

YEAR 12 Trial Exam Paper

2015

MATHEMATICAL METHODS (CAS)

Written examination 1

Worked solutions

This book presents:

- worked solutions, giving you a series of points to show you how to work through the questions
- mark allocations
- tips on how to approach the questions

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Question 1a.**Worked solution**

$$\frac{d}{dx}(e^x \sin(2x)) = e^x \sin(2x) + 2e^x \cos(2x)$$

Mark allocation: 2 marks

- 1 mark for evidence of using the product rule
- 1 mark for the correct answer

**Tip**

- *Always be on the look-out for product rule or quotient rule questions.*

Question 1b.**Worked solution**

$$f(x) = e^{\cos(x)}$$

$$f'(x) = -\sin(x)e^{\cos(x)}$$

$$f'\left(\frac{\pi}{2}\right) = -\sin\left(\frac{\pi}{2}\right)e^{\cos\left(\frac{\pi}{2}\right)} = -1 \times e^0 = -1$$

Mark allocation: 2 marks

- 1 mark for the correct derivative $f'(x)$
- 1 mark for the correct answer

**Tip**

- *Remember to re-read the question before moving on. Many students differentiate correctly but then forget to evaluate.*

Question 1c.**Worked solution**

Average value of a function is

$$\begin{aligned}
 & \frac{1}{b-a} \int_a^b f(x) dx \\
 &= \frac{1}{k-1} \int_1^k \frac{1}{3+2x} dx \\
 &= \frac{1}{k-1} \times \frac{1}{2} [\log_e |3+2x|]_1^k \\
 &= \frac{1}{2k-2} [\log_e (3+2k) - \log_e (5)] \\
 &= \frac{1}{2k-2} \log_e \left(\frac{3+2k}{5} \right)
 \end{aligned}$$

Setting $\frac{1}{2k-2} \log_e \left(\frac{3+2k}{5} \right) = \frac{1}{10} \log_e (3)$, gives $k = 6$.

Mark allocation: 3 marks

- 1 mark for setting up the correct integral with terminals from 1 to k
- 1 mark for obtaining a log equation in terms of k
- 1 mark for the answer $k = 6$

Question 2

Worked solution

To have a unique solution implies the two lines are not parallel. Therefore, they have *different* gradients.

Rearranging the lines gives

$$2y = ax - a \Rightarrow y = \frac{a}{2}x - \frac{a}{2}$$

and $y = -5x + 7$

So $\frac{a}{2} \neq -5, a \neq -10.$

$\therefore a \neq -10$

$$a \in \mathbb{R} \setminus \{-10\}$$

An alternative solution involves using matrices and calculating the determinant.

Let $A = \begin{bmatrix} a & -2 \\ 5 & 1 \end{bmatrix}$

$$\det A = ad - bc = a + 10$$

Let $\det A \neq 0 \Rightarrow a + 10 \neq 0$

So $a \neq -10.$

$$\therefore a \in \mathbb{R} \setminus \{-10\}.$$

Mark allocation: 2 marks

- 1 mark for equating gradients or forming the determinant
- 1 mark for the correct answer



Tip

- Remember there are three ways this question can be asked: unique solution, no solution or infinite solutions. Be sure to be able to answer any of these.

Question 3a.**Worked solution**

Swap x and y to get $x = \frac{1}{3} \log_e \left(\frac{y-2}{3} \right)$.

Now, rearrange to make y the subject.

$$3x = \log_e \left(\frac{y-2}{3} \right)$$

$$e^{3x} = \frac{y-2}{3}$$

$$3e^{3x} = y - 2$$

$$y = 3e^{3x} + 2$$

$$\text{So } f^{-1}(x) = 3e^{3x} + 2.$$

Mark allocation: 2 marks

- 1 mark for swapping x and y
- 1 mark for the correct answer, which must be written as f^{-1}

