

The Mathematical Association of Victoria
MATHEMATICAL METHODS (CAS)
 SOLUTIONS: Trial Exam 2014

Written Examination 2

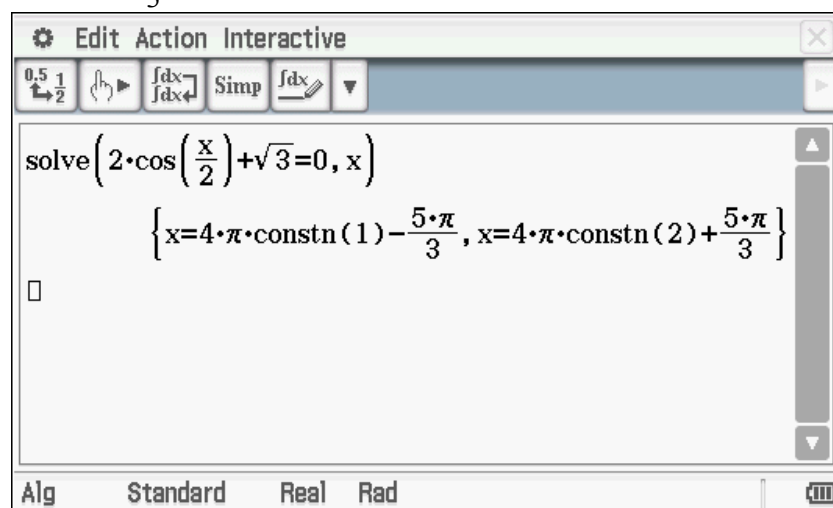
SECTION 1

1. C 2. D 3. C 4. D 5. B 6. A 7. B 8. A 9. A 10. E 11. E
 12. D 13. C 14. C 15. A 16. B 17. E 18. B 19. C 20. D 21. A 22. E

Question 1

$$2\cos\left(\frac{x}{2}\right) + \sqrt{3} = 0$$

$$x = 4\pi n \pm \frac{5\pi}{3}, n \in \mathbb{Z} \quad \text{C}$$



Question 2

Range of $y = -\sin(a(x+b))$ is $[-1, 1]$

Range of $y = -4\sin(a(x+b))$ is $[-4, 4]$

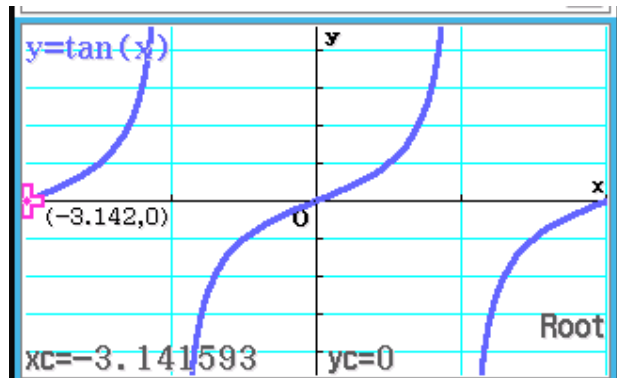
Range of the graph of $f(x) = -4\sin(a(x+b)) - b$ is $[-4-b, 4-b]$ **D**

Question 3

Consider $f: [-\pi, 2\pi] \rightarrow \mathbb{R}$, where $f(x) = 2\pi \tan(x - \pi) + 3$.

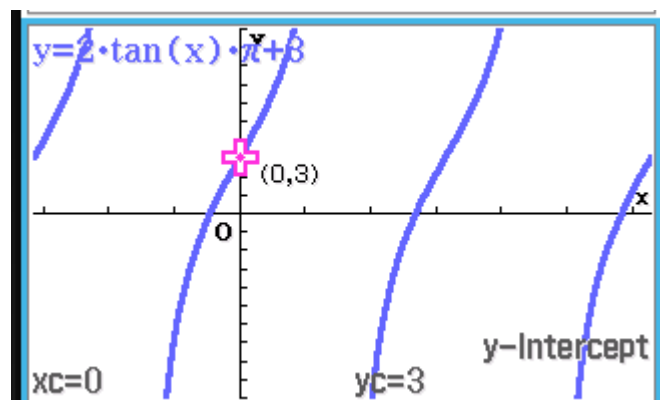
The horizontal translation in a positive direction of π will make no difference to the vertical asymptotes of the graph of $y = 2\pi \tan(x) + 3$.

