

Specific instructions to students

- Answer all of the questions in the spaces provided.
- Show all workings in questions where more than one mark is available.
- An exact value must be provided in questions where a numerical answer is required, unless otherwise specified.

QUESTION 1

For the function $f(x) = e^x \sqrt{x}$, find $f'(x)$.

$$f(x) = e^x \sqrt{x} = e^x x^{\frac{1}{2}}$$

Using the product rule,

$$\begin{aligned} f'(x) &= x^{\frac{1}{2}} e^x + e^x \frac{1}{2} x^{-\frac{1}{2}} \\ &= \sqrt{x} e^x + e^x \frac{1}{2\sqrt{x}} = e^x \left(\sqrt{x} + \frac{1}{2\sqrt{x}} \right) \end{aligned}$$

2 marks

QUESTION 2

a For the function $f(x) = e^x x^2$, find $f'(x)$.

$$f(x) = e^x x^2$$

Using the product rule,

$$f'(x) = e^x x^2 + 2e^x x$$

2 marks

b Hence find $f'(2)$.

$$f'(x) = e^x x^2 + 2e^x x$$

$$\text{so } f'(2) = 4e^2 + 4e^2 = 8e^2$$

1 mark

(Total: 3 marks)

QUESTION 3

a For the function $f(x) = \frac{\cos(2x)}{x+2}$, find $f'(x)$.

$$f(x) = \frac{\cos(2x)}{x+2}$$

Using the quotient rule,

$$f'(x) = \frac{-2(x+2)\sin(2x) - \cos(2x)}{(x+2)^2}$$

2 marks

b Hence find $f'(\pi)$.

$$\begin{aligned} f'(x) &= \frac{-2(x+2)\sin(2x) - \cos(2x)}{(x+2)^2} \\ &= \frac{-2(\pi+2)\sin(2\pi) - \cos(2\pi)}{(\pi+2)^2} \\ &= \frac{-1}{(\pi+2)^2} \end{aligned}$$

1 mark

(Total: 3 marks)

QUESTION 4

a For the function $y = \frac{\sin(2x)}{(x^2-2)^2}$, find $\frac{dy}{dx}$.

$$y = \frac{\sin(2x)}{(x^2-2)^2}$$

Using the quotient and chain rules,

$$\begin{aligned} \frac{dy}{dx} &= \frac{2(x^2-2)^2 \cos(2x) - \sin(2x) \times 2(x^2-2) \times 2x}{(x^2-2)^4} \\ &= \frac{2(x^2-2)\cos(2x) - 4x\sin(2x)}{(x^2-2)^3} \end{aligned}$$

3 marks

b Hence find $\frac{dy}{dx}$ at $x = \pi$

At $x = \pi$,

$$\begin{aligned} \frac{dy}{dx} &= \frac{2(\pi^2-2)\cos(2\pi) - 4\pi\sin(2\pi)}{(\pi^2-2)^3} \\ &= \frac{2(\pi^2-2)}{(\pi^2-2)^3} = \frac{2}{(\pi^2-2)^2} \end{aligned}$$

1 mark

(Total: 4 marks)

QUESTION 5

The volume $V(t)$ litres of liquid in a drum at time t minutes is described by the formula

$$V(t) = 3t^3 + \frac{1}{t}, t > 0.$$

- a Find the average rate of change of liquid, in litres/min, from $t = 1$ to $t = 2$.

$$\begin{aligned}\text{Average rate of change} &= \frac{V(2) - V(1)}{2 - 1} \\ &= 24.5 - 4 \\ &= 20.5 \text{ litres/min}\end{aligned}$$

2 marks

- b Find the rate of change of liquid, in litres/min, when $t = 1$.

$$\begin{aligned}V(t) &= 3t^3 + \frac{1}{t} = 3t^3 + t^{-1} \\ V'(t) &= 9t^2 - t^{-2} = 9t^2 - \frac{1}{t^2} \\ V'(1) &= 9 - 1 = 8 \text{ litres/min}\end{aligned}$$

2 marks

- c Find at what time, in mins, there is a minimum amount of liquid in the drum.

$$\begin{aligned}\text{Let } V'(t) &= 9t^2 - \frac{1}{t^2} = 0 \text{ for minimum.} \\ 9t^2 &= \frac{1}{t^2} \\ t^4 &= \frac{1}{9} \\ \text{Select the +ve solution. } t &= \frac{1}{\sqrt{3}} \text{ mins (can see from} \\ &\text{the graph that this is a minimum)}\end{aligned}$$

2 marks

(Total: 6 marks)

QUESTION 6

The equation of a graph is $f(x) = 2x^3 + 1 - kx^2$, where k is a constant.

