

The Mathematical Association of Victoria
MATHEMATICAL METHODS (CAS)
SOLUTIONS: Trial Exam 2015

Written Examination 1

Question 1

a. Let $y = xe^{2x}$

Using the Product Rule

$$\frac{dy}{dx} = (e^{2x} \times 1) + (x \times 2e^{2x}) \quad 1M$$

$$\frac{dy}{dx} = e^{2x} + 2xe^{2x} \quad 1A$$

b. Find $\int (xe^{2x}) dx$

From **part a.** we know that $\int (e^{2x} + 2xe^{2x}) dx = xe^{2x} + c \quad 1M$

$$\int (e^{2x}) dx + \int (2xe^{2x}) dx = xe^{2x} + c$$

$$\int (2xe^{2x}) dx = xe^{2x} - \int (e^{2x}) dx + c$$

$$\int (2xe^{2x}) dx = xe^{2x} - \frac{1}{2}e^{2x} + c_1$$

$$\int (xe^{2x}) dx = \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + c_1 \quad 1A$$

Question 2

$$(k-1)x + 2y = 1$$

$$x + (k-1)y = -k$$

$$\text{Let } A = \begin{bmatrix} k-1 & 2 \\ 1 & k-1 \end{bmatrix}$$

$$\det(A) = (k-1)^2 - 2 = 0 \quad 1M$$

$$(k-1)^2 - 2 = 0$$

$$(k-1)^2 = 2$$

$$k-1 = \pm\sqrt{2}$$

$$k = 1 + \sqrt{2} \text{ or } k = 1 - \sqrt{2}$$

For a unique solution

$$k \in \mathbb{R} \setminus \{1 \pm \sqrt{2}\} \quad 1A$$

OR

$$y = \frac{1 - (k-1)x}{2} \quad (1)$$

$$y = \frac{-k - x}{k-1} \quad (2)$$

If there is a unique solution, the lines must not be parallel hence have different gradients.
 Therefore, equate gradients to find the case where the lines are parallel.

$$\frac{-(k-1)}{2} = \frac{-1}{k-1} \quad \mathbf{1M}$$

$$-(k-1)^2 = -2$$

$$(k-1)^2 = 2$$

$$\therefore k = 1 + \sqrt{2} \text{ or } k = 1 - \sqrt{2}$$

For a unique solution

$$k \in \mathbb{R} \setminus \{1 \pm \sqrt{2}\} \quad \mathbf{1A}$$

Question 3

a. $f: \left[-\frac{1}{3}, \infty\right) \rightarrow \mathbb{R}, f(x) = \log_e(3x+2)$ and $g: [0, \infty) \rightarrow \mathbb{R}, g(x) = |x-1|$.

$$g(f(x)) = |\log_e(3x+2) - 1| \quad \mathbf{1M}$$

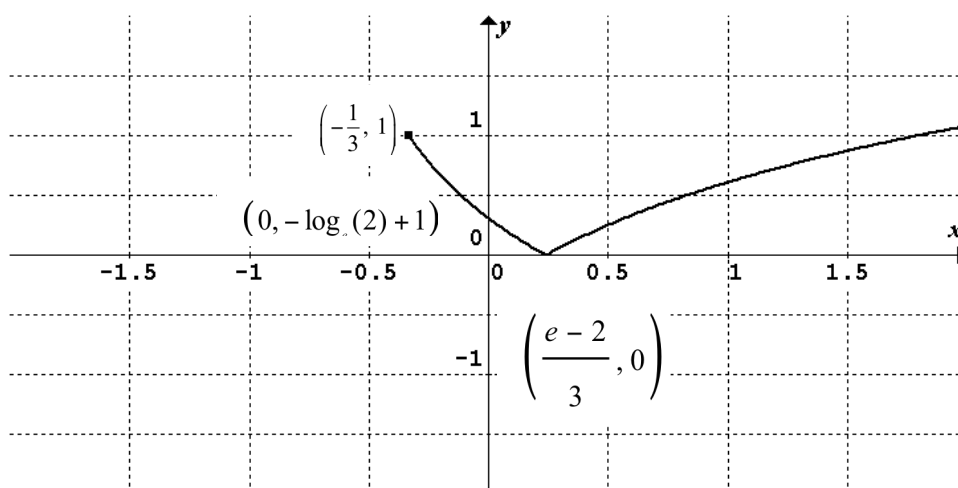
$$\text{Dom } g(f(x)) = \text{dom } f(x) = \left[-\frac{1}{3}, \infty\right) \quad \mathbf{1A}$$

b. $g(f(x)) = |\log_e(3x+2) - 1|$

Shape **1A**

Correct intercepts $\left(\frac{e-2}{3}, 0\right)$ and $(0, -\log_e(2)+1)$ **1A**