$y = 2\pi \tan(x - \pi) + 3$ $y = 2 \cdot \tan(x) \cdot \pi + 3$	where the constants $2\pi, 3$ do not alter asymptotes
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The graph of $y = \tan(x)$ from $-\pi$ to π shows expected asymptotes at $x = \pm \frac{\pi}{2}$.

For domain $[-\pi, 2\pi]$ we add the asymptote $x = \frac{3\pi}{2}$



The equations of asymptotes are $x = \frac{3\pi}{2}, x = \pm \frac{\pi}{2}$ C

Question 4

$$y = \sin(x)$$

$$\text{Then } y = \sin\left(x + \frac{\pi}{2}\right)$$

$$\text{Then } y = \sin\left(x + \frac{\pi}{2}\right) + \frac{1}{2}$$

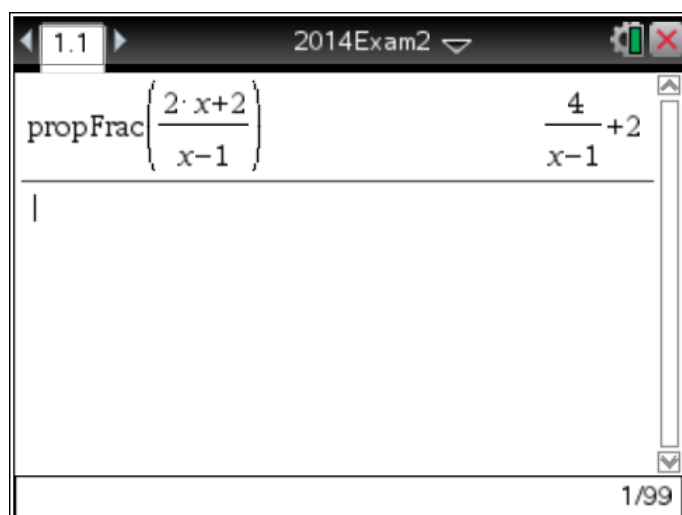
Then $y = \sin\left(\frac{x}{2} + \frac{\pi}{2}\right) + \frac{1}{2}$

Using complementary functions this now becomes $y = \cos\left(\frac{x}{2}\right) + \frac{1}{2}$

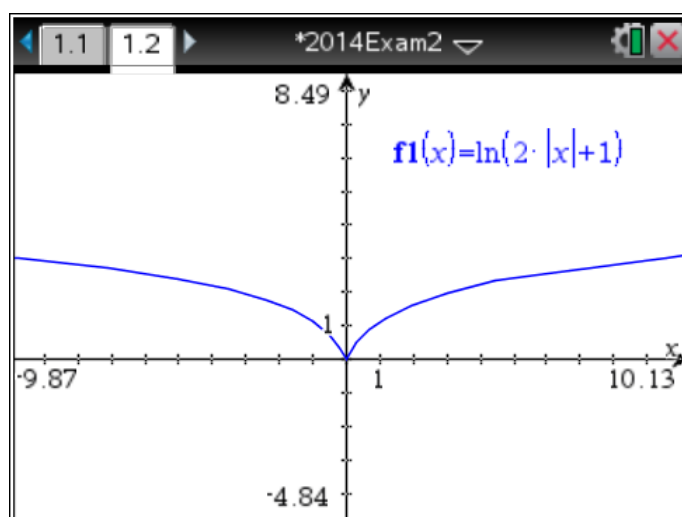
D**Question 5**

$$f(x) = \frac{2x+2}{x-1} = 2 + \frac{4}{x-1}$$

Domain $R \setminus \{1\}$, Range $R \setminus \{2\}$

B**Question 6**

The graph of $f(x) = \log_e(2|x|+1)$ has no asymptotes.

