

STUDENT NUMBER

LETTER

Figures

Words

MATHEMATICAL METHODS

Written examination 1

Reading time: 15 minutes

Writing time: 1 hour

QUESTION AND ANSWER BOOK

Number of questions	Number of questions to be answered	Marks
11	11	40

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners, rulers.
- Students are NOT permitted to bring into the examination room: notes of any kind, blank sheets of paper, white out liquid/tape or a calculator of any type.

Materials supplied

- Question and answer book of 11 pages, with a detachable sheet of miscellaneous formulas.
- Working space is provided throughout the book.

Instructions

- Detach the formula sheet from the back of this book during reading time.
- Write your **student number** in the space provided above on this page.
- All written responses must be in English.
- Unless otherwise indicated, the diagrams in this book are not drawn to scale.

At the end of the examination

- You may keep this question book.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.

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QUESTION 1

Let $f(x) = (x^2 + x) \log_e(x + 1)$. Find $f'(x)$.

$$\begin{aligned}f(x) &= (x^2 + x) \log_e(x + 1) \\f'(x) &= \log_e(x + 1) \times (2x + 1) + (x^2 + x) \times \frac{1}{x + 1} \\&= (2x + 1) \log_e(x + 1) + \frac{x(x + 1)}{x + 1} \\&= (2x + 1) \log_e(x + 1) + x\end{aligned}$$

2 marks

QUESTION 2

Let $y = \frac{e^{x^2-1}}{\cos(2x-\pi)}$. Find $\frac{dy}{dx}$ at $x = \pi$

$$\begin{aligned}y &= \frac{e^{x^2-1}}{\cos(2x-\pi)} \\\frac{dy}{dx} &= \frac{\cos(2x-\pi) \times 2xe^{x^2-1} + 2\sin(2x-\pi) \times e^{x^2-1}}{\cos^2(2x-\pi)} \\\text{At } x &= \pi \\\frac{dy}{dx} &= \frac{\cos(\pi) \times 2\pi e^{\pi^2-1} + 2\sin(\pi) \times e^{\pi^2-1}}{\cos^2(\pi)} \\&= \frac{-1 \times 2\pi e^{\pi^2-1} - 0}{1} \\&= -2\pi e^{\pi^2-1}\end{aligned}$$

2 marks

QUESTION 3

Let $\Pr(A) = 0.1$, $\Pr(B) = 0.3$

a If A and B are independent events, find $\Pr(A \cup B)$.

$$\begin{aligned}\Pr(A \cup B) &= \Pr(A) + \Pr(B) - \Pr(A \cap B) \\\text{And } \Pr(A \cap B) &= \Pr(A) \times \Pr(B) = 0.1 \times 0.3 = 0.03 \\\Pr(A \cup B) &= 0.1 + 0.3 - 0.03 \\&= 0.37\end{aligned}$$

1 mark

- b A is the event of winning a particularly difficult football game. B is the event of it being wet on the day of the game. If A and B are no longer independent events, the probability of winning the game on a wet day reduces to 0.05. Find the probability that, if the game is won, it is a wet day.

$$\Pr(A | B) = 0.05$$

$$\Pr(A | B) = \frac{\Pr(A \cap B)}{\Pr(B)}$$

$$\text{So } 0.05 = \frac{\Pr(A \cap B)}{\Pr(B)}$$

$$\Rightarrow \Pr(A \cap B) = 0.05 \times 0.3 = 0.015$$

$$\text{Now } \Pr(B | A) = \frac{\Pr(A \cap B)}{\Pr(A)}$$

$$= \frac{0.015}{0.1}$$

$$= 0.15$$

The probability that, if the game is won, it is a wet day is 0.15.

