TEST 4

Technology active end-of-year examination
Functions and graphs, Algebra, Calculus
Section A: 16 marks
Section B: 30 marks
Suggested writing time: 70 minutes

Section A: Multiple-choice questions

Specific instructions to students

- A correct answer scores 1 mark, and an incorrect answer scores 0.
- Marks are not deducted for incorrect answers.
- No marks are given if more than one letter is shaded on the answer sheet.
- Choose the alternative that most correctly answers the question and mark your choice on the multiple-choice answer section at the bottom of each page, as shown in the example below.

1 A B C D **USE PENCIL ONLY**

- Use pencil only.
- Working space is provided under those questions that require working out.

QUESTION 1

If $y = 3\log_e\left(\sqrt{x}\right)$, $\dfrac{dy}{dx} =$

A $\dfrac{3}{2x}$

B $\dfrac{3}{2\sqrt{x}}$

C $\dfrac{3}{x}$

D $3\sqrt{x}\log_e\left(\sqrt{x}\right)$

E $\dfrac{3}{2}$

$y = 3\log_e\left(\sqrt{x}\right) = 3\log_e\left(x^{\frac{1}{2}}\right) = \dfrac{3}{2}\log_e(x)$

$\dfrac{dy}{dx} = \dfrac{3}{2x}$

$\dfrac{d}{dx}\left(3\cdot\ln\left(\sqrt{x}\right)\right)$

$\dfrac{3}{2\cdot x}$

QUESTION 2

Find $f'(x)$ if $f(x) = e^x - 3e^{-x}$

A $f'(x) = e^x - 3e^{-x}$

B $f'(x) = e^x + 3e^{-x}$

C $f'(x) = e^x + 3$

D $f'(x) = -2e^{-x}$

E $f'(x) = 6e^{2x}$

$f(x) = e^x - 3e^{-x}$

$f'(x) = e^x + 3e^{-x}$

QUESTION 3

Find $f'(\pi)$ if $f(x) = 4x^3 \sin(2x)$

A $4x^2(3\sin(2x) + 2x\cos(2x))$

B $4\pi^2(3\pi + 2\pi^2)$

C 0

D $4\pi^2$

E $8\pi^3$

$f(x) = 4x^3\sin(2x)$

$f'(\pi) = 8\pi^3$

diff$(4\cdot x^3\cdot\sin(2\cdot x), x, 1, \pi)$

$8\cdot\pi^3$

QUESTION 4

If $f'(x) = 3 - e^{2x-2}$, the anti-derivative function $f(x)$ is equal to

A $3x - 2e^{2x-2} + c$

B $3x - \dfrac{1}{2}e^{2x-2}$

C $3x - \dfrac{1}{2}e^{2x-2} + c$

D $\dfrac{1}{2}e^{2x-2} + c$

E $-2e^{2x-2} + c$

$f'(x) = 3 - e^{2x-2}$

$f(x) = 3x - \dfrac{1}{2}e^{2x-2} + c$

$\int 3 - e^{2\cdot x - 2}\,dx$

$\dfrac{-e^{2\cdot x-2}}{2} + 3\cdot x$

QUESTION 5

For $f(x) = \begin{cases} x+1, & x \geq 0 \\ e^x, & x < 0 \end{cases}$ $\quad f'(x) =$

A $\begin{cases} x, & x \geq 0 \\ e^x, & x < 0 \end{cases}$

B $\begin{cases} 1, & x \geq 0 \\ e^x, & x < 0 \end{cases}$

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C $\begin{cases} x, & x > 0 \\ e^x, & x < 0 \end{cases}$

D $\begin{cases} 1, & x > 0 \\ e^x, & x < 0 \end{cases}$

E $\begin{cases} 1, & e^x > 0 \\ e^x, & e^x < 0 \end{cases}$

$f(x) = \begin{cases} x+1, & x \geq 0 \\ e^x, & x < 0 \end{cases}$

$f'(x) = \begin{cases} 1, & x > 0 \\ e^x, & x < 0 \end{cases}$

QUESTION 6

For $f(x) = \begin{cases} x+1, & x \geq 0 \\ e^x, & x < 0 \end{cases}$ $\qquad f'(-1) =$

A $\begin{cases} 0, & x \geq 0 \\ e^{-1}, & x < 0 \end{cases}$

B $\begin{cases} 1, & x \geq 0 \\ e^{-1}, & x < 0 \end{cases}$

C e^{-1}

D 0

E 1

$f(x) = \begin{cases} x+1, & x \geq 0 \\ e^x, & x < 0 \end{cases}$

$x = -1$ is in the domain $x < 0$.

