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GROUP**

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**MATHS METHODS (CAS) 3 & 4
TRIAL EXAMINATION 1
SOLUTIONS
2015**

Question 1 (5 marks)

a. $\frac{d}{dx}(4x^3 - x)^6 = 6(4x^3 - x)^5 \times (12x^2 - 1)$ (Chain rule)

(1 mark) – $6(4x^3 - x)^5$

(1 mark) – $(12x^2 - 1)$

b. $g(x) = \frac{\log_e(2x)}{1+2x}$

$$g'(x) = \frac{(1+2x) \times \frac{2}{2x} - 2 \log_e(2x)}{(1+2x)^2}$$

(quotient rule) (1 mark) – correct numerator

(1 mark) – correct denominator

$$g'\left(\frac{1}{2}\right) = \frac{(1+1) \times 2 - 2 \log_e(1)}{(1+1)^2}$$

$$= \frac{4 - 2 \times 0}{2^2}$$

since $\log_e(1) = 0$

$$= 1$$

(1 mark)

Question 2 (2 marks)

$$\int \frac{2}{(3x-5)^4} dx$$

$$= \frac{2}{3 \times -3} \times (3x-5)^{-3} + c$$

$$= \frac{-2}{9(3x-5)^3} + c$$

(1 mark) for $\frac{1}{(3x-5)^3}$

(1 mark) for $\frac{-2}{9}$

Question 3 (2 marks)

$$3^{2x} - 8 \times 3^x = 9$$

$$3^{2x} - 8 \times 3^x - 9 = 0$$

Let $a = 3^x$

$$a^2 - 8a - 9 = 0$$

$$(a - 9)(a + 1) = 0$$

$$a = 9 \text{ or } a = -1$$

Sub back

$$3^x = 9 \quad \text{or} \quad 3^x = -1$$

$$3^x = 3^2 \quad \text{no real solution exists for } x$$

$$x = 2$$

So $x = 2$.

(1 mark)

(1 mark)

Question 4 (2 marks)Method 1

$$\log_5(x^3) + 2\log_5(x) = 15$$

$$\log_5(x^3 \times x^2) = 15$$

$$\log_5(x^5) = 15$$

$$5^{15} = x^5$$

$$(x^5)^{\frac{1}{5}} = (5^{15})^{\frac{1}{5}}$$

$$x = 5^3$$

$$x = 125$$

(1 mark)

(1 mark)

Method 2

$$\log_5(x^3) + 2\log_5(x) = 15$$

$$3\log_5(x) + 2\log_5(x) = 15$$

$$5\log_5(x) = 15$$

$$\log_5(x) = 3$$

$$5^3 = x$$

$$x = 125$$

(1 mark)

(1 mark)

Question 5 (5 marks)

a. $f(x) = (x+2)\cos(x)$
 $f'(x) = (x+2) \times -\sin(x) + 1 \times \cos(x)$ (product rule)
 So $f'(x) = -(x+2)\sin(x) + \cos(x)$

(1 mark) for first term
 (1 mark) for second term

When $x = -\frac{\pi}{6}$,

$$f'\left(-\frac{\pi}{6}\right) = -\left(-\frac{\pi}{6} + 2\right)\sin\left(-\frac{\pi}{6}\right) + \cos\left(-\frac{\pi}{6}\right)$$