A

Question 7

$$(x-1)(x^2 + ax + b) = 0$$

$x = 1$ is one solution

$$x = \frac{-a \pm \sqrt{a^2 - 4b}}{2} \text{ will give another solution if}$$

$$a^2 - 4b = 0, a^2 = 4b$$

$$x = -\frac{a}{2} \text{ is the solution}$$

$$-\frac{a}{2} \neq 1, a \neq -2 \quad \mathbf{B}$$

Question 8

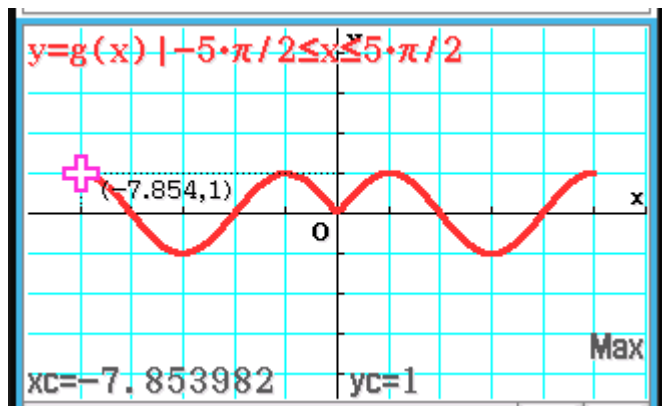
$$h(x) = g(f(x)) = (\sqrt{x+2})^2 + 5 = x + 7$$

$$\text{Dom}(h) = \text{Dom}(f), [-2, \infty) \quad \mathbf{A}$$

Question 9

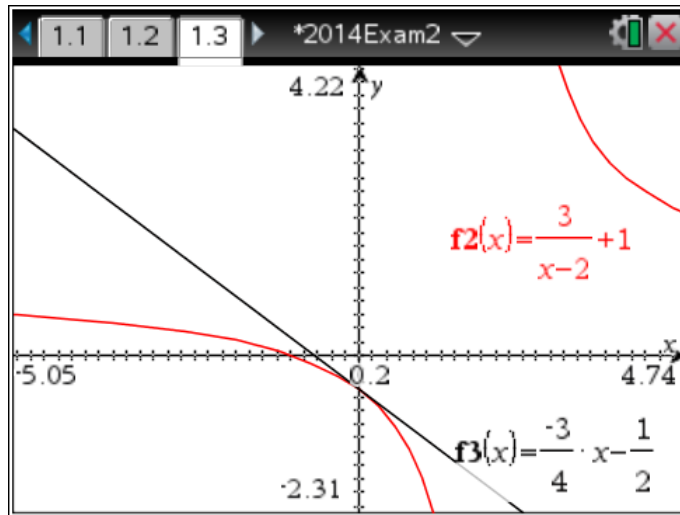
$g: \left[-\frac{5\pi}{2}, \frac{5\pi}{2}\right] \rightarrow \mathbb{R}$, where $g(x) = \sin|x|$, is **not** differentiable at cusp and endpoints.

$$\text{At } x = \pm \frac{5\pi}{2}, x = 0 \quad \mathbf{A}$$



Question 10

The tangent line will be above the graph of $f(x) = \frac{3}{x-2} + 1$ when $x < 2$.

E**Question 11**

$$g : \mathbb{R} \setminus \{-1\} \rightarrow \mathbb{R}, g(x) = \frac{1}{x+1}$$

Using First Principles

$$\begin{aligned} & \lim_{h \rightarrow 0} \left(\frac{f(x+h) - f(x)}{h} \right) \\ &= \lim_{h \rightarrow 0} \left(\frac{\frac{1}{x+h+1} - \frac{1}{x+1}}{h} \right) \\ &= \lim_{h \rightarrow 0} \left(\frac{x+1 - (x+h+1)}{h(x+1)(x+h+1)} \right) \\ &= \lim_{h \rightarrow 0} \left(\frac{-1}{x^2 + xh + 2x + h + 1} \right) \end{aligned}$$

E

Define $f(x) = \frac{1}{x+1}$

$$\frac{f(x+h)-f(x)}{h} = \frac{\frac{1}{h \cdot (x+h+1)} - \frac{1}{h \cdot (x+1)}}{1}$$

comDenom $\left(\frac{1}{h \cdot (x+h+1)} - \frac{1}{h \cdot (x+1)}\right)$

$$= \frac{-1}{x^2 + h \cdot x + 2 \cdot x + h + 1}$$

⚠ Domain of the result might be larger than the... 1/3

Question 12

f is the graph of $h(x) = (x+1)^3(x-1)$ translated 3 units up.