$f'(x) = e^x$

$f'(-1) = e^{-1}$

QUESTION 7

$P(x) = ax^3 + bx^2 - cx - 1$ is a cubic polynomial and has factors of $(x - 1)$ and $(x + 3)$. The function $f(x) = ax^3 + bx^2 - cx - 1$ has a stationary point at $x = 0$. The values of a, b and c are

A $a = 0, b = \frac{5}{9}, c = \frac{7}{9}$

B $a = \frac{2}{9}, b = \frac{7}{9}, c = 0$

C $a = \frac{2}{9}, b = \frac{7}{9}, c = c$

D no solution possible

E $a = 1, b = -2, c = 0$

$P(x) = ax^3 + bx^2 - cx - 1$ has factors of $(x - 1)$ and $(x + 3)$, and a stationary point at $x = 0$.

$\Rightarrow f(1) = 0, f(-3) = 0$ and $f'(0) = 0$.

Solving simultaneously,

```
define f(x)=ax^3+bx^2-cx-1
                                done
define g(x)=d/dx(f(x))
                                done
{f(1)=0
{f(-3)=0
{g(0)=0 |a,b,c
                  {a=2/9, b=7/9, c=0}
```

$a = \frac{2}{9}, b = \frac{7}{9}, c = 0$

QUESTION 8

$f(x) = 3\cos(x)$ and $g(x) = 2\log_e(x)$ for maximal domains. The function $f(g(x))$ can be defined as

A $f(g(x)) = 3\cos(2\log_e(x)), x \in R$

B $f(g(x)) = 2\log_e(3\cos(x)), x \in R$

C $f(g(x)) = 3\cos(2\log_e(x)), x \in (0, \infty)$

D $f(g(x)) = 2\log_e(\cos(x)) + 2\log_e(3), x \in [-1, 1]$

E $f(g(x)) = 2\log_e(\cos(x)) + 2\log_e(3), x \in [0, \infty)$

$f(x) = 3\cos(x)$ and $g(x) = 2\log_e(x)$

To find dom $f(g(x))$,

test ran $(g) \subseteq$ dom (f)

$\boldsymbol{R} \subseteq \boldsymbol{R}$

dom $f(g(x))$ = dom $g(x) = (0, \infty)$

Rule: $f(g(x)) = 3\cos(2\log_e(x))$ with domain $x \in (0, \infty)$.

QUESTION 9

The graph of $f: (1, \infty) \to \boldsymbol{R}, f(x) = \frac{1}{(x-1)^2}$ has its inverse $f^{-1}(x)$ reflected over the y-axis. The image of $f^{-1}(x)$ after this transformation is

A $y = \frac{1}{(x-1)^2}$

B $y = \frac{1}{\sqrt{x}} + 1$

C $y = -\frac{1}{\sqrt{x}} + 1$

D $y = \frac{1}{\sqrt{-x}} + 1$

E $y = -\frac{1}{\sqrt{-x}} + 1$

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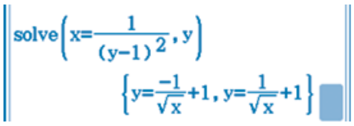
$f\colon (1, \infty) \to \boldsymbol{R}, f(x) = \dfrac{1}{(x-1)^2}$

The inverse is found by swapping x and y and selecting the +ve solution for the domain given.



$f^{-1}(x) = \dfrac{1}{\sqrt{x}} + 1$

The image of $f^{-1}(x)$ after being reflected over the y-axis is

$y = \dfrac{1}{\sqrt{-x}} + 1$

QUESTION 10

If $T\colon \boldsymbol{R}^2 \to \boldsymbol{R}^2$, $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is applied to the function $y = \log_e(x)$, the image function can be expressed as

A $y = 2\log_e(x-1) - 1$

B $y = 2\log_e(x-2) + 1$

C $y = 2\log_e(x-2) - 1$

D $y = 0.5\log_e(1-x) + 1$

E $y = 2\log_e(1-x) - 1$

$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ to $y = \log_e(x)$

$\begin{bmatrix} -x+1 \\ 2y-1 \end{bmatrix} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$