$$= \left(\frac{\pi}{6} - 2\right) \times -\frac{1}{2} + \frac{\sqrt{3}}{2}$$

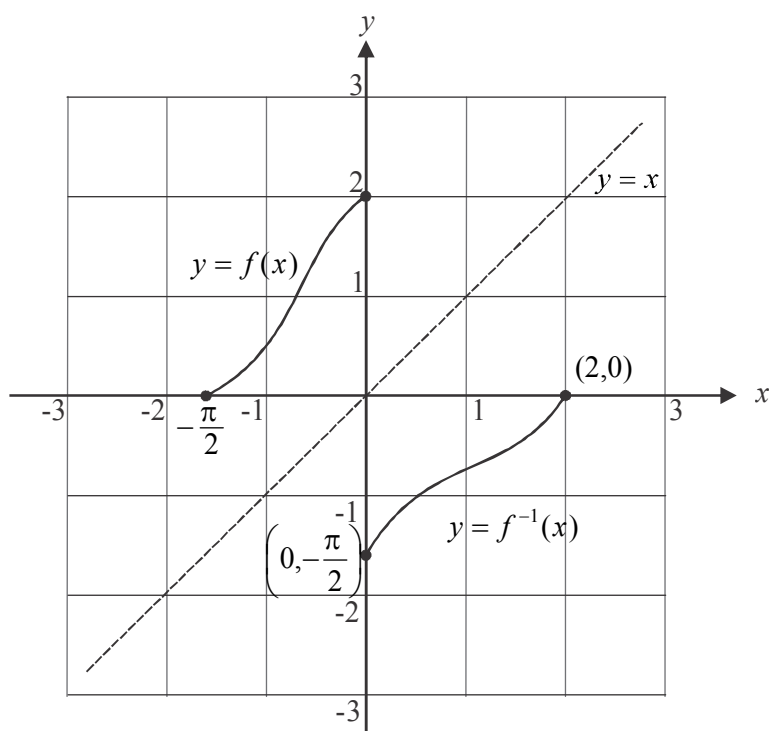
$$= -\frac{\pi}{12} + 1 + \frac{\sqrt{3}}{2}$$

S	A
T	C

So the gradient of the graph at $x = -\frac{\pi}{6}$ is $-\frac{\pi}{12} + 1 + \frac{\sqrt{3}}{2}$.

(1 mark)

b.



The graph of f^{-1} is a reflection of the graph of f in the line $y = x$.

The endpoints of f are $(-\frac{\pi}{2}, 0)$ and $(0, 2)$.

The endpoints of f^{-1} are therefore $(0, -\frac{\pi}{2})$ and $(2, 0)$.

(1 mark) – correct shape
 (1 mark) – correct endpoints

Question 6 (3 marks)

Let (x', y') be an image point.

$$\begin{aligned} \begin{bmatrix} x' \\ y' \end{bmatrix} &= \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -2 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{x}{2} \\ -3y \end{bmatrix} + \begin{bmatrix} -2 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{x}{2} - 2 \\ -3y + 1 \end{bmatrix} \end{aligned}$$

(1 mark)

$$\text{So } x' = \frac{x}{2} - 2 \quad \text{and} \quad y' = -3y + 1$$

$$x' + 2 = \frac{x}{2} \quad 3y = 1 - y'$$

$$x = 2(x' + 2) \quad y = \frac{1 - y'}{3}$$

$$\text{So } y = e^{\frac{x-4}{2}} - 1$$

$$\text{becomes } \frac{1 - y'}{3} = e^{\frac{2(x'+2)-4}{2}} - 1$$

(1 mark)

$$\frac{1 - y'}{3} = e^{\frac{2x'+4-4}{2}} - 1$$

$$\frac{1 - y'}{3} = e^{x'} - 1$$

$$1 - y' = 3e^{x'} - 3$$

$$-y' = 3e^{x'} - 4$$

$$y' = 4 - 3e^{x'}$$

The image equation is $y = 4 - 3e^x$.

So $a = 4$ and $b = -3$.

(1 mark)

Question 7 (5 marks)

- a. The graphs will intersect when

$$f(x) = g(x)$$

$$2\sin\left(\frac{\pi x}{9}\right) = \sqrt{3}$$

$$\sin\left(\frac{\pi x}{9}\right) = \frac{\sqrt{3}}{2}$$

$$\frac{\pi x}{9} = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{7\pi}{3}, \dots$$

$$x = \frac{\pi}{3} \times \frac{9}{\pi}, \frac{2\pi}{3} \times \frac{9}{\pi}, \frac{7\pi}{3} \times \frac{9}{\pi}, \dots$$

$$= 3, 6, 21, \dots$$

**(1 mark)**

Since f and g are each defined for $x \in [0, 9]$, then $x = 3$ or 6 .

Points of intersection are $(3, \sqrt{3})$ and $(6, \sqrt{3})$.

(1 mark)

- b. Do a quick sketch.

