutions

SECTION 1

1	A	B	C	D	E
2	A	B	C	D	E
3	A	B	C	D	E
4	A	B	C	D	E
5	A	B	C	D	E
6	A	B	C	D	E
7	A	B	C	D	E
8	A	B	C	D	E
9	A	B	C	D	E
10	A	B	C	D	E
11	A	B	C	D	E

12	A	B	C	D	E
13	A	B	C	D	E
14	A	B	C	D	E
15	A	B	C	D	E
16	A	B	C	D	E
17	A	B	C	D	E
18	A	B	C	D	E
19	A	B	C	D	E
20	A	B	C	D	E
21	A	B	C	D	E
22	A	B	C	D	E

SECTION 1**Question 1 B**

The square-root function is defined when the argument is zero or positive. Therefore:

- $\sqrt{x+2}$ is defined when $x+2 \geq 0$; that is, $x \geq -2$.
- $\sqrt{x+1}$ is defined when $x+1 \geq 0$; that is, $x \geq -1$.
- $\sqrt{x-1}$ is defined when $x-1 \geq 0$; that is, $x \geq 1$.

Hence for all three square roots to be defined, it is necessary that $x \geq 1$.

However, notice that $\sqrt{x-1}$ is in the denominator of $\frac{1}{\sqrt{x-1}}$, hence when $\sqrt{x-1} = 0$, $\frac{1}{\sqrt{x-1}}$ is undefined.

$\sqrt{x-1} = 0$ when $x = 1$, so we exclude when $x = 1$ from the domain and conclude that $\frac{\sqrt{x+2}}{\sqrt{x+1}} + \frac{1}{\sqrt{x-1}}$ is only defined when $x > 1$.

Therefore the required maximal domain is $(1, \infty)$.

Question 2 D

$$y'(x) = 3x^2$$

$$\begin{aligned} y'(1) &= 3 \times 1^2 \\ &= 3 \end{aligned}$$

Since $m_1 = 3$:

$$m_1 m_2 = -1$$

$$3m_2 = -1$$

$$m_2 = -\frac{1}{3}$$

The gradient of the normal is $-\frac{1}{3}$.

Given that a line through the point (x_1, y_1) with gradient m is $y - y_1 = m(x - x_1)$, substitute $(x_1, y_1) = (1, 2)$ and gradient $m = -\frac{1}{3}$.

Thus $y - 2 = -\frac{1}{3}(x - 1)$, which is equivalent to $y - 2 = \frac{1}{3}(1 - x)$.

Question 3 B

Use CAS to find the derivative of $y = x^2(x - a) + b \Rightarrow \frac{dy}{dx} = 3x^2 - 2ax$

Turning points occur when $3x^2 - 2ax = 0$.

$$\therefore x = 0 \text{ or } x = \frac{2a}{3}$$

Since (2, 10) is a turning point:

$$2 = \frac{2a}{3}$$

$$a = 3$$

Hence the equation becomes $y = x^2(x - 3) + b$.

Since the turning point (2, 10) lies on the graph:

$$10 = 2^2(2 - 3) + b$$

$$b = 14$$

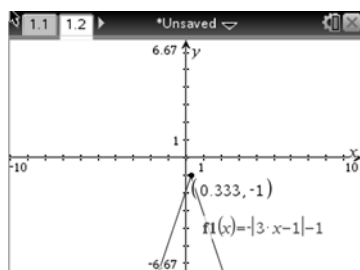
So $y = x^2(x - 3) + 14$.

Local maximum occurs when $x = 0$, so the value of y when $x = 0$ is given by $y = 0^2(0 - 3) + 14$.
Coordinates of the local maximum are thus (0, 14).

The slope of the line between the turning points is $\frac{10 - 14}{2 - 0} = -2$.

Question 4 D

Using CAS to sketch the graph:



It can be seen that the graph is continuous and has a vertex at $\left(\frac{1}{3}, -1\right)$, so **A** and **B** are true statements.