So

$x = 1 - x_1$

$y = 0.5(y_1 + 1)$

So $y = \log_e(x) \Rightarrow 0.5(y_1 + 1) = \log_e(1 - x_1)$

$y_1 + 1 = 2\log_e(1 - x_1)$

$y = 2\log_e(1 - x) - 1$

The image is $y = 2\log_e(1-x) - 1$

QUESTION 11

$\int_0^1 (3x^2 + 2x - 2)^2\,dx =$

A 2.1333

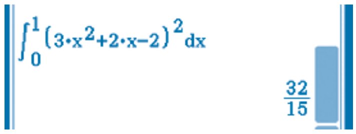
B $\dfrac{32}{15}$

C 0

D $\dfrac{1112}{15}$

E 1.262

$\int_0^1 (3x^2 + 2x - 2)^2\,dx = \dfrac{32}{15}$



QUESTION 12

If $f(x) = -\sqrt{x+1} + 2$ and $g(x) = e^x$, both for maximal domains, then the domain of the composite function $g(f(x))$ is

A $[-1, \infty)$

B $\boldsymbol{R}$

C $(0, \infty)$

D $[2, \infty)$

E $(-\infty, 2]$

$f(x) = -\sqrt{x+1} + 2$ and $g(x) = e^x$

To find the domain of $g(f(x))$,

test ran $(f) \subseteq$ dom (g)

$(-\infty, 2] \subseteq \boldsymbol{R}$

dom $g(f(x))$ = dom $f(x) = [-1, \infty)$

QUESTION 13

The functionality equation $f(x) + f(y) = f(xy)$ is satisfied by

A $f(x) = 2e^x$

B $f(x) = \dfrac{3}{x}$

C $f(x) = 2\log_e(x)$

D $f(x) = x$

E $f(x) = 2x^2$

$f(x) + f(y) = f(xy)$ looks like it suits a log equation.

Test C: $f(x) = 2\log_e(x)$

LHS $= f(x) + f(y)$

$= 2\log_e(x) + 2\log_e(y)$

$= 2(\log_e(x) + \log_e(y))$

$= 2\log_e(xy)$

RHS $= f(xy) = 2\log_e(xy) =$ LHS

So the rule is $f(x) = 2\log_e(x)$

QUESTION 14

The average rate of change between the points $(0, -1)$ and $(2, -5)$ is

A 2

B 1

C -2

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D -1

E -3

Average rate of change between the points $(0, -1)$ and $(2, -5)$

$= \frac{-5+1}{2-0} = \frac{-4}{2} = -2$

QUESTION 15

The average rate of change between the points $x = -\pi$ and $x = 2\pi$ on the curve with the equation $y = \cos^2(x)$ is

A $\frac{1}{3\pi}$

B $-\frac{1}{3\pi}$

C $-\frac{\pi}{2}$

D $\frac{1}{\pi}$

E 0

$y = \cos^2(x)$ has points $(-\pi, 1)$ and $(2\pi, 1)$

Average rate of change $= \frac{1-1}{2\pi+\pi} = 0$

QUESTION 16

The gradient of the chord PQ for the function $f(x) = x^2 + 3x$ from the x-coordinates of P, where $x = 1$ and Q where $x = 1 + h$, can be expressed as

A $\lim\limits_{h\to 0}\frac{f(1+h)-f(1)}{h}$

B $\frac{f(1)-f(1+h)}{h}$

C $\lim\limits_{x\to h}\frac{f(1+h)-f(1)}{h}$

D $\frac{(1+h)^2+3(1+h)}{h}$

E $5+h$

Gradient of the chord PQ, for $f(x) = x^2 + 3x$ from $x = 1$ to $x = 1 + h$ is

$= \frac{f(1+h)-f(1)}{h}$

$= \frac{(1+h)^2+3(1+h)-(1+3)}{h}$

$= \frac{2h+h^2+3h}{h}$

$= 5 + h$

ONE ANSWER PER LINE **USE PENCIL ONLY**

1	■	B	C	D	E	9	A	B	C	■	E
2	A	■	C	D	E	10	A	B	C	D	■
3	A	B	C	D	■	11	A	■	C	D	E
4	A	B	■	D	E	12	■	B	C	D	E
5	A	B	C	■	E	13	A	B	■	D	E
6	A	B	■	D	E	14	A	B	■	D	E
7	A	■	C	D	E	15	A	B	C	D	■
8	A	B	■	D	E	16	A	B	C	D	■

Section B: Extended-response questions

Specific instructions to students

- Answer all of the questions in the spaces provided.
- Show all workings in questions where more than one mark is available.
- An exact value must be provided in questions where a numerical answer is required, unless otherwise specified.

QUESTION 17

a Find $\frac{dy}{dx}$ if $y = 3x\cos(x)$.

