

TEST 2

Functions and graphs
Technology-free end-of-year examination
Total marks: 25
Suggested writing time: 40 minutes

Specific instructions to students

- Answer all of the questions in the spaces provided.
- Show all workings in questions where more than one mark is available.
- An exact value must be provided in questions where a numerical answer is required, unless otherwise specified.

QUESTION 1

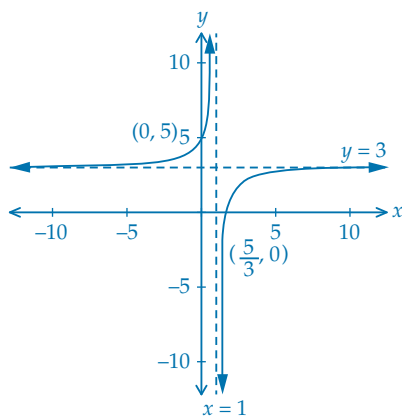
- a State the maximal domain for the function

$$y = -\frac{2}{x-1} + 3$$

$$x \in \mathbb{R} \setminus \{1\}$$

1 mark

- b Hence sketch the graph of $y = -\frac{2}{x-1} + 3$, labelling equations of asymptotes and axial intercepts with their coordinates.



2 marks
(Total: 3 marks)

QUESTION 2

- a The coordinates of A and B are $(2, 3)$ and $(4, -1)$ respectively. Find the equation of the line joining A and B .

Use the equation of the line $y - y_1 = m(x - x_1)$.

$$\text{Gradient of the line} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-1 - 3}{4 - 2} = -2$$

Using $(2, 3)$, $y - y_1 = m(x - x_1)$ gives

$$y - 3 = -2(x - 2)$$

$$y = -2x + 7$$

1 mark

- b Hence find the equation of the line perpendicular to the line found in part a, which passes through the midpoint of A and B .

$$m_T = -2 \text{ so } m_N = \frac{1}{2}$$

$$\text{Midpoint of } A \text{ and } B = \left(\frac{2+4}{2}, \frac{3-1}{2} \right) = (3, 1)$$

Using $(3, 1)$, $y - y_1 = m(x - x_1)$ gives

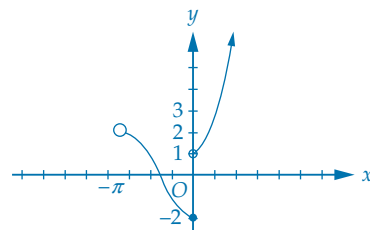
$$y - 1 = \frac{1}{2}(x - 3)$$

$$2y = x - 1$$

2 marks
(Total: 3 marks)

QUESTION 3

- a Sketch the graph of $y = \begin{cases} -2\cos(x), & -\pi < x \leq 0 \\ e^x, & x > 0 \end{cases}$



2 marks

- b State the value of $f(0)$.

$$f(0) = -2$$

1 mark
(Total: 3 marks)

QUESTION 4

Find the image of the point $(1, -6)$ under each of the following transformations:

a reflection in the y -axis

By matrix methods, the solutions are:

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -6 \end{bmatrix} = \begin{bmatrix} -1 \\ -6 \end{bmatrix}$$

So $(1, -6) \rightarrow (-1, -6)$, i.e., positive x becomes negative x .

1 mark

b dilation by factor 2 from the x -axis

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -6 \end{bmatrix} = \begin{bmatrix} 1 \\ -12 \end{bmatrix}$$

So $(1, -6) \rightarrow (1, -12)$; i.e. each y value is multiplied by 2.

1 mark

c dilation by factor 3 from y -axis

$$\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -6 \end{bmatrix} = \begin{bmatrix} 3 \\ -6 \end{bmatrix}$$

So $(1, -6) \rightarrow (3, -6)$; i.e., each x value is multiplied by 3.

