

2015 VCAA Math Methods CAS Exam 2 Solutions

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CAS should be used whenever possible to speed up the solution process.

SECTION 1

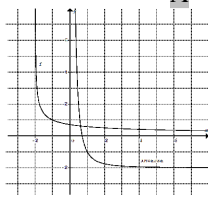
1	2	3	4	5	6	7	8	9	10	11
A	A	C	B	E	C	C	D	B	D	A

12	13	14	15	16	17	18	19	20	21	22
E	E	D	B	D	D	A	C	B	D	B

Q1 Period = $\frac{2\pi}{3}$, the range is $[-2-3, 2-3]$, i.e. $[-5, -1]$ **A**

Q2 **A**

$$x = \frac{1}{\sqrt{y+2}}, y = \frac{1}{x^2} - 2, \therefore f^{-1}(x) = \frac{1}{x^2} - 2$$



Q3 **C**

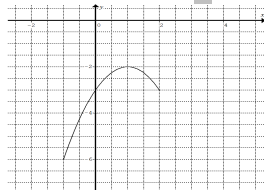
$$Q4 \quad y = x^2, \quad m_t = \frac{dy}{dx} = 2x. \text{ At } (2, 4), \quad m_t = 2(2) = 4.$$

$$\text{Consider } (3, 8) \text{ and } (2, 4), \quad m = \frac{8-4}{3-2} = 4 = m_t$$

Q5 Reflection in the line $y = x$ **E**

$$Q6 \quad P(3) = 3^3 - a \cdot 3^2 - 4(3) + 4 = 10, \therefore a = 1$$

Q7 **C**



$$Q8 \quad \text{Area under the straight line} = \frac{1}{2} p^2 = \frac{25}{8}, \therefore p = \frac{5}{2}$$

$$Q9 \quad \frac{1}{6}(a-2) = 1, \therefore a = 8, \quad E(X) = \int_2^8 \frac{x}{6} dx = \left[\frac{x^2}{12} \right]_2^8 = 5$$

$$Q10 \quad np = 2, \quad npq = \frac{4}{3}, \therefore n = 6, \quad p = \frac{1}{3} \text{ and } q = \frac{2}{3}$$

$$\Pr(X=1) = {}^6C_1 \left(\frac{1}{3}\right)^1 \left(\frac{2}{3}\right)^5$$

$$Q11 \quad \sqrt{x^3+1} = \sqrt{8\left(\frac{x}{2}\right)^3+1}$$

$$Q12 \quad \Pr(\text{at least one}) = 1 - \Pr(\text{none}) = 1 - \frac{1}{{}^8C_3} = \frac{55}{56}$$

$$Q13 \quad \int_0^1 ae^x dx + ae \cdot 1 = 1, \quad [ae^x]_0^1 + ae = 1, \quad 2ae - a = 1, \quad a = \frac{1}{2e-1}$$

$$Q14 \quad p + 2p + 3p + 4p + 5p = 1, \therefore p = \frac{1}{15}$$

$$\text{Mean of } X = 1 \times \frac{1}{15} + 2 \times \frac{2}{15} + 3 \times \frac{3}{15} + 4 \times \frac{4}{15} + 5 \times \frac{5}{15} = \frac{11}{3}$$

$$Q15 \quad \int_0^5 (2g(x) + ax) dx = \int_0^5 2g(x) dx + \int_0^5 ax dx$$

$$= 2 \int_0^5 g(x) dx + \left[\frac{ax^2}{2} \right]_0^5 = 2 \times 20 + \frac{25a}{2} = 90, \therefore a = 4$$

$$Q16 \quad f'(x) = \int g(x) dx, \quad \max^{m-1} = \frac{bx^{n+1}}{n+1}$$

$$\therefore \frac{b}{a} = m(n+1)x^{m-n-2} \quad \therefore m-n-2=0 \quad \therefore \frac{b}{a} = m(n+1) \text{ is an integer}$$

since m and n are positive integers. **D**

Q17 By CAS $y = x^3 - 3x^2$ has a local max at $(0, 0)$ and a local min at $(2, -4)$. To have three distinct x -intercepts, $y = x^3 - 3x^2$ must translate upwards by less than 4 units. Hence $c \in (0, 4)$. **D**

$$Q18 \quad f(x) = x^2 \text{ satisfies } |f(x+y) - f(x-y)| = 4\sqrt{f(x)f(y)}.$$

$$\text{L.H.S.} = |(x+y)^2 - (x-y)^2| = |4xy| = 4|xy|$$

$$\text{R.H.S.} = 4\sqrt{x^2y^2} = 4\sqrt{(xy)^2} = 4|xy| = \text{L.H.S.} \quad \text{A}$$