3 marks
(Total: 4 marks)

QUESTION 4

A probability density function is defined below.

$$f(x) = \begin{cases} \frac{x}{8} + k, & -1 \leq x \leq 1, \\ 0, & \text{otherwise,} \end{cases} \quad \text{where } k \in \mathbb{R}$$

- a Find k .

$$f(x) = \begin{cases} \frac{x}{8} + k, & -1 \leq x \leq 1, \\ 0, & \text{otherwise,} \end{cases} \quad \text{where } k \in \mathbb{R}$$

$$\text{For a PDF, } \int_{-1}^1 \left(\frac{x}{8} + k \right) dx = 1$$

$$\left[\frac{x^2}{16} + kx \right]_{-1}^1 = 1$$

$$\left(\frac{1}{16} + k \right) - \left(\frac{1}{16} - k \right) = 1$$

$$2k = 1$$

$$k = \frac{1}{2}$$

2 marks

- b Find the mean of the probability density function.

$$\begin{aligned}
 \text{Mean} = E(X) &= \int_{-1}^1 x \left(\frac{x}{8} + k \right) dx \quad \text{where } k = \frac{1}{2} \\
 &= \int_{-1}^1 \left(\frac{x^2}{8} + \frac{x}{2} \right) dx \\
 &= \left[\frac{x^3}{24} + \frac{x^2}{4} \right]_{-1}^1 \\
 &= \left(\frac{1}{24} + \frac{1}{4} \right) - \left(-\frac{1}{24} + \frac{1}{4} \right) \\
 &= \frac{1}{12}
 \end{aligned}$$

2 marks

- c Find the median of the probability density function.

$$\begin{aligned}
 \int_{-1}^m \left(\frac{x}{8} + \frac{1}{2} \right) dx &= \frac{1}{2} \quad \text{where } m \text{ is the median} \\
 \left[\frac{x^2}{16} + \frac{x}{2} \right]_{-1}^m &= \frac{1}{2} \\
 \left(\frac{m^2}{16} + \frac{m}{2} \right) - \left(\frac{1}{16} - \frac{1}{2} \right) &= \frac{1}{2} \\
 \frac{m^2}{16} + \frac{m}{2} + \frac{7}{16} &= \frac{1}{2} \\
 m^2 + 8m + 7 &= 8 \\
 m^2 + 8m - 1 &= 0 \\
 m &= \frac{-8 \pm \sqrt{64 + 4}}{2} = \frac{-8 \pm 2\sqrt{17}}{2} = -4 \pm \sqrt{17} \\
 \text{Select the } m \text{ value that is within the domain.} \\
 m &= -4 + \sqrt{17}
 \end{aligned}$$

2 marks

(Total: 6 marks)

QUESTION 5

- a Find $\frac{d}{dx}(x^2 \log_e(x))$.

$$\begin{aligned}
 \frac{d}{dx}(x^2 \log_e(x)) &= 2x \log_e(x) + x^2 \times \frac{1}{x} \\
 &= 2x \log_e(x) + x
 \end{aligned}$$

1 mark

b Hence find $\int_1^2 4x \log_e(x) dx$.

$$\begin{aligned}\frac{d}{dx}(x^2 \log_e(x)) &= 2x \log_e(x) + x \\ \Rightarrow \int (2x \log_e(x) + x) dx &= x^2 \log_e(x) + \frac{1}{2}x^2 + c \\ \Rightarrow \int_1^2 (2x \log_e(x) + x) dx &= [x^2 \log_e(x) + \frac{1}{2}x^2]_1^2 \\ \int_1^2 (2x \log_e(x)) dx + \int_1^2 x dx &= [x^2 \log_e(x) + \frac{1}{2}x^2]_1^2 \\ \int_1^2 (2x \log_e(x)) dx &= [x^2 \log_e(x) + \frac{1}{2}x^2]_1^2 - \int_1^2 x dx \\ \int_1^2 (4x \log_e(x)) dx &= 2[x^2 \log_e(x) + \frac{1}{2}x^2]_1^2 - 2 \int_1^2 x dx \\ &= 2(4 \log_e(2) + 2) - 2 \left[\frac{x^2}{2} \right]_1^2 \\ &= 8 \log_e(2) + 4 - (4 - 1) \\ \int_1^2 (4x \log_e(x)) dx &= 8 \log_e(2) + 1\end{aligned}$$

3 marks
(Total: 4 marks)