$$f(x) = (x+1)^3(x-1) + 3$$

D**Question 13**

$$g - f = -x^4 + x^2 + 7 - (x^2 + 2x + 3) = -x^4 + x^2 - 2x + 4$$

Solve $\frac{d}{dx}(-x^4 + x^2 - 2x + 4) = 0$ for x

Maximum value is $\frac{3}{4} + 4$

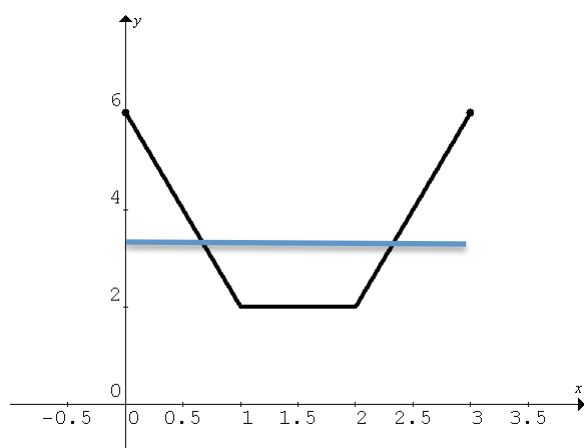
C

solve $\left(\frac{d}{dx}(-x^4 + x^2 + 7 - (x^2 + 2 \cdot x + 3)) = 0, x\right)$

$$x = -\frac{2}{3}$$

$$-x^4 + x^2 + 7 - (x^2 + 2 \cdot x + 3) \Big|_{x = -\frac{2}{3}} = \frac{3 \cdot 2^3}{4} + 4$$

2/99

Question 14

The area above the line $y = \frac{10}{3}$ equals the area below the line.

The average value is $\frac{10}{3}$. **C**

Question 15

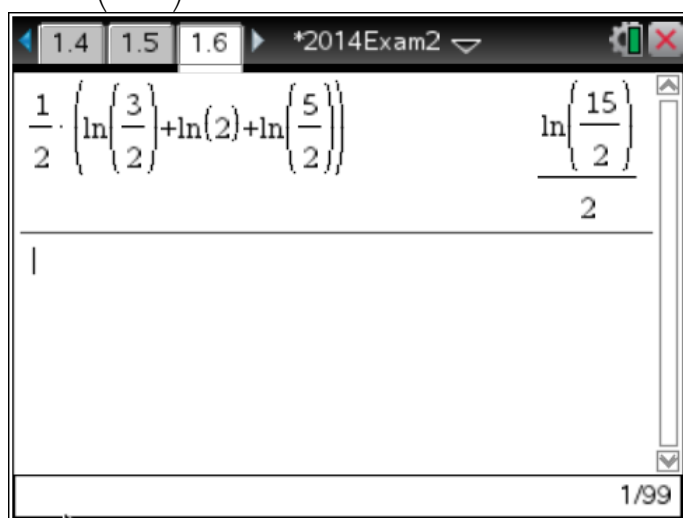
$$y = \log_e(x-1)$$

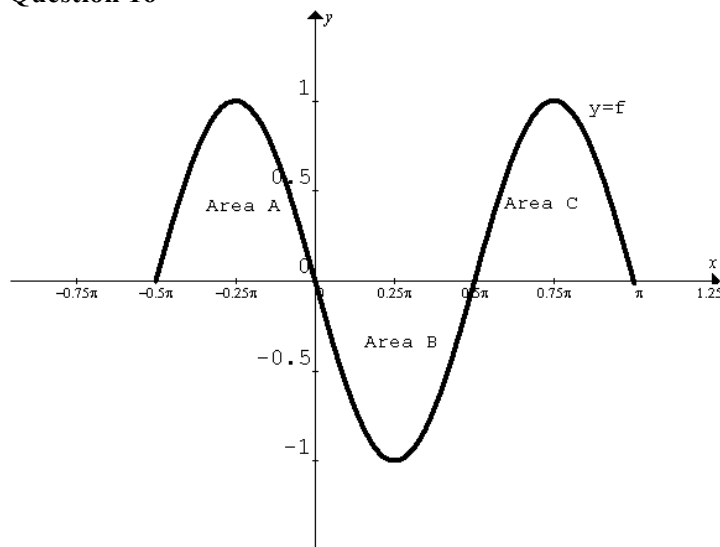
$$\text{Area} = \frac{1}{2}(\log_e(1.5) + \log_e(2) + \log_e(2.5))$$