$$Q19 \quad \text{Let } F(t) = \int \left(\sqrt{t^2 + 4} \right) dt$$

$$\therefore f(x) = \int_0^x \left(\sqrt{t^2 + 4} \right) dt = [F(t)]_0^x = F(x) - F(0)$$

$$\therefore f'(x) = F'(x) = \sqrt{x^2 + 4}, \quad f'(-2) = \sqrt{(-2)^2 + 4} = 2\sqrt{2}$$

$$Q20 \quad f(x-1) = x^2 - 2x + 3$$

$$\text{Replace } x \text{ with } x+1: f(x+1-1) = (x+1)^2 - 2(x+1) + 3$$

$$\therefore f(x) = x^2 + 2 \quad \text{B}$$

$$Q21 \quad \text{Let } ax^2 = mx + c, \quad ax^2 - mx - c = 0$$

$$\text{No intersections: } \Delta < 0, \quad (-m)^2 - 4a(-c) < 0, \quad m^2 + 4ac < 0$$

$$\text{If } a > 0, \quad c < -\frac{m^2}{4a}; \text{ if } a < 0, \quad c > -\frac{m^2}{4a} \quad \text{D}$$

$$Q22 \quad \text{Let } f(x) = -a|x| \text{ where } a \in \mathbb{R}^+.$$

$$\therefore g(-f(x)) = g(a|x|) = g(|a|-|x|) \geq 0 \text{ and } g(-f(x)) \text{ is even.} \quad \text{B}$$

SECTION 2

Q1a $f(x) = \frac{1}{5}(x-2)^2(5-x)$

$f'(x) = \frac{2}{5}(x-2)(5-x) - \frac{1}{5}(x-2)^2 = \frac{3}{5}(x-2)(4-x)$

Q1bi At $P\left(1, \frac{4}{5}\right)$, $m = f'(1) = -\frac{9}{5}$

$y - \frac{4}{5} = -\frac{9}{5}(x-1)$, $9x + 5y = 13$ or $y = -\frac{9}{5}x + \frac{13}{5}$

Q1bii $9x + 5y = 13$ When $x = 0$, $y = \frac{13}{5}$, $\therefore S\left(0, \frac{13}{5}\right)$

When $y = 0$, $x = \frac{13}{9}$, $\therefore Q\left(\frac{13}{9}, 0\right)$

Q1c Distance $PS = \sqrt{(0-1)^2 + \left(\frac{13}{5} - \frac{4}{5}\right)^2} = \frac{\sqrt{106}}{5}$

Q1d Use CAS to solve simultaneous equations

$y = \frac{1}{5}(x-2)^2(5-x)$ and $9x + 5y = 13$ to find the second point of intersection at $x = 7$.

Area of the shaded region $= \int_1^7 \left(\frac{1}{5}(x-2)^2(5-x) - \left(-\frac{9}{5}x + \frac{13}{5} \right) \right) dx$
 $= 21.6$ square units by CAS

Q2a $y = 60 - \frac{3}{80}x^2$, $\frac{dy}{dx} = -\frac{3}{40}x$

At $A(-40, 0)$, $m = -\frac{3}{40}(-40) = 3$, $\tan \theta = 3$, $\theta \approx 72^\circ$

Q2b $y = \frac{x^3}{25600} - \frac{3x}{16} + 35$, $\frac{dy}{dx} = \frac{3x^2}{25600} - \frac{3}{16}$

The turning point of $\frac{dy}{dx}$ is $\left(0, -\frac{3}{16}\right)$.

$\therefore$ the max downward slope is $-\frac{3}{16}$.

Q2c Vertical distance $D = 60 - \frac{3}{80}x^2 - \left(\frac{x^3}{25600} - \frac{3x}{16} + 35 \right)$

Let $\frac{dD}{dx} = 0$, $-\frac{3}{40}x - \frac{3x^2}{25600} + \frac{3}{16} = 0$

$\therefore u = x \approx 2.49$ m (2.49031)

and $v = \frac{2.49031^3}{25600} - \frac{3 \times 2.49031}{16} + 35 \approx 34.53$ m

Q2d $P(-2.49031, w)$,

$w = \frac{(-2.49031)^3}{25600} - \frac{3(-2.49031)}{16} + 35 \approx 35.47$ m

$D_{MN} = 60 - \frac{3}{80}(2.49031)^2 - \left(\frac{2.49031^3}{25600} - \frac{3 \times 2.49031}{16} + 35 \right) \approx 25.23$ m

$D_{PQ} = 60 - \frac{3}{80}(-2.49031)^2 - \left(\frac{(-2.49031)^3}{25600} - \frac{3(-2.49031)}{16} + 35 \right) \approx 24.30$ m