QUESTION 6

a Show that $P(x) = 3x^3 - x^2 - 2$ has only one real factor.

$$\begin{aligned}P(x) &= 3x^3 - x^2 - 2 \\ P(1) &= 3 - 1 - 2 = 0, \text{ so } x - 1 \text{ is a factor.} \\ \text{By synthetic division} \\ 1 &\begin{array}{r|rrrr} & 3 & -1 & 0 & -2 \\ & & 3 & 2 & 2 \\ \hline & 3 & 2 & 2 & 0 \end{array} \\ \text{The quadratic factor is } &(3x^2 + 2x + 2). \\ \Delta &= 2^2 - 4 \times 3 \times 2 = -20 \\ \text{So there are no real factors for the quadratic.} \\ \therefore (x - 1) &\text{ is the only real factor.}\end{aligned}$$

2 marks

- b Find the equation of the tangent to the curve $y = 3x^3 - x^2 - 2$ at $x = 1$.

$$y = 3x^3 - x^2 - 2$$

$$\text{So } \frac{dy}{dx} = 9x^2 - 2x$$

Gradient at $x = 1$ is 7.

Use the equation of a line $y - y_1 = m(x - x_1)$, where $m = \text{gradient of curve}$.

Using point $(1, 0)$,

$$y - 0 = 7(x - 1)$$

The equation of the tangent is

$$y = 7x - 7$$

2 marks

- c Hence, find the coordinates of the point where the tangent to the curve $y = 3x^3 - x^2 - 2$ at $x = 1$ intersects again with the curve $y = 3x^3 - x^2 - 2$.

$$\text{Upper curve: } y = 3x^3 - x^2 - 2$$

$$\text{Lower curve: } y = 7x - 7$$

Point of intersection

$$\text{Equate } 3x^3 - x^2 - 2 = 7x - 7$$

$$3x^3 - x^2 - 7x + 5 = 0$$

We already know that one point of intersection is $(1, 0)$.

By synthetic division

$$\begin{array}{r|rrrr} 1 & 3 & -1 & -7 & 5 \\ & & 3 & 2 & -5 \\ \hline & 3 & 2 & -5 & 0 \end{array}$$

The quadratic factor is $(3x^2 + 2x - 5)$.

$$\text{So } y = (x - 1)(x - 1)(3x + 5)$$

$$\text{2nd point of intersection is } \left(-\frac{5}{3}, -\frac{56}{3}\right).$$

Alternate solution

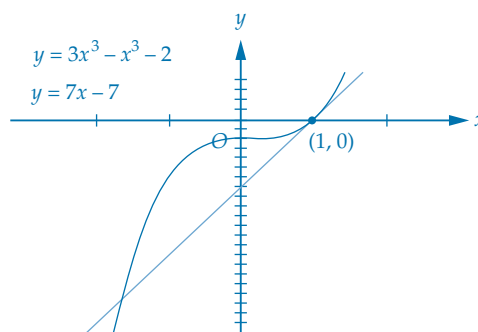
We know that $x = 1$ meets at the tangent to the curve, so we expect a repeated factor.

$$\text{So } y = (x - 1)^2(\text{linear factor})$$

$$= (x^2 - 2x + 1)(\text{linear factor})$$

By observation of $y = 3x^3 - x^2 - 7x + 5$, the linear factor must be $(3x + 5)$.

$$\text{The 2nd point of intersection is } \left(-\frac{5}{3}, -\frac{56}{3}\right).$$



2 marks

(Total: 6 marks)

QUESTION 7

Consider the graph of $y = 2 \sin\left(x - \frac{\pi}{2}\right) + 1$.

- a** Find the x -intercepts of the graph for the domain $x \in [0, 2\pi]$.

