

Specific instructions to students

- Answer all of the questions in the spaces provided.
- Show all workings in questions where more than one mark is available.
- An exact value must be provided in questions where a numerical answer is required, unless otherwise specified.

QUESTION 1

$\Pr(A)$ and $\Pr(B)$ are independent events, where $\Pr(A) = 0.4$ and $\Pr(B) = 0.2$.

a Complete the Karnaugh map below.

	A	A'	
B	0.08	0.12	0.2
B'	0.32	0.48	0.8
	0.4	0.6	1

Independent events, so $\Pr(A \cap B) = \Pr(A) \times \Pr(B)$
 $= 0.4 \times 0.2 = 0.08$

2 marks

b Find $\Pr(A | B)$.

$$\begin{aligned}\Pr(A | B) &= \frac{\Pr(A \cap B)}{\Pr(B)} \\ &= \frac{0.08}{0.2} = \frac{8}{20} \\ &= 0.4\end{aligned}$$

2 marks

c Find $\Pr(B | A)$.

$$\begin{aligned}\Pr(B | A) &= \frac{\Pr(B \cap A)}{\Pr(A)} \\ &= \frac{0.08}{0.4} = \frac{8}{40} \\ \Pr(B | A) &= 0.2\end{aligned}$$

1 mark

(Total: 5 marks)

QUESTION 2

$\Pr(A)$ and $\Pr(B)$ are mutually exclusive, where $\Pr(A) = 0.4$ and $\Pr(B) = 0.5$.

a Complete the Karnaugh map below.

	A	A'	
B	0	0.5	0.5
B'	0.4	0.1	0.5
	0.4	0.6	1

Mutually exclusive events, so $\Pr(A \cap B) = 0$

2 marks

b Find $\Pr(A | B)$.

$$\begin{aligned}\Pr(A | B) &= \frac{\Pr(A \cap B)}{\Pr(B)} \\ &= \frac{0}{0.5} \\ &= 0\end{aligned}$$

1 mark

c Find $\Pr(A \cup B)$.

$$\begin{aligned}\Pr(A \cup B) &= \Pr(A) + \Pr(B) \quad (\text{if mutually exclusive}) \\ \Pr(A \cup B) &= 0.9\end{aligned}$$

2 marks

(Total: 5 marks)

QUESTION 3

I toss a biased coin three times. For this coin, $\Pr(H) = 0.6$ and $\Pr(T) = 0.4$.

What is the probability that I will get two heads and one tail?

$$\begin{aligned}\Pr(\text{HHT}) \text{ or } \Pr(\text{HTH}) \text{ or } \Pr(\text{THH}) \\ &= 3 \times 0.6^2 \times 0.4 \\ &= 0.432\end{aligned}$$

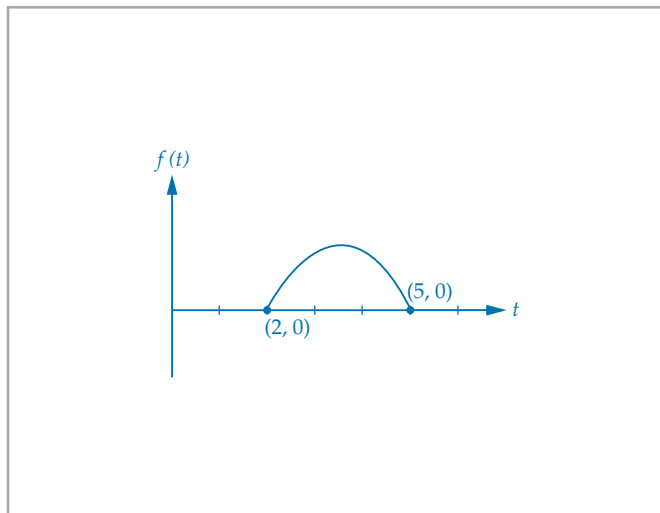
2 marks

QUESTION 4

The contagious period for a certain virus is between 2 and 5 days after contact with the virus. The probability density function that describes the probability that symptoms will appear after t days is

$$f(t) = \begin{cases} \frac{2}{9}(t-5)(2-t), & 2 \leq t \leq 5 \\ 0, & \text{otherwise} \end{cases}$$

- a Sketch a graph of the PDF.



2 marks

- b Find the probability that symptoms will appear within 3 days after contact with the virus.

$$\begin{aligned} \Pr(T \leq 3) &= \int_2^3 \frac{2}{9}(t-5)(2-t) dt \\ &= -\frac{2}{9} \int_2^3 (t^2 - 7t + 10) dt \\ &= -\frac{2}{9} \left[\frac{t^3}{3} - \frac{7t^2}{2} + 10t \right]_2^3 \\ &= -\frac{2}{9} \left(9 - \frac{63}{2} + 30 - \frac{8}{3} + 14 - 20 \right) \\ &= -\frac{2}{9} \left(-\frac{7}{6} \right) \\ \Pr(T \leq 3) &= \frac{7}{27} \end{aligned}$$

2 marks

- c Find the modal time in which symptoms will appear after contact with the virus.