$y = 3x\cos(x)$

$\frac{dy}{dx} = 3\cos(x) - 3x\sin(x)$

1 mark

b Hence find an expression for $\int x\sin(x)\,dx$.

From part **a**,

$\int 3\cos(x) - 3x\sin(x)\,dx = 3x\cos(x)$

This gives

$3\int x\sin(x)dx = 3\int\cos(x)dx - 3x\cos(x)$

$\int x\sin(x)dx = \int\cos(x)dx - x\cos(x)$

$= \sin(x) - x\cos(x)$

3 marks

c Evaluate $\int_0^{\pi} x\sin(x)dx$.

$\int x\sin(x)dx = \sin(x) - x\cos(x)$

$\int_0^{\pi} x\sin(x)dx = [\sin(x) - x\cos(x)]_0^{\pi}$

$= (\sin(\pi) - \pi\cos(\pi)) - (0-0)$

$= \pi$

1 mark

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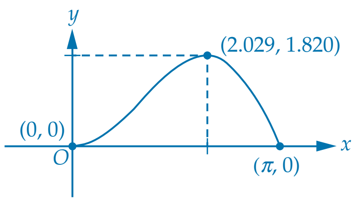
Consider the curve of $f(x) = x \sin(x)$.

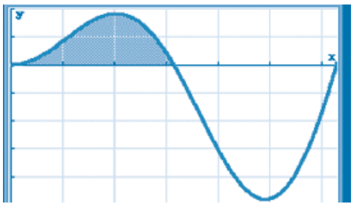
d Sketch the graph of $f: [0, \pi] \to \boldsymbol{R}, f(x) = x \sin(x)$, labelling the coordinates of the endpoints and the local maximum point, correct to 3 decimal places.



2 marks

e Find the area under the curve $f(x) = x \sin(x)$ for $x \in [0, \pi]$.

Graph the curve $f(x) = x \sin(x)$ for the domain $x \in [0, \pi]$.



The area required is all above the x-axis, so area $= \pi$.

1 mark

f Solve the equation $f'(x) = 0$ for $x \in [0, \pi]$, giving your answer correct to 3 decimal places.

$f(x) = x \sin(x)$

$f'(x) = \sin(x) + x \cos(x) = 0$

gives

solve(x·cos(x)+sin(x)=0 | 0≤x≤π, x)
{x=0, x=2.028757838}

Within the domain, turning points are at $x = 0$, $x = 2.029$

So the local maximum is at $x = 2.029$, matching the graph in part **d**.

2 marks

(Total: 10 marks)

QUESTION 18

A particular rock concert emits sound (measured in decibels) that ranges from 90 dB to quite a high level. The function that models the sound at the concert follows the function $f(t) = 10 \sin(\pi t) + 100$, where $t \geq 0$, t hours after the start of the concert at 7 p.m.

a What is the average noise level, in dB, at the rock concert from 7 p.m. to 11 p.m.?

$f(t) = 10 \sin(\pi t) + 100$

$t = 0$ to $t = 4$

Average value $= \dfrac{1}{4-0}\displaystyle\int_0^4 (10\sin(\pi t) + 100)\,dt = 100$ dB

$\int_0^4 \frac{1}{4}\cdot(10\cdot\sin(\pi x)+100)dx$ 100

1 mark

b The concert is so good that the band returns for encores, and the concert finishes at 11.45 p.m. What is the average sound, in dB, correct to 2 decimal places, at the rock concert from 7 p.m. to 11.45 p.m.?

$t = 0$ to $t = 4.75$

Average value $= \dfrac{1}{4.75-0}\displaystyle\int_0^{4.75} (10\sin(\pi t) + 100)\,dt$

$= 101.14$ dB

$\int_0^{4.75} \frac{1}{4.75}\cdot(10\cdot\sin(\pi x)+100)dx$ 101.1439768

1 mark

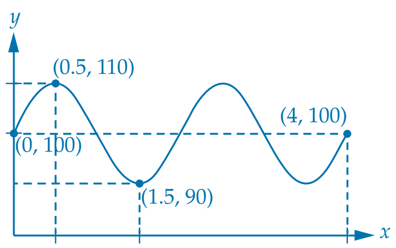
c For $f(t) = 10 \sin(\pi t) + 100$, what is the amplitude and period of the function?

Amplitude = 10

Period $= \dfrac{2\pi}{\pi} = 2$

2 marks

d Sketch the graph of $f(t) = 10 \sin(\pi t) + 100$ for the domain $x \in [0, 4]$, showing the coordinates of the endpoints and the maximum and minimum turning points.