For the graph of $y = f(x)$,

$$\text{period} = 2\pi \div \frac{\pi}{9} = 18$$

but $d_f = [0, 9]$.

$$\text{Area} = \int_3^6 (f(x) - g(x)) dx$$

$$= \int_3^6 \left(2\sin\left(\frac{\pi x}{9}\right) - \sqrt{3} \right) dx$$

$$= \left[-2 \div \frac{\pi}{9} \cos\left(\frac{\pi x}{9}\right) - \sqrt{3}x \right]_3^6$$

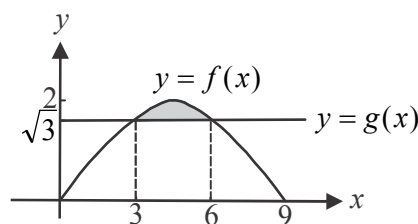
$$= \left[\frac{-18}{\pi} \cos\left(\frac{\pi x}{9}\right) - \sqrt{3}x \right]_3^6$$

$$= \left(\frac{-18}{\pi} \cos\left(\frac{2\pi}{3}\right) - \sqrt{3} \times 6 \right) - \left(\frac{-18}{\pi} \cos\left(\frac{\pi}{3}\right) - \sqrt{3} \times 3 \right)$$

$$= \frac{-18}{\pi} \times \frac{-1}{2} - 6\sqrt{3} + \frac{18}{\pi} \times \frac{1}{2} + 3\sqrt{3}$$

$$= \frac{9}{\pi} - 6\sqrt{3} + \frac{9}{\pi} + 3\sqrt{3}$$

$$\text{Area} = \frac{18}{\pi} - 3\sqrt{3} \text{ square units.}$$

**(1 mark)****(1 mark)**

Question 8 (4 marks)

- a. Since we have a probability density function,

$$\int_0^a \frac{1}{x+1} dx = 1 \quad (1 \text{ mark})$$

$$\left[\log_e |x+1| \right]_0^a = 1$$

$$\log_e(a+1) - \log_e(1) = 1, \quad a > 0$$

$$\log_e(a+1) = 1 \quad \text{since } \log_e(1) = 0$$

$$e^1 = a+1$$

$$a = e - 1$$

(1 mark)

b. $\Pr(X < m | X < 1)$
 $= \frac{\Pr(X < m \cap X < 1)}{\Pr(X < 1)} \quad (\text{conditional probability formula})$

$$= \frac{\Pr(X < m)}{\Pr(X < 1)}$$

(1 mark)

Now, $\Pr(X < m) = 0.5$ since m is the median.

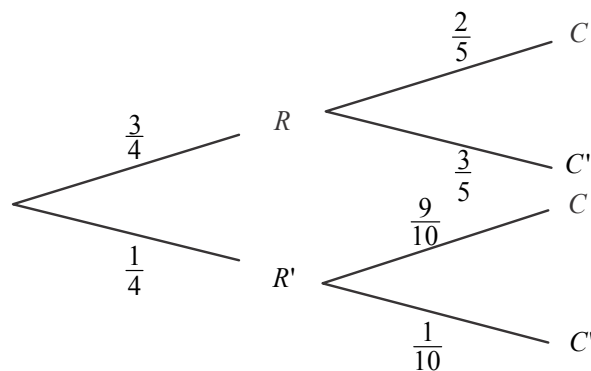
$$\begin{aligned} \Pr(X < 1) &= \int_0^1 \frac{1}{x+1} dx \\ &= \left[\log_e |x+1| \right]_0^1 \\ &= \log_e(2) - \log_e(1) \\ &= \log_e(2) \quad \text{since } \log_e(1) = 0 \end{aligned}$$

$$\begin{aligned} \text{So } \frac{\Pr(X < m)}{\Pr(X < 1)} &= \frac{0.5}{\log_e(2)} \\ &= \frac{1}{2 \log_e(2)} \quad \text{or} \quad \frac{1}{\log_e(4)} \end{aligned}$$

(1 mark)

Question 9 (6 marks)**a.** $\Pr(C, C') + \Pr(C', C)$ **(1 mark)**

$$\begin{aligned}
 &= \frac{9}{10} \times \frac{1}{10} + \frac{1}{10} \times \frac{9}{10} \\
 &= \frac{9}{100} + \frac{9}{100} \\
 &= \frac{18}{100} \\
 &= \frac{9}{50}
 \end{aligned}$$

(1 mark)**b. i.****(1 mark)**

$$\begin{aligned}
 \Pr(C) &= \Pr(R, C) + \Pr(R', C) \\
 &= \frac{3}{4} \times \frac{2}{5} + \frac{1}{4} \times \frac{9}{10} \\
 &= \frac{6}{20} + \frac{9}{40} \\
 &= \frac{21}{40}
 \end{aligned}$$