Q2e Let $D = 60 - \frac{3}{80}x^2 - \left(\frac{x^3}{25600} - \frac{3x}{16} + 35 \right) = 0$

By CAS, $x_E = -23.71$ and $x_F = 28.00$

Q2f Area of the shaded region

$= \int_{-23.71}^{28.00} \left(60 - \frac{3}{80}x^2 - \left(\frac{x^3}{25600} - \frac{3x}{16} + 35 \right) \right) dx \approx 870$ m² by CAS

Q3ai $\Pr(X > 7) = \int_7^{8.3} (x-6)^2(8-x)dx = 0.6875 = \frac{11}{16}$ by CAS

Q3aii Binomial distribution: $n = 3$, $p = \frac{11}{16}$

$\Pr(X = 1) = {}^3C_1 \left(\frac{11}{16} \right)^1 \left(\frac{5}{16} \right)^2 = \frac{825}{4096}$

Q3b Mean $= \int_6^{8.3} \frac{3}{4}(x-6)^2(8-x)x dx = 7.2$ cm

Q3c Normal distribution: $\mu = 74$, $\sigma = 9$

$\Pr(X < 85 | X > 74) = \frac{\Pr(74 < X < 85)}{\Pr(X > 74)} \approx 0.778$

Q3di Binomial distribution: $n = 3$, $p = 0.03$, $q = 0.97$

$\Pr(X \geq 1) = 1 - \Pr(X = 0) = 1 - 0.97^4 \approx 0.1147$

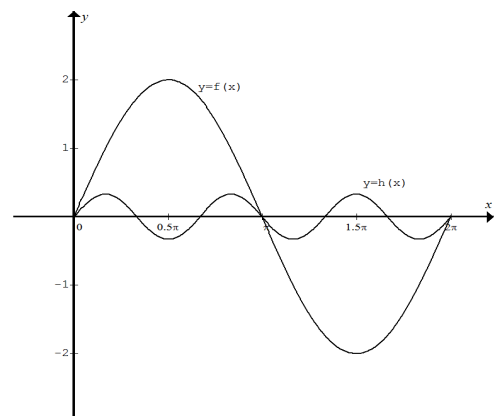
Q3dii Binomial distribution: $n?$, $p = 0.03$, $q = 0.97$

$1 - \Pr(X = 0) = 1 - 0.97^n > 0.5$

$\therefore 0.97^n < 0.5$, $n > 22.75$, $\therefore n = 23$

Q4a Area of shaded region $= 2 \times \int_0^\pi 2 \sin(x) dx = 4 \int_0^\pi \sin(x) dx$, $a = 4$

Q4b



Q4c Dilate from the x -axis by a factor of $\frac{1}{6}$, then dilate from the y -axis by a factor of $\frac{1}{3}$.

Q4di If n is even, the calculation is similar to part a.

$$\begin{aligned}\text{Area of shaded region} &= 2 \times \int_0^\pi m \sin(x) dx = 2m \int_0^\pi \sin(x) dx \\ &= 2m[-\cos(x)]_0^\pi = 4m = 4m + \frac{0}{n^2}\end{aligned}$$

If n is odd, area of shaded region

$$= 2 \left(\int_0^\pi m \sin(x) dx - \int_0^{\frac{\pi}{n}} \frac{1}{n} \sin(nx) dx \right) = 4m + \frac{-4}{n^2}$$

Q5ai $S(t) = 2e^{\frac{t}{3}} + 8e^{\frac{-2t}{3}}$, $0 \leq t \leq 5$

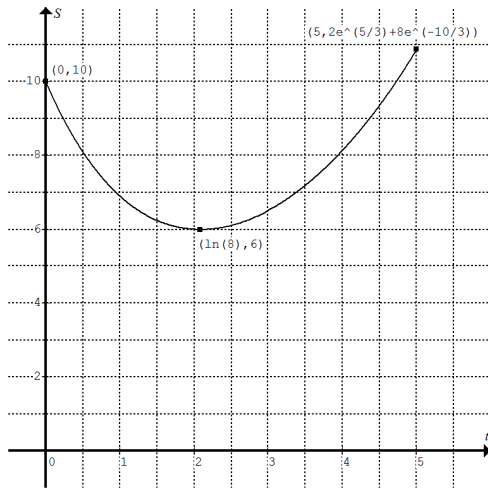
$S(0) = 10$, $S(5) = 2e^{\frac{5}{3}} + 8e^{\frac{-10}{3}}$

Q5aii $\frac{dS}{dt} = \frac{2}{3}e^{\frac{t}{3}} - \frac{16}{3}e^{\frac{-2t}{3}}$ Let $\frac{dS}{dt} = 0$ to find $S_{\min}$.