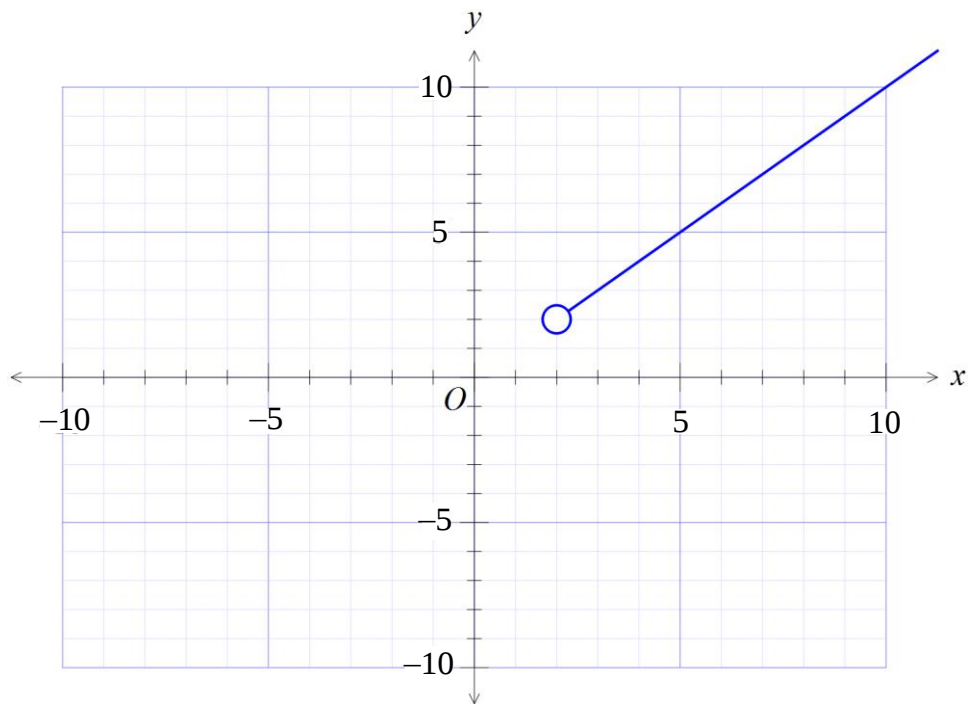
**Tip**

- Remember to use correct notation – the question asked for f^{-1} , so make sure you use this in your answer.

Question 3b.**Worked solution**

The graph of $y = f^{-1}(f(x))$ will be the graph of $y = x$ for the domain of $f(x)$, i.e. $x \in (2, \infty)$.

So the graph will be

**Mark allocation: 1 mark**

- 1 mark for correctly drawn graph (It must have an open circle at $x = 2$.)

**Tip**

- Remember to find the domain of f . This is important as the graph of $y = f^{-1}(f(x))$ will exist only for the domain of f .

Question 3c.**Worked solution**

The matrix $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ describes a transformation that produces a reflection in both the x - and y -axes.

So the new equation is $y = -\frac{1}{3} \log_e \left(\frac{-2-x}{3} \right)$.