(1 mark)**ii.** $\Pr(R | C') = \frac{\Pr(R \cap C')}{\Pr(C')}$ (conditional probability formula) **(1 mark)**

$$\begin{aligned}
 &= \frac{\frac{3}{4} \times \frac{3}{5}}{1 - \Pr(C)} \quad \text{(using the tree diagram)}
 \end{aligned}$$

$$= \frac{9}{20} \div \left(1 - \frac{21}{40}\right) \quad \text{(from part i.)}$$

$$= \frac{9}{20} \div \frac{19}{40}$$

$$= \frac{9}{20} \times \frac{40}{19}$$

$$= \frac{18}{19}$$

(1 mark)

Question 10 (6 marks)

a. i. $f(x) = \sqrt{u-x}$

$$= (u-x)^{\frac{1}{2}}$$

$$f'(x) = \frac{1}{2}(u-x)^{-\frac{1}{2}} \times -1$$

$$= \frac{-1}{2(u-x)^{\frac{1}{2}}}$$

$$= \frac{-1}{2\sqrt{u-x}}$$

When the gradient of the tangent equals -1 ,

$$\frac{-1}{2\sqrt{u-x}} = -1 \quad (1 \text{ mark})$$

$$1 = 2\sqrt{u-x}$$

$$\frac{1}{2} = \sqrt{u-x}$$

$$\left(\frac{1}{2}\right)^2 = u-x$$

$$x = u - \frac{1}{4}$$

So $f(x) = \sqrt{u - \left(u - \frac{1}{4}\right)}$

$$= \sqrt{\frac{1}{4}}$$

$$= \frac{1}{2}$$

The y -coordinate of P is $\frac{1}{2}$ when the gradient of the tangent is -1 .

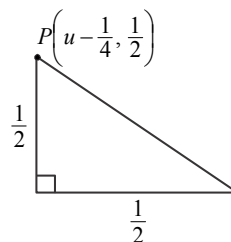
(1 mark)

ii. Method 1

Since the gradient of the tangent is -1 , and

since the y coordinate of P is $\frac{1}{2}$,

then the horizontal distance will be $\frac{1}{2}$ unit.



(1 mark)

Method 2

Equation of tangent

$$y - \frac{1}{2} = -1 \left(x - \left(u - \frac{1}{4} \right) \right)$$

$$y = -x + u - \frac{1}{4} + \frac{1}{2}$$

$$y = -x + u + \frac{1}{4}$$

When $y = 0$, $x = u + \frac{1}{4}$

So the tangent intersects the x -axis at $\left(u + \frac{1}{4}, 0 \right)$ and P is $\left(u - \frac{1}{4}, \frac{1}{2} \right)$.

So the horizontal distance is $u + \frac{1}{4} - \left(u - \frac{1}{4} \right) = \frac{1}{2}$ unit.

(1 mark)

b. Let area of rectangle $OMQN$ be A

$$A = OM \times MQ$$

$$= x \times f(x)$$

$$= x\sqrt{u-x}$$

$$= x(u-x)^{\frac{1}{2}}$$

$$\frac{dA}{dx} = 1 \times \sqrt{u-x} + x \times \frac{1}{2}(u-x)^{-\frac{1}{2}} \times -1$$

$$= \sqrt{u-x} - \frac{x}{2\sqrt{u-x}} \quad (1 \text{ mark})$$

$$\frac{dA}{dx} = 0 \text{ for max.}$$

$$\sqrt{u-x} - \frac{x}{2\sqrt{u-x}} = 0$$

$$\sqrt{u-x} = \frac{x}{2\sqrt{u-x}}$$

$$x = 2(u-x)$$

$$x = 2u - 2x$$

$$3x = 2u$$

$$x = \frac{2u}{3} \quad (1 \text{ mark})$$

Substitute

$$x = \frac{2u}{3} \text{ into } A = x\sqrt{u-x}$$

$$A = \frac{2u}{3} \times \sqrt{u - \frac{2u}{3}}$$

$$= 2 \times \frac{u}{3} \times \sqrt{\frac{u}{3}}$$

$$= 2 \left(\frac{u}{3} \right)^{\frac{3}{2}}$$

Maximum area is $2 \left(\frac{u}{3} \right)^{\frac{3}{2}}$ square units.

(1 mark)