The graph is completely below the y -axis, therefore **C** is also true.

The gradient of the graph is positive to the left of $x = \frac{1}{3}$ and negative to the right of $x = \frac{1}{3}$, but is not defined at $x = 0$. Therefore **D** is false.

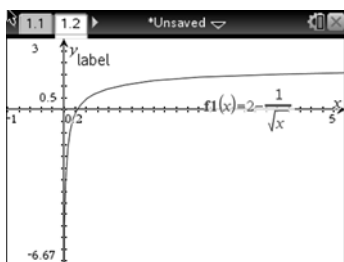
Question 5 C

To find the inverse, interchange x and y :

$\text{solve}\left(x=2-\frac{1}{\sqrt{y}}, y\right) \quad y=\frac{1}{(x-2)^2}$

The range of f will be the domain of f^{-1} .

From the graph, the range of f is $(-\infty, 2)$, so the domain of f^{-1} is $(-\infty, 2)$.

**Question 6 C**

$\text{solve}\left(4 \cdot \sin(x) \cdot \cos(x) - \sqrt{2} = 0, x\right) | 0 < x < 2 \cdot \pi$
 $x = \frac{\pi}{8} \text{ or } x = \frac{3 \cdot \pi}{8} \text{ or } x = \frac{9 \cdot \pi}{8} \text{ or } x = \frac{11 \cdot \pi}{8}$

We observe that the only solutions in the required interval are $\frac{\pi}{8}$, $\frac{3\pi}{8}$, $\frac{9\pi}{8}$ and $\frac{11\pi}{8}$.

Question 7 E

Using CAS:

Define $f(x)=\cos(x)$ Done
 Define $g(x)=e^{-x}$ Done
 Define $h(x)=g(f(x))$ Done
 $\frac{d}{dx}(h(x)) \quad -\sin(x) \cdot e^{-\cos(x)}$

Question 8 B

Using CAS:

$\frac{1}{\sin(3 \cdot x)} \quad \int_0^{\frac{\pi}{4}} \sin(3 \cdot x) \, dx \quad \frac{2 \cdot (\sqrt{2} + 2)}{3 \cdot \pi}$

Question 9 D

Using the rule for linear approximation: $f(x + h) \approx hf'(x) + f(x)$

$$f(x) = e^{-2x}$$

$$= \frac{1}{e^{2x}}$$

$$\frac{1}{e^{1.96}} = \frac{1}{e^{2 \times 0.98}}, \text{ so we require } f(0.98).$$

Therefore we use $x = 1$ and $h = -0.02$.

Applying these values to the rule, we get $f(1 - 0.02) \approx 0.02f'(1) + f(1)$.

Question 10 C

We require the graph of the antiderivative of h , so h is a gradient function. Since the graph of $h \geq 0$ for all values of x , the graph of the antiderivative must have a gradient ≥ 0 . Hence **C** is the only correct option.

Question 11 A

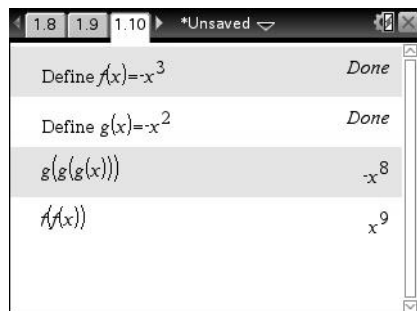
$$\Pr(X > 13.5) = \Pr\left(Z > \frac{x - \mu}{\sigma}\right)$$

$$= \Pr\left(Z > \frac{13.5 - 12}{0.75}\right)$$

$$= (Z > 2), \text{ which is equal to } \Pr(Z < -2) \text{ by symmetry}$$

Question 12 D

Using CAS:



These screenshots demonstrate that $f(f(x))$ does not equal $g(g(g(x)))$ in general, but only when $x = 0$ or 1 .