Mark allocation: 1 mark

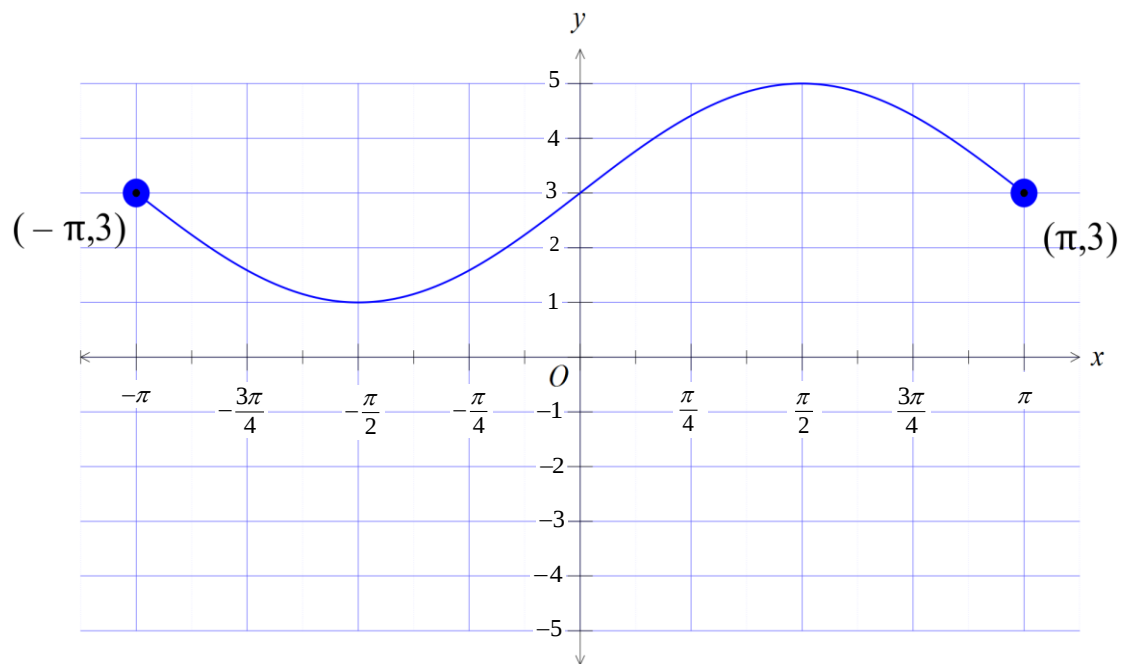
- 1 mark for the correct answer

Question 4a.**Worked solution**

The transformations:

- a dilation of factor 2 from the x -axis
- a translation of +3 units up

give a new equation of $y = 2\sin(x) + 3$.

**Mark allocation: 2 marks**

- 1 mark for one cycle of either a sin or cos curve that is centred around $y = 3$
- 1 mark if all aspects are correct, including correctly labelled end points

Question 4b.**Worked solution**

$$\frac{dy}{dx} = 2 \cos(x)$$

When $x = 0$, $\frac{dy}{dx} = 2$, so the gradient of the normal is $-\frac{1}{2}$.

The equation of the normal to the curve is

$$\therefore y - 3 = -\frac{1}{2}(x - 0)$$

$$y = -\frac{1}{2}x + 3$$

Mark allocation: 2 marks

- 1 mark for obtaining the gradient of the normal to the curve
- 1 mark for obtaining the line equation

**Tip**

- *Again, re-read the question. Don't forget to find the equation, not just the gradient!*

Question 5a.**Worked solution**

Max will need to drink cola, lemonade, lemonade and then cola.

So the $\Pr(CLLC) = 1 \times 0.7 \times 0.4 \times 0.6$

$$= \frac{7}{10} \times \frac{2}{5} \times \frac{3}{5} = \frac{21}{125}$$

$$= 0.168$$

Mark allocation: 2 marks

- 1 mark for writing CLLC for writing a product that involves 4 terms
- 1 mark for the correct answer, written as either 0.168 or $\frac{21}{125}$

Question 5b.**Worked solution**

$$\begin{array}{cc} C_i & L_i \\ C_{i+1} & \begin{bmatrix} 0.3 & 0.6 \\ 0.7 & 0.4 \end{bmatrix} \\ L_{i+1} & \end{array}$$