Solve $2 \sin\left(x - \frac{\pi}{2}\right) + 1 = 0$ for $x \in [0, 2\pi]$.

$$2 \sin\left(x - \frac{\pi}{2}\right) = -1 \Rightarrow \sin\left(x - \frac{\pi}{2}\right) = -\frac{1}{2}$$

Reference angle $= \frac{\pi}{6}$

$$\begin{aligned}\left(x - \frac{\pi}{2}\right) &= -\frac{\pi}{6}, \pi + \frac{\pi}{6}, 2\pi - \frac{\pi}{6} \\ &= -\frac{\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}\end{aligned}$$

$$\begin{aligned}x &= -\frac{\pi}{6} + \frac{\pi}{2}, \frac{7\pi}{6} + \frac{\pi}{2}, \frac{11\pi}{6} + \frac{\pi}{2} \\ &= \frac{2\pi}{6}, \frac{10\pi}{6}, \frac{14\pi}{6}\end{aligned}$$

$$= \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3} \text{ where } \frac{7\pi}{3} \text{ is now out of the domain } [0, 2\pi]$$

$$x = \frac{\pi}{3}, \frac{5\pi}{3}$$

2 marks

- b** Hence find the x -intercepts of the graph of $y = 2 \sin\left(x - \frac{\pi}{2}\right) + 1$ after it is translated $\frac{\pi}{4}$ units in the positive direction of the x -axis and then reflected over the x -axis.

$$x = \frac{\pi}{3}, \frac{5\pi}{3}$$

Translated $\frac{\pi}{4}$ units in the positive direction of the x -axis:

$$\begin{aligned}x &= \frac{\pi}{3} + \frac{\pi}{4}, \frac{5\pi}{3} + \frac{\pi}{4} \\ &= \frac{7\pi}{12}, \frac{23\pi}{12}\end{aligned}$$

Reflected over the x -axis, intercepts remain the same.

$$x = \frac{7\pi}{12}, \frac{23\pi}{12}$$

2 marks

(Total: 4 marks)

QUESTION 8

Sam finds that, on average, he solves an equation correctly 7 out of 10 times. In a particular examination, Sam is faced with 4 such equations. What is the probability that Sam will solve at least one of these equations correctly?

$$\begin{aligned}\text{Bi}(n, p) &= \text{Bi}(4, 0.7) \\ \Pr(X \geq 1) &= 1 - \Pr(X = 0) \\ 1 - \Pr(X = 0) &= 1 - {}^4C_0(0.3)^4(0.7)^0 \\ &= 1 - (0.3)^4 \\ &= 1 - 0.0081 \\ &= 0.9919\end{aligned}$$

2 marks

QUESTION 9

The curve $y = 2x^2 + 3$ for $x \geq 0$ is added to the curve $y = px^2 - 6$ for $x < 2$ to create the function $f(x) = 2x^2 + 3 + px^2 - 6$ where p is a real constant.

- a State the domain for which $f(x)$ exists.

$$\begin{aligned}\text{Intersection of domains } x \geq 0 \text{ and } x < 2. \\ x \in [0, 2)\end{aligned}$$

1 mark

- b Find the value(s) of p for which the inverse, $f^{-1}(x)$, exists.

$$\begin{aligned}f(x) &= 2x^2 + 3 + px^2 - 6 \\ y &= (2 + p)x^2 - 3 \\ \text{Interchange } x \text{ and } y \text{ to find the inverse.} \\ x &= (2 + p)y^2 - 3 \\ y^2 &= \sqrt{\frac{x+3}{p+2}} \\ y &= \pm \sqrt{\frac{x+3}{p+2}} \\ \text{Select the +ve branch because of the domain of } f(x). \\ f^{-1}(x) &= \sqrt{\frac{x+3}{p+2}} \\ f^{-1}(x) \text{ exists for } p+2 > 0. \\ \therefore p &> -2\end{aligned}$$

2 marks

(Total: 3 marks)

QUESTION 10

Consider the functions with maximal domains: $f(x) = x^2$ and $g(x) = \log_e(x)$.

- a State, with a reason, if $f(g(x))$ exists.

For $f(g(x))$, test $\text{ran}(\text{inner}) \subseteq \text{dom}(\text{outer})$.

$$\mathbf{R} \subseteq \mathbf{R}$$

$\therefore f(g(x))$ exists.