Correct endpoint $\left(-\frac{1}{3}, 1\right)$ **1A**



Question 4

a. $h: [0, 14] \rightarrow \mathbb{R}, h(t) = 2 \sin\left(\frac{\pi}{30}(t+1)\right) + 2$

$$h'(t) = \frac{\pi}{15} \cos\left(\frac{\pi}{30}(t+1)\right) \quad \mathbf{1A}$$

b. Given $\frac{dV}{dt} = 2 \text{ cm}^3/\text{s}$

$$\frac{dV}{dh} = \frac{dV}{dt} \times \frac{dt}{dh} \quad \mathbf{1M}$$

$$\frac{dV}{dh} = 2 \times \frac{15}{\pi \cos\left(\frac{\pi}{30}(t+1)\right)}$$

When $h = 3$,

$$2 \sin\left(\frac{\pi}{30}(t+1)\right) + 2 = 3$$

$$\sin\left(\frac{\pi}{30}(t+1)\right) = \frac{1}{2}$$

$$\frac{\pi}{30}(t+1) = \frac{\pi}{6}$$

$$t = 4 \quad \mathbf{1A}$$

$$\frac{dV}{dh} = 2 \times \frac{15}{\pi \cos\left(\frac{\pi}{30}(4+1)\right)}$$

$$\frac{dV}{dh} = \frac{30}{\pi \cos\left(\frac{\pi}{6}\right)}$$

$$\frac{dV}{dh} = \frac{20\sqrt{3}}{\pi} \text{ cm}^3/\text{cm} \quad \mathbf{1A}$$

Question 5

a. $f(x) = \frac{1}{2} \log_e(x(x+1)) \log_e(2x-1)$

Using the product and chain rules

$$f'(x) = \left[\log_e(2x-1) \times \frac{1}{2x(x+1)} \times (2x+1) \right] + \left[\frac{1}{2} \log_e(x(x+1)) \times \frac{2}{2x-1} \right]$$

$$\therefore f'(x) = \left[\frac{2x+1}{2x(x+1)} \log_e(2x-1) \right] + \left[\frac{1}{2x-1} \log_e(x(x+1)) \right] \quad \mathbf{1A}$$

b. $f'(2) = \left[\frac{5}{12} \log_e(3) \right] + \left[\frac{1}{3} \log_e(6) \right] \quad \mathbf{1M}$

in the form of $\log_e(a^m b^n)$

$$\therefore f'(2) = \left[\log_e \left(3^{\frac{5}{12}} \right) \right] + \left[\log_e \left(6^{\frac{1}{3}} \right) \right]$$

$$\therefore f'(2) = \log_e \left(3^{\frac{5}{12}} 6^{\frac{1}{3}} \right) \quad \mathbf{1A}$$

$$= \log_e \left(3^{\frac{5}{12}} 2^{\frac{2}{3}} \right) \quad \mathbf{1A}$$

Question 6

$$2\sqrt{3} \cos(2x) = -3$$

$$\cos(2x) = -\frac{3}{2\sqrt{3}} = -\frac{\sqrt{3}}{2}$$

$$2x = \frac{5\pi}{6}, \frac{7\pi}{6} \dots \quad \mathbf{1A}$$

$$x = \frac{5\pi}{12}, \frac{7\pi}{12} \dots$$

$$x = \frac{5\pi}{12} + k\pi, k \in \mathbb{Z} \quad \mathbf{1A}$$

$$x = \frac{7\pi}{12} + k\pi, k \in \mathbb{Z} \quad \mathbf{1A}$$

Question 7

a. $f(x+h) \approx hf'(x) + f(x)$

$$f(x) = 2(x-1)^{\frac{1}{3}}$$

Let $x = 28$, $f(28) = 6 \quad \mathbf{1A}$

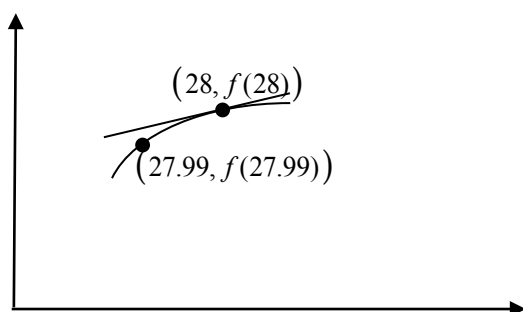
$$f'(x) = \frac{2}{3}(x-1)^{-\frac{2}{3}}, f'(28) = \frac{2}{27} \quad \mathbf{1A}$$