Question 13 A

Since $f(x)$ is an antiderivative of $f'(x) = 2^{3x+1}$, using CAS:

CAS interface showing the integral of $2^{3x+1} dx$ resulting in $\frac{2 \cdot 8^x}{3 \cdot \ln(2)}$.

We can use CAS to confirm that $\ln(8) = 3\ln(2)$.

CAS interface showing $\ln(8)$ and $3 \cdot \ln(2)$ are equal.

Question 14 C

By the chain rule, $h'(x) = f'(g(x))g'(x)$.

So if $g'(a) = 0$, then $h'(a) = 0$.

Hence if g has a stationary point at $x = a$, h will have a stationary point at $x = a$.

Question 15 D

The chain rule applies here: $\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$

Since the diameter equals the height of the cone, then $2r = h$, where h is the depth of the water and r is the radius of the water surface. This relation will still hold as the level of the water drops because of similar triangles.

$$\begin{aligned} \text{Hence } V &= \frac{1}{3}\pi r^2 h \\ &= \frac{1}{3}\pi \left(\frac{h}{2}\right)^2 h \\ &= \frac{1}{12}\pi h^3 \end{aligned}$$

Using CAS to find the derivative:

CAS interface showing the definition of $v = \frac{1}{12} \cdot \pi \cdot h^3$ and its derivative $\frac{d}{dh}(v) = \frac{h^2 \cdot \pi}{4}$.

Since water leaks out at a rate proportional to the depth, $\frac{dV}{dt} = kh$.

Substituting into the chain rule:

$$\begin{aligned} kh &= \frac{h^2 \pi}{4} \times \frac{dh}{dt} \\ \frac{dh}{dt} &= \frac{4k}{\pi h} \end{aligned}$$

Question 16 D

Using mapping coordinates:

$(x, y) \rightarrow (3x, y)$ under a dilation of factor 3 from y -axis, and $(x, y) \rightarrow (3x, -y)$ under a reflection on the x -axis

$x = 3x'$ gives $x' = \frac{x}{3}$, and $y = -y'$ gives $y' = y$.

The image equation becomes $-y = \cos\left(3 \times \frac{x}{3}\right) + 1$

$$y = -\cos(x) - 1$$

Since there is a dilation from the y -axis, the domain is affected: $\left[0, \frac{\pi}{3}\right] \rightarrow \left[0 \times 3, \frac{\pi}{3} \times 3\right]$

Hence the domain of the image function is $[0, \pi]$.

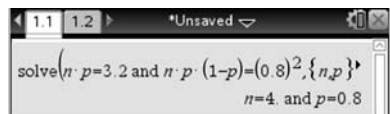
Question 17 B

$$\begin{aligned} \int_1^4 3(f(x) - 1)dx &= \int_1^4 3(f(x) - 3)dx \\ &= \int_1^4 3f(x)dx - \int_1^4 3dx \\ &= \int_1^4 3f(x)dx - [3x]_1^4 \\ &= 3 \times 6 - (12 - 3) \\ &= 9 \end{aligned}$$

Question 18 B

Since distribution is binomial, $\mu = np$ and $\sigma^2 = np(p - 1)$.

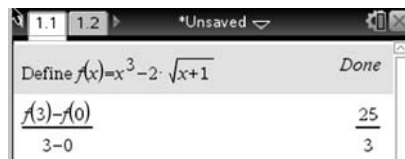
Solving simultaneously gives:



solve($n \cdot p = 3.2$ and $n \cdot p \cdot (1 - p) = (0.8)^2$, $\{n, p\}$)
 $n = 4$ and $p = 0.8$

Question 19 D

The average rate of change over this interval is given by $\frac{f(3) - f(0)}{3 - 0}$.