$$\Pr(L \text{ long term}) = \frac{0.7}{1.3} = \frac{7}{13}$$

Mark allocation: 1 mark

- 1 mark for the correct answer

Question 6a.**Worked solution**

Swap x and y , giving

$$x = \frac{1}{2} \log_e(y - k)$$

$$2x = \log_e(y - k)$$

$$e^{2x} = y - k$$

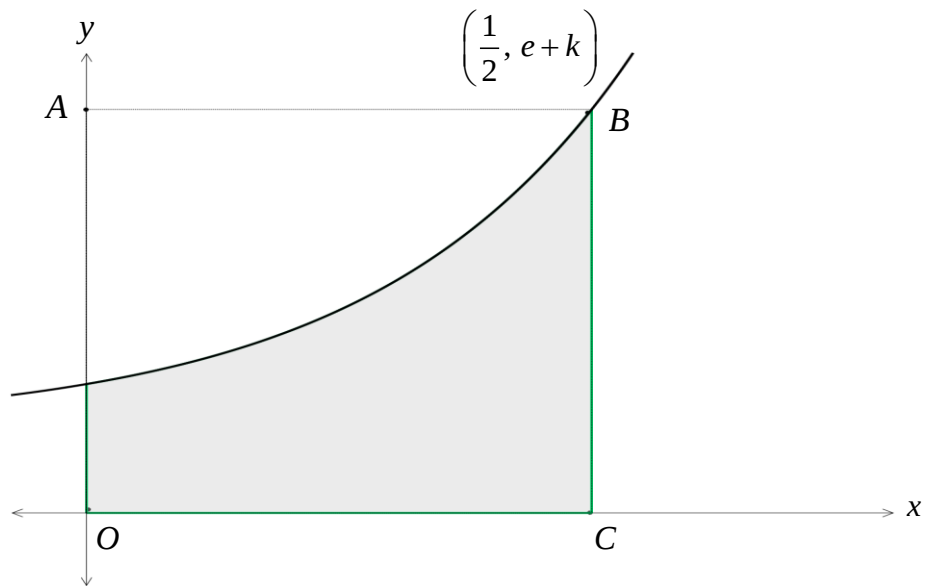
$$y = e^{2x} + k$$

Mark allocation: 1 mark

- 1 mark for using the correct method, leading to the correct answer

Question 6b.**Worked solution**

Using the inverse found in part **a**, consider the equivalent rectangle.



Area of the rectangle is $\frac{1}{2}(e+k)$.

Therefore, the area under the curve is equal to $\frac{1}{4}(e+k)$.

$$\text{So } \int_0^{\frac{1}{2}} e^{2x} + k \, dx = \frac{1}{4}(e+k).$$

$$\begin{aligned} \text{LHS} &= \left[\frac{1}{2} e^{2x} + kx \right]_0^{\frac{1}{2}} \\ &= \left(\frac{1}{2} e + \frac{1}{2} k \right) - \left(\frac{1}{2} + 0 \right) \\ &= \frac{1}{2} e + \frac{1}{2} k - \frac{1}{2} \end{aligned}$$

$$\Rightarrow \frac{1}{2} e + \frac{1}{2} k - \frac{1}{2} = \frac{1}{4}(e+k)$$

$$\Rightarrow \frac{1}{2} k - \frac{1}{4} k = \frac{1}{2} - \frac{1}{4} e$$

$$\Rightarrow \frac{1}{4} k = \frac{1}{4}(2-e)$$

$$\Rightarrow k = (2-e)$$

Mark allocation: 4 marks

- 1 mark for finding the area of the equivalent rectangle
- 1 mark for setting up the integral as $\int_0^{\frac{1}{2}} e^{2x} + k \, dx$
- 1 mark for obtaining the integral $\left[\frac{1}{2} e^{2x} + kx \right]_0^{\frac{1}{2}}$
- 1 mark for the correct answer

**Tip**

- *This is a 'hence' question so you must make use of your answer to part a.*

Question 7a.**Worked solution**

$$\sin(3x) = \frac{-1}{\sqrt{2}}, \quad \text{3rd and 4th quadrants}$$

$$3x = \frac{5\pi}{4}, \frac{7\pi}{4}$$

$$3x = \frac{5\pi}{4} + 2k\pi, \frac{7\pi}{4} + 2k\pi$$

$$x = \frac{5\pi}{12} + \frac{2k\pi}{3}, \frac{7\pi}{12} + \frac{2k\pi}{3}, k \in \mathbb{Z} \quad \text{or} \quad x = \frac{(8k-1)\pi}{12}, \frac{(8k+5)\pi}{12}, k \in \mathbb{Z}$$

Mark allocation: 2 marks

- 1 mark for using $\frac{\pi}{4}$ as the basic angle
- 1 mark for the correct answer

**Tip**

- For trigonometry general solution questions, you must always write $k \in \mathbb{Z}$.