Mode = t -value where max exists
Mode = 3.5

2 marks

(Total: 6 marks)

QUESTION 5

- a Find the anti-derivative with respect to x of e^{2x-1} .

$$\int e^{2x-1} dx = \frac{1}{2} e^{2x-1} + c$$

1 mark

- b Find the anti-derivative with respect to x of $f'(x) = e^{2x-1}$ if $f(1) = 2$.

$$\begin{aligned} \int e^{2x-1} dx &= \frac{1}{2} e^{2x-1} + c \\ \text{Given } f(1) &= 2, \\ \Rightarrow 2 &= \frac{1}{2} e^1 + c \\ \Rightarrow c &= 2 - \frac{1}{2} e \\ \text{So } \int e^{2x-1} dx &= \frac{1}{2} e^{2x-1} + 2 - \frac{1}{2} e \end{aligned}$$

2 marks

(Total: 3 marks)

QUESTION 6

Consider the following functions.

$$f: D \rightarrow \mathbf{R}, f(x) = \log_e(x-1) \text{ and}$$

$$g: \mathbf{R} \rightarrow \mathbf{R}, g(x) = x^2$$

- a For the maximal domain of $f(x)$, find D .

$$\begin{aligned} f(x) &= \log_e(x-1) \text{ exists for } (1, \infty). \\ D &= (1, \infty) \end{aligned}$$

1 mark

- b For $f(g(x))$ to be defined, find the rule and domain of $f(g(x))$.

For $f(g(x))$ to exist, test $\text{ran}(g) \subseteq \text{dom}(f)$.

$$[0, \infty) \not\subseteq (1, \infty)$$

For $f(g(x))$ to exist, we need to restrict the range of g to $(1, \infty)$.

$$\text{Domain } f(g(x)) = \text{dom } g(x) = (-\infty, -1) \cup (1, \infty)$$

$$\text{Rule: } f(g(x)) = \log_e(x^2 - 1)$$

$$\text{Domain } f(g(x)) = (-\infty, -1) \cup (1, \infty)$$

3 marks

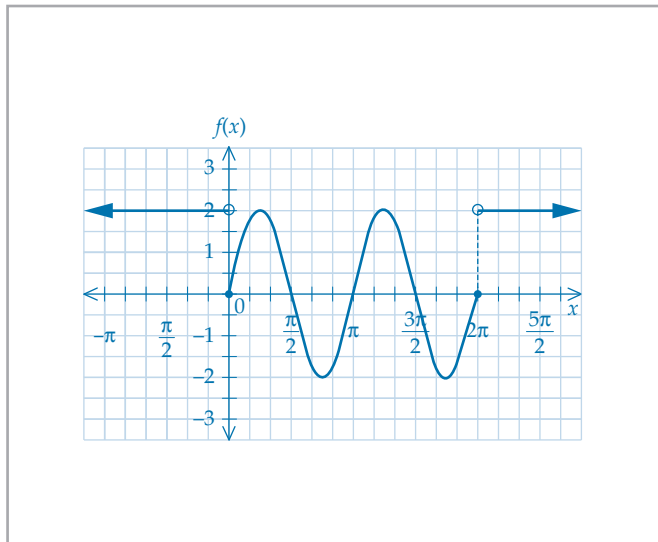
(Total: 4 marks)

QUESTION 7

Given the hybrid function

$$f(x) = \begin{cases} -2 \sin\left(2\left(x - \frac{\pi}{2}\right)\right), & 0 \leq x \leq 2\pi \\ 2, & x < 0 \cup x > 2\pi \end{cases}$$

a Sketch the graph of $f(x)$



2 marks

b State the domain for which $f'(x)$ is defined.

$$f'(x) \text{ is defined for } \mathbb{R} \setminus \{0, 2\pi\}.$$

1 mark

c With the help of your graph, solve the equation $f(x) = 2$.

$$f(x) = 2 \text{ for } x < 0, x > 2\pi \text{ and } x = \frac{\pi}{4}, \frac{5\pi}{4}$$

2 marks

(Total: 5 marks)

QUESTION 8

Solve the equation $\log_e(2x) + \log_e(x) = 5$.

$$\log_e(2x) + \log_e(x) = 5$$

$$\Rightarrow \log_e(2x^2) = 5$$

$$\Leftrightarrow e^5 = 2x^2$$

$$x^2 = \frac{e^5}{2}$$

$$\Rightarrow x = \pm \sqrt{\frac{e^5}{2}} \quad \text{reject negative solution}$$

$$x = \sqrt{\frac{e^5}{2}}$$

2 marks

QUESTION 9

A discrete random variable is defined as follows.

x	0	1	2	3
$\Pr(X = x)$	0.2	$2a$	0.3	a

a Find a .

$$0.2 + 2a + 0.3 + a = 1$$

$$3a + 0.5 = 1$$

$$a = \frac{1}{6}$$

1 mark

b Hence find the mean of the distribution.

$$\begin{aligned} E(X) &= (0 \times 0.2) + (1 \times \frac{1}{3}) + (2 \times 0.3) + (3 \times \frac{1}{6}) \\ &= \frac{43}{30} \end{aligned}$$

2 marks

(Total: 3 marks)