1 mark

(Total: 3 marks)

QUESTION 5

a Find the rule and domain for the inverse of the function

$$f: (-\infty, 2] \rightarrow \mathbf{R}, f(x) = 2(x-2)^2 + 1$$

To find f^{-1} , swap x and y in $y = 2(x-2)^2 + 1$.

$$x = 2(y-2)^2 + 1$$

$$(y-2)^2 = \frac{x-1}{2}$$

$$y = 2 \pm \sqrt{\frac{x-1}{2}} \quad \text{select lower branch}$$

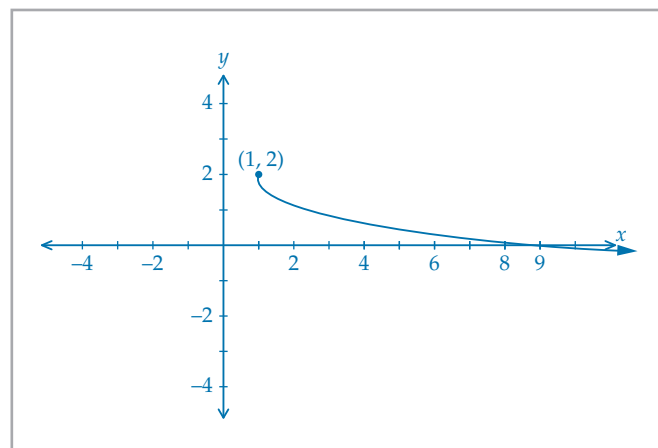
$$\text{Rule: } f^{-1}(x) = 2 - \sqrt{\frac{x-1}{2}}$$

$$\text{Domain: } f^{-1}(x) = \text{range } f(x) = [1, \infty)$$

$$f^{-1}: [1, \infty) \rightarrow \mathbf{R}, f^{-1}(x) = 2 - \sqrt{\frac{x-1}{2}}$$

2 marks

b Sketch the graph of $f^{-1}(x)$



2 marks

(Total: 4 marks)

QUESTION 6

Solve for x in the equation $e^{2x} - 2e^x - 3 = 0$.

$$e^{2x} - 2e^x - 3 = 0$$

$$\text{Let } A = e^x$$

$$\text{This gives } A^2 - 2A - 3 = 0$$

$$(A-3)(A+1) = 0$$

$$A = 3, A = -1$$

$$\text{So } e^x = 3, e^x = -1 \text{ (not a possible solution)}$$

$$e^x = 3 \Leftrightarrow x = \log_e(3)$$

$$\text{Answer: } x = \log_e(3)$$

2 marks

QUESTION 7

Solve for x in the equation

$$2 \log_e(x+2) - 2 \log_e(2) = 3 \log_e(4)$$

$$2 \log_e(x+2) - 2 \log_e(2) = 3 \log_e(4)$$

$$\Leftrightarrow \log_e(x+2)^2 = 2 \log_e(2) + 3 \log_e(2^2)$$

$$\Leftrightarrow \log_e(x+2)^2 = 2 \log_e(2) + 6 \log_e(2)$$

$$\Leftrightarrow \log_e(x+2)^2 = 8 \log_e(2) = \log_e(2^8)$$

Hence, we have

$$(x+2)^2 = 2^8$$

$$x^2 + 4x + 4 = 2^8$$

$$x^2 + 4x - 252 = 0$$

$$(x+18)(x-14) = 0 \rightarrow x = -18, x = 14$$

$$x = -18 \text{ gives } \log_e(-16), \text{ which doesn't exist.}$$

$$\text{Answer: } x = 14$$

2 marks

QUESTION 8

For $f: (1, \infty) \rightarrow \mathbf{R}$, $f(x) = -\log_e(x-1)$ and $g: [-1, 10] \rightarrow \mathbf{R}$,
 $g(x) = e^x$

- a** find the rule and domain for $(f+g)(x)$

The **sum** is defined as $(f+g)(x) = f(x) + g(x)$

Rule: $(f+g)(x) = -\log_e(x-1) + e^x$

Domain $(f+g)(x) =$ intersection of domain of $f(x)$
and $g(x)$.

Domain $(f+g)(x) = x \in (1, 10]$

2 marks

- b** find the rule and domain for $f(g(x))$ to exist.

$f(g(x))$ exists if $\text{ran}(g) \subseteq \text{dom}(f)$.

$\text{ran}(g) \subseteq \text{dom}(f) \rightarrow [e^{-1}, e^{10}] \not\subset (1, \infty)$

Restrict the range of g so that $\text{ran}(g) \subseteq \text{dom}(f)$.

$(e^0, e^{10}] \subseteq (1, \infty)$

Rule: $f(g(x)) = -\log_e(e^x - 1)$

Domain $f(g(x)) = \text{restricted dom}(g(x)) = (0, \infty)$

3 marks

(Total: 5 marks)