$$= \frac{1}{2} \log_e\left(\frac{3}{2} \times 2 \times \frac{5}{2}\right)$$

$$= \frac{1}{2} \log_e\left(\frac{15}{2}\right)$$

$$= \log_e\left(\sqrt{\frac{15}{2}}\right) \quad \mathbf{A}$$

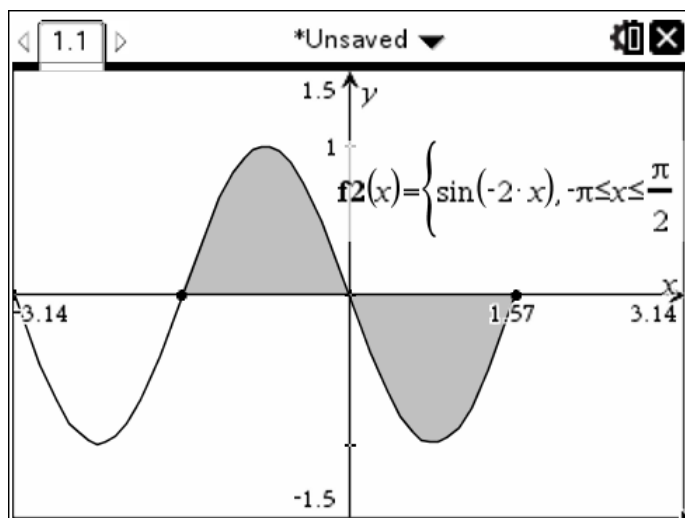


Question 16

The graph of f has been reflected in the x and y axes.

Area A = Area B

Hence $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (-f(-x)) dx = 0$

B**Question 17**

To find p .

Solve $2p^2 + 0.1 + 0.02 + p + p^2 = 1$ where $p > 0$

$$\{2 \cdot p^2 + 0.1 + 0.02 + p + p^2 = 1, p\}$$

$$\left\{ p = -\frac{11}{15}, p = \frac{2}{5} \right\}$$

gives $p = \frac{2}{5} = 0.4$

$$E(X) = 1.98$$

$$\text{Var}(X) = 2.4196$$

The standard deviation of X is closest to 1.5555 E

```

1*0.1+2*0.02+3*0.4+4*0.4^2
1.98
0.1+4*0.02+9*0.4+16*0.4^2
6.34
ans-1.98^2
2.4196
sqrt(2.4196)
1.555506348

```

Question 18

$$\text{Bi}(n, p) = \text{Bi}(50, 0.05)$$

$$\Pr(X < 2) = \Pr(X \leq 1) = 0.2794 \quad \text{B}$$

```

binomialCDF(0, 1, 50, 0.05)
0.2794317523

```

Question 19

$$S_0 = \begin{bmatrix} 0.7 \\ 0.3 \end{bmatrix} \begin{matrix} \leftarrow W \\ \leftarrow L \end{matrix}$$

$$T = \begin{bmatrix} 0.55 & 0.65 \\ 0.45 & 0.35 \end{bmatrix}$$

$$\text{Need } S_3 = T^3 S_0$$

```

[0.55 0.65]
[0.45 0.35]^3 [0.7]
[0.3]
[0.5908]
[0.4092]

```

$$S_3 = \begin{bmatrix} 0.5908 \\ 0.4092 \end{bmatrix} \begin{matrix} \leftarrow W \\ \leftarrow L \end{matrix} \quad \text{C}$$