Define $f(x) = x^3 - 2 \cdot \sqrt{x+1}$ Done
 $\frac{f(3) - f(0)}{3 - 0} = \frac{25}{3}$

Question 20 E

Since $\Pr(A) = a$, $\Pr(A') = 1 - a$. Similarly, $\Pr(B') = 1 - b$.

Since A and B are independent, $\Pr(A' \cap B') = \Pr(A') \times \Pr(B')$.

$$\begin{aligned}\text{Using the addition rule: } \Pr(A' \cup B') &= \Pr(A') + \Pr(B') - \Pr(A' \cap B') \\ &= (1 - a) + (1 - b) - (1 - a)(1 - b) \\ &= 2 - a - b - (1 - a - b + ab) \\ &= 1 - ab\end{aligned}$$

Question 21 B

We observe that the system of equations are non-linear, since $y - 4x = x^2$ is equivalent to $y = x^2 + 4x$, and so y is a quadratic function of x .

The graph of the equation $2y - 2ax = -b$ will be from a family of straight lines. We know that in general, a line can intersect the parabola in 0, 1 or 2 places. It will touch in 1 place only if it is a tangent to the parabola or if it is a vertical line.

Making y the subject in both equations leads to $y = x^2 + 4x$ and $y = ax - \frac{b}{2}$.

Eliminating y from these equations gives:

$$x^2 + 4x = ax - \frac{b}{2}$$

$$x^2 + (4 - a)x + \frac{b}{2} = 0$$

The discriminant of this equation is $(4 - a)^2 - 4 \times \frac{b}{2} = (4 - a)^2 - 2b$.

The equation will have one solution when the discriminant is zero.

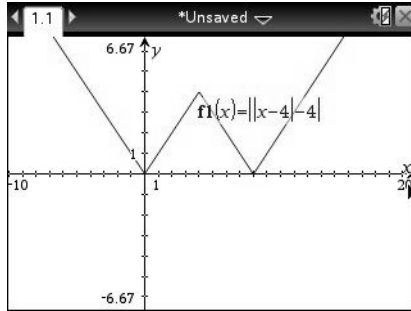
$$(4 - a)^2 - 2b = 0$$

$$(4 - a)^2 = 2b$$

$$a = 4 \pm \sqrt{2b}$$

Question 22 D

Use CAS to plot a graph of $y = ||x - 4| - 4|$.



Inspection of the graph reveals that it has a minimum value of 0 when $x = 0$ and $x = 8$. There is also a local maximum of 4 when $x = 4$.

We now consider the lines $y = k$ for various values of k and, in particular, consider how many intercepts it will have with the graph of $y = ||x - 4| - 4|$.

The solutions for x of the simultaneous equations $y = ||x - 4| - 4|$ and $y = k$ will be the same as the solutions of $||x - 4| - 4| = k$.

To sum up the cases:

$k < 0$	0 solutions
$k = 0$	2 solutions
$0 < k < 4$	4 solutions
$k = 4$	3 solutions
$k > 4$	2 solutions

SECTION 2

Question 1 (12 marks)

a. $\int_0^4 (4x - x^2) dx = \frac{32}{3} \text{ units}^2$

A1

A screenshot of a CAS calculator window titled "1.1" and "*Unsaved". The input field shows the integral $\int_0^4 (4x - x^2) dx$. The output field shows the result $\frac{32}{3}$.

b. Solving:

$$y = x$$

$$y = 4x - x^2$$

simultaneously gives the coordinates of the points of intersection.

A screenshot of a CAS calculator window titled "1.1" and "*Unsaved". The input field shows the command `solve({y=x, y=4-x^2}, {x,y})`. The output field shows the solutions $x=0 \text{ and } y=0 \text{ or } x=3 \text{ and } y=3$.

The required points are (0, 0) and (3, 3).