1 mark

- b State, with a reason, if $g(f(x))$ exists.

For $g(f(x))$, test $\text{ran}(\text{inner}) \subseteq \text{dom}(\text{outer})$.

$$[0, \infty) \not\subseteq (0, \infty)$$

$\therefore g(f(x))$ does not exist.

1 mark

- c Define $h'(x)$ if $h(x) = f(g(x))$

$h(x) = f(g(x))$, which exists

$$h(x) = (\log_e(x))^2$$

$$h'(x) = 2 \log_e(x) \times \frac{1}{x}$$

$$= \frac{2}{x} \log_e(x)$$

Domain $f(g(x)) = \text{domain } g(x) = (0, \infty)$.

Domain $h'(x) = (0, \infty)$

$$h': (0, \infty) \rightarrow \mathbf{R}, h'(x) = \frac{2}{x} \log_e(x)$$

2 marks

(Total: 4 marks)

QUESTION 11

From a sample of 60 Year 12 students, 45 said they like chocolate. Estimate the probability of Year 12 students liking chocolate and the variance of the sampling distribution.

$$p \approx \hat{p} = \frac{45}{60} = \frac{3}{4}$$

$$\text{Var}(\hat{p}) \approx \frac{\hat{p}(1-\hat{p})}{n} = \frac{\frac{3}{4}\left(1-\frac{3}{4}\right)}{60} = \frac{\frac{3}{16}}{60}$$

$$\text{Var}(\hat{p}) = \frac{1}{320}$$

The probability of Year 12 students liking chocolate is about 0.75, with a variance of $\frac{1}{320}$.

3 marks

Mathematical Methods Formulas

Mensuration

area of trapezium:	$\frac{1}{2}(a+b)h$
curved surface area of a cylinder:	$2\pi rh$
volume of a cylinder:	$\pi r^2 h$
volume of a cone:	$\frac{1}{3}\pi r^2 h$
volume of a pyramid:	$\frac{1}{3}Ah$
volume of a sphere:	$\frac{4}{3}\pi r^3$
area of a triangle:	$\frac{1}{2}bc \sin A$

Calculus

$\frac{d}{dx}(x^n) = nx^{n-1}$	$\int x^n dx = \frac{1}{n+1}x^{n+1} + c, n \neq -1$
$\frac{d}{dx}(e^{ax}) = ae^{ax}$	$\int e^{ax} dx = \frac{1}{a}e^{ax} + c$
$\frac{d}{dx}(\log_e(x)) = \frac{1}{x}$	$\int \frac{1}{x} dx = \log_e x + c$
$\frac{d}{dx}(\sin(ax)) = a \cos(ax)$	$\int \sin(ax) dx = -\frac{1}{a} \cos(ax) + c$
$\frac{d}{dx}(\cos(ax)) = -a \sin(ax)$	$\int \cos(ax) dx = \frac{1}{a} \sin(ax) + c$
$\frac{d}{dx}(\tan(ax)) = \frac{a}{\cos^2(ax)} = a \sec^2(ax)$	
product rule: $\frac{d}{dx}(uv) = u \frac{dv}{dx} + v \frac{du}{dx}$	quotient rule: $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$
chain rule: $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$	approximation: $f(x+h) \approx f(x) + hf'(x)$

Probability

$\Pr(A) = 1 - \Pr(A')$	$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$
$\Pr(A B) = \frac{\Pr(A \cap B)}{\Pr(B)}$	variance: $\text{Var}(X) = \sigma^2 = E((X - \mu)^2) = E(X^2) - \mu^2$
mean: $\mu = E(X)$	

Probability distribution		Mean	Variance
Discrete	$\Pr(X = x) = p(x)$	$\mu = \sum x p(x)$	$\sigma^2 = \sum (x - \mu)^2 p(x)$
Continuous	$\Pr(a < X < b) = \int_a^b f(x) dx$	$\mu = \int_{-\infty}^{\infty} x f(x) dx$	$\sigma^2 = \int_{-\infty}^{\infty} (x - \mu)^2 f(x) dx$