Question 20

P and Q mutually exclusive when $\Pr(P \cap Q) = 0$

$$\text{Given } \Pr(P \cap Q) + \frac{1}{2} \Pr(P' \cap Q) = \frac{3}{10}$$

So

$$0 + \frac{1}{2} \Pr(P' \cap Q) = \frac{3}{10}$$

$$\therefore \Pr(P' \cap Q) = \frac{3}{5}$$

	P	P'	
Q	0	$\frac{3}{5}$	$\frac{3}{5}$
Q'	$\Pr(P \cap Q')$	$\Pr(P' \cap Q')$	$\frac{2}{5}$
	—	—	1

$$\Pr(P \cap Q') + \Pr(P' \cap Q') = \frac{2}{5} \quad \mathbf{D}$$

Question 21

$N(\mu, \sigma^2)$

$$\Pr(X > 80) = \frac{85}{1000} \quad \text{and} \quad \Pr(X < 10) = 0.01$$

$$\Pr(Z > 1.37220) = \frac{85}{1000} \quad \text{and} \quad \Pr(Z < -2.32635) = 0.01$$

invNormCDF("R", $\frac{85}{1000}$, 1, 0)
1.372203809
invNormCDF("L", 0.01, 1, 0)
-2.326347874

$$\text{Solve using } z = \frac{x - \mu}{\sigma}$$

$$\begin{cases} 1.37220 = \frac{80-m}{s} \\ -2.32635 = \frac{10-m}{s} \end{cases} \quad m, s$$

$$\{m=54.02928175, s=18.9263\}$$

Giving $\mu = 54.029$

A

Question 22

$$f(x) = \begin{cases} |\sin(x)| & \frac{3\pi}{2} \leq x \leq 2\pi \\ 0 & \text{elsewhere} \end{cases}$$

$$\text{Variance} = \int_{\frac{3\pi}{2}}^{2\pi} x^2 f(x) dx - (E(X))^2$$

$$\text{Variance} = \int_{\frac{3\pi}{2}}^{2\pi} x^2 f(x) dx - \left(\int_{\frac{3\pi}{2}}^{2\pi} x f(x) dx \right)^2 = 0.14159 \quad \text{E}$$

$$f(x) = \begin{cases} |\sin(x)|, & \frac{3\pi}{2} \leq x \leq 2\pi \\ 0, & \frac{3\pi}{2} \geq x \geq 2\pi \end{cases}$$

done

$$\int_{\frac{3\pi}{2}}^{2\pi} x^2 \cdot f(x) dx$$

28.05363964

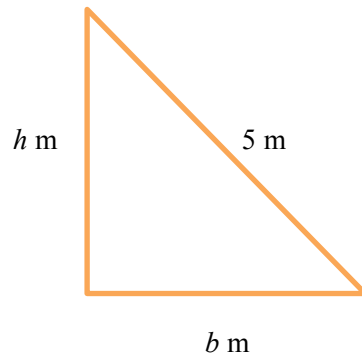
$$\text{ans} - \left(\int_{\frac{3\pi}{2}}^{2\pi} x \cdot f(x) dx \right)^2$$

0.1415926519

SECTION 2

Question 1 (10 marks)

a. i.



$$h^2 + b^2 = 25$$

$$h = \sqrt{25 - b^2}, \quad h > 0$$

$$\frac{dh}{db} = \frac{-2b}{2\sqrt{25 - b^2}} = \frac{-b}{\sqrt{25 - b^2}} \quad \mathbf{1M}$$

$$\frac{db}{dt} = 2 \text{ m/s}$$

$$\frac{dh}{dt} = \frac{dh}{db} \times \frac{db}{dt} \quad \mathbf{1M}$$

$$\frac{dh}{dt} = \frac{-b}{\sqrt{25 - b^2}} \times 2$$

$$b = 3 \text{ m}$$

$$\frac{dh}{dt} = \frac{-3}{\sqrt{25 - 9}} \times 2 = \frac{-6}{\sqrt{16}} = \frac{-3}{2} \text{ m/s}$$