A1

c. $\text{area} = \int_0^3 (4x - x^2 - x) dx$

$$= 4.5 \text{ units}^2$$

M1

A1

A screenshot of a CAS calculator window titled "1.1" and "*Unsaved". The input field shows the integral $\int_0^3 (4x - x^2 - x) dx$. The output field shows the result $\frac{9}{2}$.

d. The required coordinates of P_m are the solutions of the following simultaneous equations:

$$y = mx$$

$$y = 4x - x^2$$

A screenshot of a CAS calculator window titled "1.1" and "*Unsaved". The input field shows the command `solve({y=m*x, y=4-x^2}, {x,y})`. The output field shows the solutions $x=(m-4) \text{ and } y=m(m-4) \text{ or } x=0 \text{ and } y=0$.

M1

As P_m is on the line $y = mx$ and is not located at the origin, it has coordinates $(4 - m, m(4 - m))$.

A1

e. $\int_0^{4-m} (4x - x^2 - mx) dx = \frac{(4-m)^3}{6} \text{ units}^2$

M1 A1

A screenshot of a CAS calculator window titled '1.1' and '*Unsaved'. It shows the integral $\int_0^{4-m} (4x - x^2 - mx) dx$ and the result $\frac{-(m-4)^3}{6}$.

f. area of $C_m = \text{area of } A - \text{area of } B_m$

M1

$$= \frac{32}{3} - \frac{(4-m)^3}{6}$$

A1

g. area of $C_m = \text{area of } B_m$ when the area of B_m is half the area of region A.

$$\frac{(4-m)^3}{6} = \frac{1}{2} \times \frac{32}{3}$$

M1

$$m = -\sqrt[3]{32} \left(\text{also } 4 - 2^{\frac{5}{3}} \right)$$

A1

A screenshot of a CAS calculator window titled '1.1' and '*Unsaved'. It shows the equation $\text{solve}\left(\frac{(4-m)^3}{6} = \frac{1}{2} \cdot \frac{32}{3}, m\right)$ and the result $m = 4 - 2 \cdot 2^{\frac{2}{3}}$.

Question 2 (14 marks)

a. $h(t)$ is a continuous function, therefore at $t = 10$, $k = h(10)$.

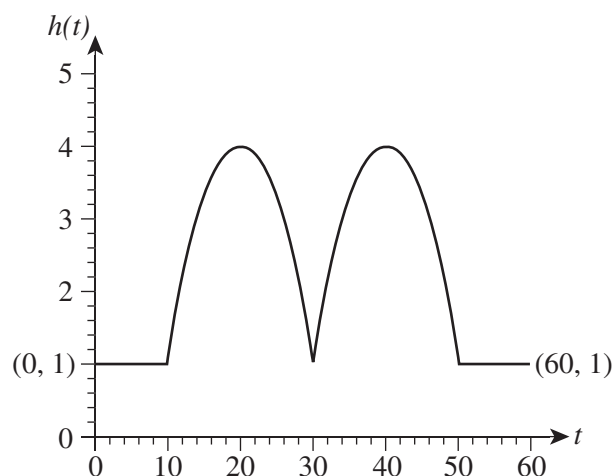
$$h(10) = 1$$

$$\therefore k = 1$$

A1

A screenshot of a CAS calculator window with tabs '1.2', '1.3', and '1.4'. It shows the definition $\text{Define } h(t) = \left| 3 \cdot \sin\left(\frac{\pi}{20} \cdot (t-10)\right) \right| + 1$ and the calculation $h(10) = 1$.

b.



correct shape/period of sine graph A1
 correct amplitude and effect of absolute value A1
 correct linear 'branches' shown A1

- c. From the graph (and due to the symmetry of the sine curve) we can determine that the maximum value occurs at $t = 20$ and $t = 40$. Since $t = 0$ is 6:00 am, the gate reaches its maximum height at 6:20 am and 6:40 am on Saturday.