$$h = -0.01$$

$$f(27.99) \approx -\frac{1}{100} \times \frac{2}{27} + 6 = 5 \frac{1349}{1350} \quad \mathbf{1A}$$

b. It will be an overestimate. $\mathbf{1A}$

The tangent to the graph of f at $x = 28$ will be above the graph of f at $x = 27.99$. $\mathbf{1A}$



Question 8

$$f(x) = \begin{cases} -(x-1)(x-2) & 1 \leq x \leq 2 \\ \frac{1}{2}x - 1 & 2 < x \leq a \\ 0 & \text{elsewhere} \end{cases}$$

$$\int_1^2 (-(x-1)(x-2)) dx$$

$$= \int_1^2 (-x^2 + 3x - 2) dx$$

$$= \left[-\frac{x^3}{3} + \frac{3x^2}{2} - 2x \right]_1^2 \quad \mathbf{1M}$$

$$= -\frac{8}{3} + 6 - 4 + \frac{1}{3} - \frac{3}{2} + 2$$

$$= -\frac{7}{3} - \frac{3}{2} + 4$$

$$= -\frac{23}{6} + 4$$

$$= \frac{1}{6} \quad \mathbf{1A}$$

$$\text{Solve } \int_2^a \left(\frac{1}{2}x - 1 \right) dx = \frac{5}{6} \text{ for } a \quad \mathbf{1M}$$

$$\left[\frac{x^2}{4} - x \right]_2^a = \frac{5}{6}$$

$$\frac{a^2}{4} - a - 1 + 2 = \frac{5}{6} \quad \mathbf{1M}$$

$$3a^2 - 12a + 2 = 0$$

$$a = \frac{12 \pm \sqrt{120}}{6}$$

$$a = \frac{6 + \sqrt{30}}{3}, a > 2 \quad \mathbf{1A}$$

Note: some students might solve for a using the following equation

$$\int_1^2 (-(x-1)(x-2)) dx + \int_2^a \left(\frac{1}{2}x - 1 \right) dx = 1$$

Question 9

$$\text{a. } \left(\frac{1}{5}\right)^3 = \frac{1}{125} \quad \mathbf{1A}$$

$$\begin{aligned} \text{b. } {}^5C_2 \left(\frac{1}{5}\right)^2 \left(\frac{4}{5}\right)^3 & \quad \mathbf{1M} \\ = 10 \times \frac{1}{25} \times \frac{64}{125} & \\ = \frac{128}{625} & \quad \mathbf{1A} \end{aligned}$$

$$\begin{aligned} \text{c. } X &\sim \text{Bi}\left(22, \frac{1}{5}\right) \\ \mu = np &= \frac{22}{5}, \sigma = \sqrt{npq} = \sqrt{22 \times \frac{1}{5} \times \frac{4}{5}} = \frac{2\sqrt{22}}{5} \quad \mathbf{1M} \\ (\mu - 2\sigma, \mu + 2\sigma) & \\ = \left(\frac{22}{5} - \frac{4\sqrt{22}}{5}, \frac{22}{5} + \frac{4\sqrt{22}}{5}\right) & \quad \mathbf{1A} \end{aligned}$$

We are approximately 95% certain that Max will get between and including 1 and 8 correct.
The data is skewed. So it is only an approximation. $\mathbf{1A}$

$$\text{d. } 1 - \Pr(X = 0) > \frac{369}{625} \quad \mathbf{1M}$$

$$\begin{aligned} {}^nC_0 \left(\frac{1}{5}\right)^0 \left(\frac{4}{5}\right)^n &< \frac{256}{625} \\ \left(\frac{4}{5}\right)^n &< \left(\frac{4}{5}\right)^4 & \mathbf{1M} \\ n &> 4 \\ n &= 5 & \mathbf{1A} \end{aligned}$$