Question 7b.**Worked solution**

The average rate of change of the function is given by

$$\begin{aligned}
 &= \frac{f(b) - f(a)}{b - a} \\
 &= \frac{f\left(\frac{\pi}{3}\right) - f\left(\frac{\pi}{4}\right)}{\frac{\pi}{3} - \frac{\pi}{4}} \\
 &= \frac{0 - \sqrt{2}}{\frac{\pi}{12}} \\
 &= \frac{-12\sqrt{2}}{\pi}
 \end{aligned}$$

Mark allocation: 2 marks

- 1 mark for correctly evaluating either $f\left(\frac{\pi}{3}\right)$ or $f\left(\frac{\pi}{4}\right)$
- 1 mark for the correct answer

**Tip**

- *Be careful! Students often confuse average rate of change for average value of the function.*

Question 8a.**Worked solution**

Using the log laws, we get

$$\log_8 (x+1)^2 + \log_8 4 = 1$$

$$\log_8 4(x+1)^2 = 1$$

$$8^1 = 4(x+1)^2$$

$$2 = (x+1)^2$$

$$\pm\sqrt{2} = (x+1)$$

$$x = -1 \pm \sqrt{2}$$

However, $x > -1$ so $x = -1 + \sqrt{2}$.

Mark allocation: 2 marks

- 1 method mark for using log laws
- 1 mark for the correct answer

Question 8b.**Worked solution**

Let $a = e^x$, as this forms the trinomial equation into a recognisable quadratic equation.

$$a^2 - 8a + 7 = 0$$

Factorising gives $(a - 1)(a - 7) = 0$.

So $a = 1$ or $a = 7$.

$$\Rightarrow e^x = 1 \text{ or } e^x = 7$$

$$\Rightarrow x = 0 \text{ or } x = \log_e 7$$

Mark allocation: 3 marks

- 1 method mark for establishing the trinomial equation
- 1 method mark for factorising the trinomial
- 1 answer mark for both correct answers

Question 9a.**Worked solution**

Converting to standard normal gives

$$\begin{aligned}\Pr(X < 30) &= \Pr\left(Z < \frac{30-60}{15}\right) \\ &= \Pr(Z < -2) \\ &= 0.0228\end{aligned}$$

Mark allocation: 1 mark

- 1 mark for the correct answer

Question 9b.**Worked solution**

Using the conditional probability rule gives

$$\begin{aligned}\Pr(X < 60 | X > 30) &= \frac{\Pr(X < 60 \cap X > 30)}{\Pr(X > 30)} \\ &= \frac{\Pr(30 < X < 60)}{\Pr(X > 30)}\end{aligned}$$

Again, converting to standard normal gives

$$\frac{\Pr(30 < X < 60)}{\Pr(X > 30)} = \frac{\Pr(-2 < Z < 0)}{\Pr(Z > -2)}$$

Using the symmetry of the normal distribution curve, $\Pr(Z > -2) = 1 - \Pr(Z < -2)$.

So the $\Pr(-2 < Z < 0) = 0.5 - 0.0228 = 0.4772$ and $\Pr(Z > 0) = 0.5$.

$$\text{Therefore, } \Pr(X < 60 | X > 30) = \frac{0.4772}{1 - 0.0228} = \frac{0.4772}{0.9772} = \frac{4772}{9772}.$$

Mark allocation: 2 marks

- 1 mark for using conditional probability
- 1 mark for the correct answer

**Tip**

- The answer must be written as a fraction and not a decimal inside a fraction.

Question 10**Worked solution**

$$\begin{aligned}
 f'(x) &= (x-a)3(x-b)^2 + (x-b)^3 \\
 &= (x-b)^2 [3(x-a) + (x-b)] \\
 &= (x-b)^2 [4x - 3a - b]
 \end{aligned}$$

For stationary points, let $f'(x) = 0$.

$$\Rightarrow (x-b)^2(4x-3a-b) = 0$$

$$\Rightarrow x = b \text{ or } 4x = 3a + b$$

$$\Rightarrow x = b \text{ or } x = \frac{3a+b}{4}$$

$$\text{So } b = 3 \text{ and } 4 = \frac{3a+3}{4}.$$

$$16 = 3a + 3$$

$$13 = 3a$$

$$a = \frac{13}{3}$$

Mark allocation: 3 marks

- 1 mark for $f'(x) = 0$
- 1 mark for $b = 3$
- 1 mark for $a = \frac{13}{3}$

END OF WORKED SOLUTIONS