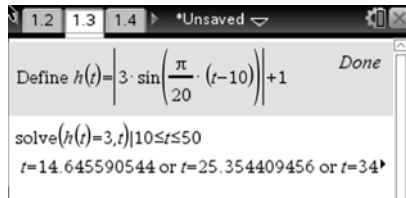
A1

- d. $h(t) = 3, t \in [0, 60]: t = 14.65, 25.35, 34.65$ and 45.35

M1

Corresponding to the hour after 9:00 am, the gate would first reach a height of 3 m at 14.65 minutes after 9:00 am. Since the delivery van arrives at 9:10 am, the driver will have to wait 4.65 minutes.

A1



- e. Each hour, the boom gate is at least 3 m high between 14.65 and 25.35 minutes past the hour, and again between 34.65 and 45.35 minutes past the hour.

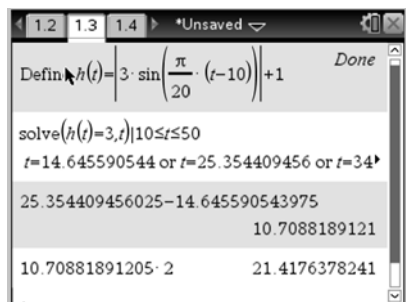
$$25.35 - 14.65 = 10.71$$

M1

Twice every hour the boom gate is over 3 m high.

$$\therefore 10.71 \times 2 = 21.42 \text{ minutes each hour}$$

A1



- f. The driver arrives at 9:10 am and cannot enter for 4.65 minutes.

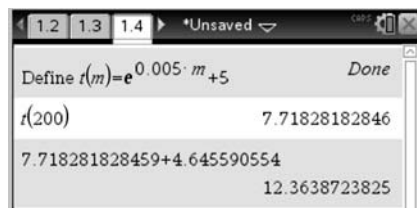
The time taken to deliver his 200 kg load = $t(200)$

$$= 7.72 \text{ minutes}$$

A1

The total time elapsed is 12.36 minutes.

A1



- g. The driver arrives at 9:28 am and cannot enter for 6.65 minutes.

The time taken to deliver his 500 kg load = $t(500)$

$$= 17.18 \text{ minutes}$$

A1

By this time the boom gate has dropped below a height of 3 m, so he cannot get out.

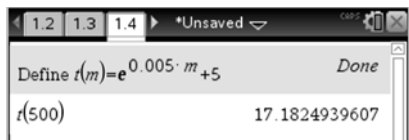
The boom gate is only above 3 m for 10.71 minutes at a time, which is not enough time for the driver to unload his delivery. He will have to wait until the boom gate next reaches a height of 3 m to exit, which will be at 14.65 minutes past the next hour.

Therefore the time he will exit = 14.65 minutes after 10:00 am

M1

$$= 10:15 \text{ am}$$

A1



Question 3 (16 marks)

- a. i. $\Pr(\text{paddles} | \text{paddles previous morning}) = 0.4$ (from question)

Therefore $\Pr(\text{paddles next 2 mornings}) = 0.4 \times 0.4$

$$= 0.16$$

A1

- ii. P = paddles, S = swims

$\Pr(\text{paddles once over next 3 mornings}) = \Pr(\text{PSS}) + \Pr(\text{SPS}) + \Pr(\text{SSP})$

M1

$$= 0.4 \times 0.6 \times 0.3 + 0.6 \times 0.7 \times 0.6 + 0.6 \times 0.3 \times 0.7$$

$$= 0.45$$

A1

- iii. The transition matrix is $T = \begin{bmatrix} 0.3 & 0.6 \\ 0.7 & 0.4 \end{bmatrix}$.

Since Timothy paddles on Monday morning, the initial state matrix is $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

$$\Pr(\text{swims on Friday}) = T^4 \times \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

M1

$$= 0.458$$

A1



b. Method 1:

$$S_{\infty} = T_{\infty} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

He trains for the paddle on 53.85% of mornings.