The tangent to the graph at $x = 1$ meets the x -axis at the point $(2, 0)$. Find the value of k .

$$f(x) = 2x^3 + 1 - kx^2$$

$$\Rightarrow f(1) = 3 - k$$

$$f'(x) = 6x^2 - 2kx$$

$$\Rightarrow f'(1) = 6 - 2k$$

The equation of the tangent using $y - y_1 = m(x - x_1)$ is:

$$y - (3 - k) = (6 - 2k)(x - 1)$$

At the point $(2, 0)$,

$$0 - (3 - k) = (6 - 2k)(2 - 1)$$

$$\Rightarrow -3 + k = 6 - 2k$$

This gives $k = 3$.

3 marks

QUESTION 7

- a The graph of $y = \frac{x^3}{3} - \frac{x^2}{4} + ax + b$ has a stationary point at $\left(2, \frac{2}{3}\right)$. Find the values of a and b .

$$y = \frac{x^3}{3} - \frac{x^2}{4} + ax + b$$

Substituting $\left(2, \frac{2}{3}\right)$ gives

$$\frac{2}{3} = \frac{8}{3} - \frac{4}{4} + 2a + b$$

$$\text{So } 2a + b = -1$$

$$\text{Also, } \frac{dy}{dx} = x^2 - \frac{x}{2} + a$$

$$x^2 - \frac{x}{2} + a = 0 \text{ at } x = 2.$$

$$\text{So } 3 + a = 0$$

$$\therefore a = -3, b = 5$$

3 marks

- b Hence, find the x -coordinate of the other stationary point.

$$y = \frac{x^3}{3} - \frac{x^2}{4} - 3x + 5$$

$$\frac{dy}{dx} = x^2 - \frac{x}{2} - 3 = 0 \quad \text{for stationary points}$$

$$2x^2 - x - 6 = 0$$

$$(2x + 3)(x - 2) = 0 \text{ gives } x = 2 \text{ and } x = -\frac{3}{2}$$

Other stationary point: $x = -\frac{3}{2}$

2 marks

(Total: 5 marks)

QUESTION 8

- a For the function $y = \frac{e^{3x}}{\sin(3x)}$, find an expression for $\frac{dy}{dx}$.

$$y = \frac{e^{3x}}{\sin(3x)}$$

Using the quotient rule,

$$\frac{dy}{dx} = \frac{\sin(3x) \times 3e^{3x} - e^{3x} \times 3 \cos(3x)}{\sin^2(3x)}$$

$$= \frac{3 \sin(3x)e^{3x} - 3 \cos(3x)e^{3x}}{\sin^2(3x)}$$

2 marks

- b Hence, find $\left\{ x : \frac{dy}{dx} = 0 \right\}$ for $x \in \left[0, \frac{\pi}{2} \right]$.

$$\frac{3 \sin(3x)e^{3x} - 3 \cos(3x)e^{3x}}{\sin^2(3x)} = 0$$

$$\Rightarrow 3e^{3x} \sin(3x) - 3e^{3x} \cos(3x) = 0$$

$$\Rightarrow 3e^{3x} (\sin(3x) - \cos(3x)) = 0$$

No solution for $3e^{3x} = 0$.

$$\therefore \sin(3x) - \cos(3x) = 0$$

$$\sin(3x) = \cos(3x)$$

$$\tan(3x) = 1$$

$$3x = \frac{\pi}{4}, \pi + \frac{\pi}{4} = \frac{\pi}{4}, \frac{5\pi}{4}$$

$$x = \frac{\pi}{12}, \frac{5\pi}{12}$$

2 marks

(Total: 4 marks)