3 marks

The level of sound, in decibels, at which a person may sustain hearing loss is 95 dB.

e For what period of time during the 4-hour concert will the decibel level be dangerous for patrons?

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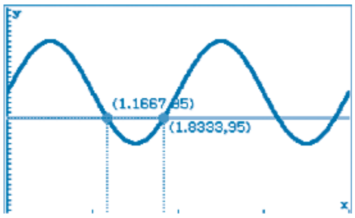
Dangerous ≥ 95

Solve $f(t) \geq 95$



Dangerous for $0 \leq t \leq 1\frac{1}{6}$ and $1\frac{5}{6} \leq t \leq 3\frac{1}{6}$ and $3\frac{5}{6} \leq t \leq 4$.

Dangerous from 7 p.m. to 8:10 p.m., 8:50 p.m. to 10:10 p.m., and 10:50 p.m. to 11 p.m.

2 marks

Normal conversation is held between 60 and 65 dB.

f Give a reason as to why normal conversation cannot be heard during the 4-hour concert.

The minimum noise level of the concert is 90 dB.

A conversation between 60 and 65 dB cannot be heard.

1 mark

g Find $\frac{d}{dt}(10 \sin(\pi t) + 100)$.

$\frac{d}{dt}(10 \sin(\pi t) + 100) = 10\pi \cos(\pi t)$

1 mark

h Hence solve the equation $f'(t) = 0$.

General solution

$10\pi \cos(\pi t) = 0$

$\Rightarrow \cos(\pi t) = 0$

$\Rightarrow \pi t = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \ldots$

$t = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \ldots$

$t = \frac{1}{2} + n, n \in \mathbf{Z}$

2 marks

The loudest recommended exposure with hearing protection is 140 dB. Even short-term exposure can cause permanent hearing loss at this level.

i Using your information from part **h**, explain why there is no possibility of permanent hearing loss for the concert patrons.

Stationary points at $t = \frac{1}{2} + n, n \in \mathbf{Z}$

1st stationary point is a maximum at $t = \frac{1}{2}, f\left(\frac{1}{2}\right) = 110$

The 3rd stationary point is a maximum at $t = \frac{1}{2} + 2 = \frac{5}{2}, f\left(\frac{5}{2}\right) = 110$, etc.

So the maximum noise level during the concert is 110, which is < 140.

So there is no hearing loss.

1 mark

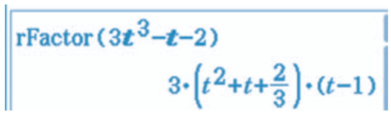
(Total: 14 marks)

QUESTION 19

The velocity of a particle is described by the function $v(t) = 3t^3 - t - 2$, where $v(t)$ is in m/s and t is in seconds, $t \geq 0$.

a By finding the discriminant of a quadratic factor in $v(t)$, show that its graph has only one t-intercept.

$v(t) = 3t^3 - t - 2$



$v(t) = (3t^2 + 3t + 2)(t - 1) = 0$ for t-intercepts.

A quadratic factor is $3t^2 + 3t + 2$

Discriminant $= 9 - 4 \times 3 \times 2 = -15$, so there are no real factors.

The only t-intercept is at $t = 1$.

2 marks

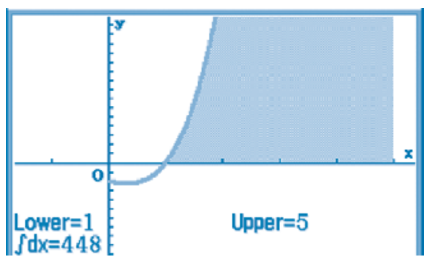
b Find when the particle momentarily stops in its journey.

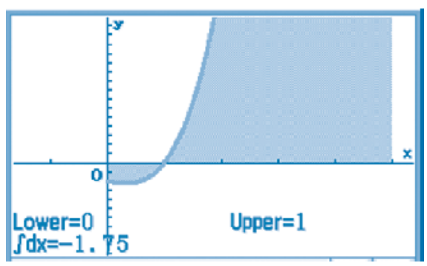
$v'(t) = 9t^2 - 1 = 0$ gives $t = \pm\frac{1}{3}$

Since $t \geq 0$, $t = \frac{1}{3}$.

1 mark

c What distance does the particle travel in the first 5 seconds?





Total distance $= 1.75 + 448 = 449.75$ m

3 marks

(Total: 6 marks)

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