A1

Method 2:

$$\frac{0.7}{0.6 + 0.7} = 0.53846$$

$$= 53.85\%$$

A1

c. i. $\Pr(\text{paddles} < 30 \text{ minutes}) = \int_{20}^{30} \frac{1}{100} (x - 20) dx$

$$= \frac{1}{20}$$

M1

A1

ii. Let Y = number of days he paddles less than 30 minutes.

This represents a binomial distribution: $Y \sim \text{Bi}\left(10, \frac{1}{20}\right)$.

M1

$$\Pr(Y = 3) = 0.010$$

A1

- d. This is conditional probability: $\Pr(X > 60 | X > 30)$.

M1

$$\begin{aligned}\frac{\Pr(X > 60 \cap X > 30)}{\Pr(X > 30)} &= \frac{\Pr(X > 60)}{\Pr(X > 30)} \\ &= \frac{\Pr(X > 60)}{1 - \Pr(X < 30)} \\ &= \frac{\int_{60}^{120} f(x) dx}{\frac{19}{20}}\end{aligned}$$

M1

Must give either correct numerator or denominator for M1.

$$\Pr(\text{paddles} > 60 \text{ minutes on a given weekend}) = \frac{9}{19}$$

A1

A screenshot of a CAS calculator window titled '1.1' and '*Unsaved'. It shows the integral $\int_{60}^{120} \left(\frac{1}{4000} (120-x) \right) dx$ being evaluated, resulting in $\frac{9}{19}$.

- e. $\Pr(X < n) = 0.8$

M1

$$\int_{20}^{40} \frac{1}{1000} (x-20) dx + \int_{40}^n \frac{1}{4000} (120-x) dx = 0.8$$

M1

A screenshot of a CAS calculator window titled '1.1' and '*Unsaved'. It shows the equation $\int_{20}^{40} \left(\frac{1}{1000} (x-20) \right) dx + \int_{40}^n \left(\frac{1}{4000} (120-x) \right) dx = 0.8$ being solved for n , resulting in $n=80$ or $n=160$.

Since $n < 120$, $n = 80$.

A1

Question 4 (8 marks)

- a. R_2 is the distance of P2 from the origin (0, 0).

The coordinates of P2 are $\left(4 \cos\left(\frac{\pi t}{4}\right), 4 \sin\left(\frac{\pi t}{4}\right) \right)$, hence using the distance formula:

$$\begin{aligned}R_2 &= \sqrt{\left(4 \cos\left(\frac{\pi t}{4}\right) - 0 \right)^2 + \left(4 \sin\left(\frac{\pi t}{4}\right) - 0 \right)^2} \\ &= 4\end{aligned}$$

A1

To find the period of the orbit we need to find the period of the functions $4 \cos\left(\frac{\pi t}{4}\right)$ and $4 \sin\left(\frac{\pi t}{4}\right)$.

$$\begin{aligned}T_2 &= \frac{2\pi}{\frac{\pi}{4}} \\ &= 8\end{aligned}$$

A1

b. i. $R_2 = kT_2^q$ gives $4 = k8^q$.

$$R_1 = kT_1^q \text{ gives } 1 = k1^q.$$

A1

ii. $k = 1$ (since $1^q = 1$ for all values of q)

Therefore $4 = 8^q$

$$q = \frac{2}{3}$$

Hence the required law is $R = T^{\frac{2}{3}}$.

A1

c. $R_1 = a + b \times 2^1$

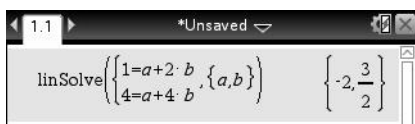
$$R_2 = a + b \times 2^2$$

M1

$$1 = a + 2b$$

$$4 = a + 4b$$

Solving simultaneously gives:



Thus $a = -2, b = \frac{3}{2}$.