$$\therefore \frac{2}{3}e^{\frac{t}{3}} - \frac{16}{3}e^{\frac{-2t}{3}} = 0, \left(\frac{2}{3}e^{\frac{t}{3}} - \frac{16}{3}e^{\frac{-2t}{3}} \right) e^{\frac{2t}{3}} = 0$$

$$\therefore e^t = 8, t = \log_e 8, \therefore c = 8 \text{ and } S_{\min} = 2e^{\frac{\log_e 8}{3}} + 8e^{\frac{-2 \log_e 8}{3}} = 6$$

Q5aiii



Q5aiv Average rate of change $= \frac{6-10}{\log_e 8 - 0} = -\frac{4}{\log_e 8}$

Q5b $V(t) = de^{\frac{t}{3}} + (10-d)e^{\frac{-2t}{3}}$, $0 \leq t \leq 5$ and $0 < d < 10$

$$\frac{dV}{dt} = \frac{d}{3}e^{\frac{t}{3}} - \frac{2(10-d)}{3}e^{\frac{-2t}{3}}$$

$V_{\min}$ occurs when $t = \log_e 9$ (i.e. $e^t = 9$) and $\frac{dV}{dt} = 0$.

$$\therefore \frac{d}{3} \left(9^{\frac{1}{3}} \right) - \frac{2(10-d)}{3} \left(9^{-\frac{2}{3}} \right) = 0 \quad \therefore d = \frac{20}{11}$$

Note: When d decreases (or increases), the turning point (local min) of $V(t)$ occurs at increased (or decreased) t values. It can also occur outside the interval $[0, 5]$ and $\therefore$ the minimum value (not necessarily the turning point) of $V(t)$ occurs at one of the endpoints.

Q5ci $\frac{dV}{dt} = \frac{d}{3}e^{\frac{t}{3}} - \frac{2(10-d)}{3}e^{\frac{-2t}{3}}$

Turning point at $t = 0$, $\frac{d}{3} - \frac{2(10-d)}{3} = 0$, $\therefore d = \frac{20}{3}$

$\therefore V_{\min}$ occurs at $t = 0$ when $d \in \left[\frac{20}{3}, 10 \right)$

Q5cii $\frac{dV}{dt} = \frac{d}{3}e^{\frac{t}{3}} - \frac{2(10-d)}{3}e^{\frac{-2t}{3}}$

Turning point at $t = 5$, $\frac{d}{3}e^{\frac{5}{3}} - \frac{2(10-d)}{3}e^{\frac{-10}{3}} = 0$, $\therefore d = \frac{20}{e^5 + 2}$

$\therefore V_{\min}$ occurs at $t = 5$ when $d \in \left(0, \frac{20}{e^5 + 2} \right]$.

Q5d $V(t) = de^{\frac{t}{3}} + (10-d)e^{\frac{-2t}{3}}$, $\frac{dV}{dt} = \frac{d}{3}e^{\frac{t}{3}} - \frac{2(10-d)}{3}e^{\frac{-2t}{3}}$

(a, m) is a local minimum, $\therefore \frac{d}{3}e^{\frac{a}{3}} - \frac{2(10-d)}{3}e^{\frac{-2a}{3}} = 0 \dots (1)$

and $m = de^{\frac{a}{3}} + (10-d)e^{\frac{-2a}{3}} = \frac{k}{2}d^{\frac{2}{3}}(10-d)^{\frac{1}{3}} \dots (2)$

From (1), $e^a = \frac{2(10-d)}{d}$

$\therefore e^{\frac{a}{3}} = \left(\frac{2(10-d)}{d} \right)^{\frac{1}{3}} \dots (3)$ and $e^{\frac{-2a}{3}} = \left(\frac{2(10-d)}{d} \right)^{-\frac{2}{3}} \dots (4)$

Substitute (3) and (4) in (2):

$$m = d \frac{2^{\frac{1}{3}}(10-d)^{\frac{1}{3}}}{d^{\frac{1}{3}}} + (10-d) \frac{2^{-\frac{2}{3}}(10-d)^{-\frac{2}{3}}}{d^{-\frac{2}{3}}} = \frac{k}{2}d^{\frac{2}{3}}(10-d)^{\frac{1}{3}}$$

$$\therefore 2^{\frac{1}{3}} + \frac{1}{2^{\frac{2}{3}}} = \frac{k}{2}, \therefore k = 2 \times 2^{\frac{1}{3}} + 2^{\frac{1}{3}} = 2^{\frac{1}{3}}(2+1) = 3 \times 2^{\frac{1}{3}}$$

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