A1

d. Substituting $n = 3$ into $R_n = -2 + \frac{3}{2} \times 2^n$ gives:

$$R_3 = -2 + \frac{3}{2} \times 2^3$$

$$= 10$$

A1

$$R = T^{\frac{2}{3}}$$

$$10 = (T_3)^{\frac{2}{3}}$$

$$T_3 = 10\sqrt[3]{10}$$

A1

Question 5 (8 marks)

- a. $f(-1) = 6$ and $f(2) = 24$.

Therefore coordinates of endpoints are $(-1, 6)$ and $(2, 24)$. A1

The stationary points are given by $f'(x) = 0$

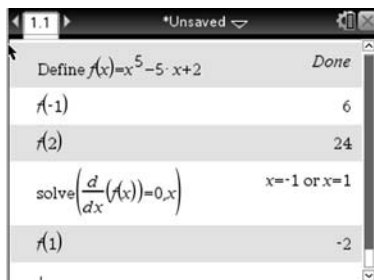
$$x = \pm 1$$

M1

Since there can be no stationary point at an endpoint, $x = -1$ is not a stationary point.

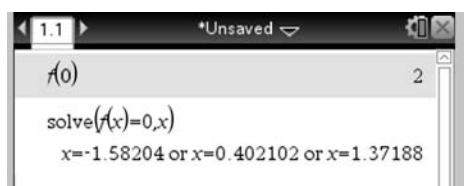
$f(1) = -2$, so there is a stationary point at $(1, -2)$. A1

No mark to be awarded if two stationary points are given.

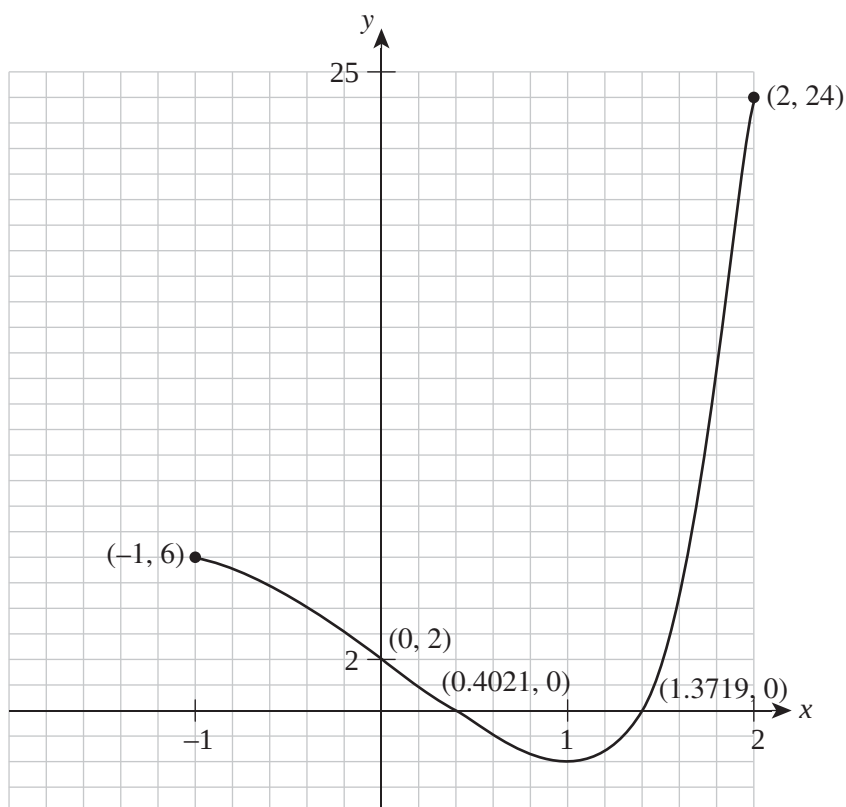


- b. y-intercept is given by $f(0)$: $(0, 2)$ A1

x-intercept is given by $f(x) = 0$: $(0.4021, 0)$ and $(1.3719, 0)$ A1



c.

*shape A1**all coordinates A1*d. range of f over given domain: $[-2, 24]$

A1