The ladder is moving $\frac{3}{2}$ m/s down the wall. $\mathbf{1A}$

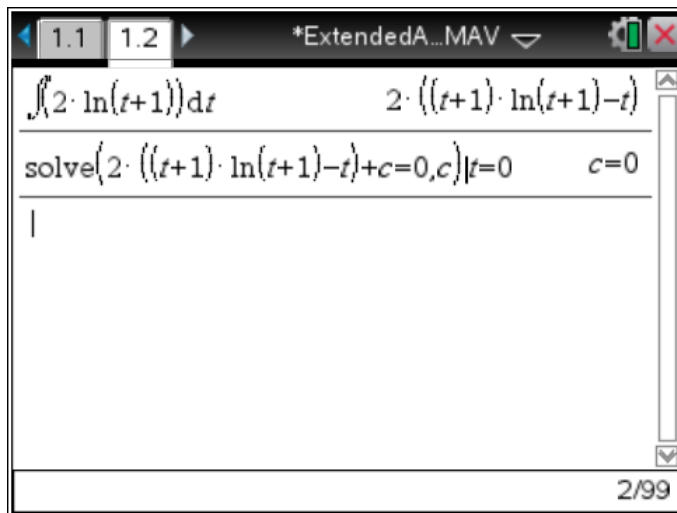
ii. $h = 2\sqrt{6} \text{ m}, h^2 = 24, b = 1 \text{ m}, \frac{db}{dt} = 2 \text{ m/s}, t = 2 \text{ s} \quad \mathbf{1A}$

b. i. $A = \int (2 \log_e(t+1)) dt$

$$A = 2(t+1) \log_e(t+1) - 2t + c \quad \mathbf{1M}$$

$$A(0) = c = 0$$

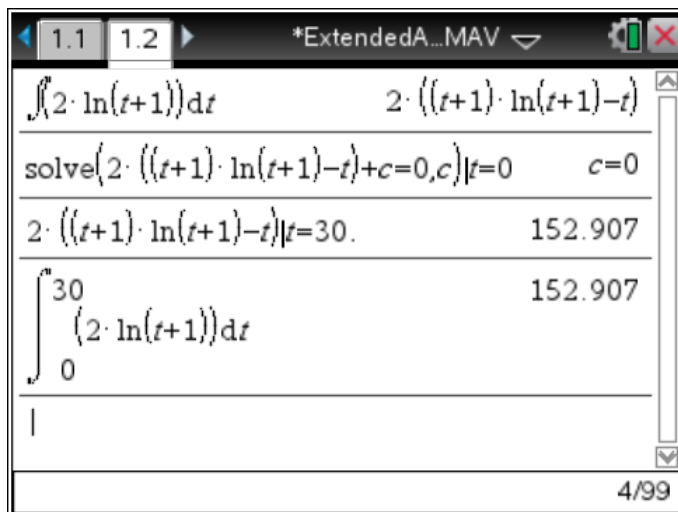
$$A = 2(t+1) \log_e(t+1) - 2t \quad \mathbf{1A}$$



ii. $A(30) = 152.9 \text{ m}^2$ correct to one decimal place **1M 1A**

OR

$$A = \int_0^{30} (2 \log_e(t+1)) dt = 152.9 \text{ m}^2 \text{ correct to one decimal place} \quad \mathbf{1M \ 1A}$$



iii. Solve $A = 2(t+1) \log_e(t+1) - 2t = 202.907...$ for t **1M**

$$t = 37.0... \text{ min}$$

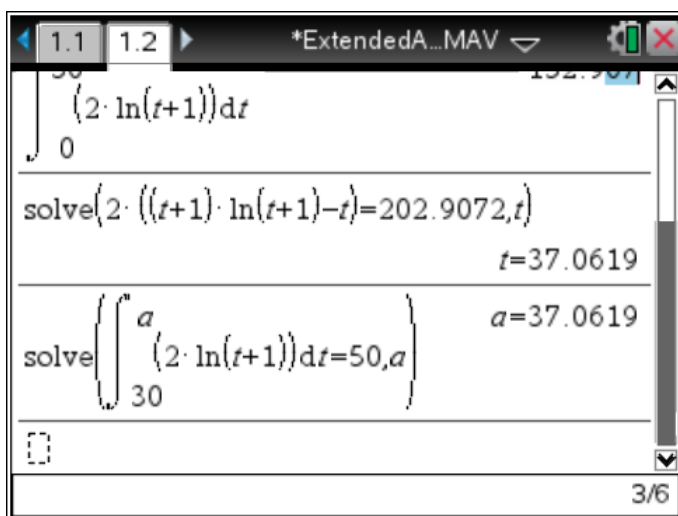
It takes an extra 7 minutes **1A**

OR

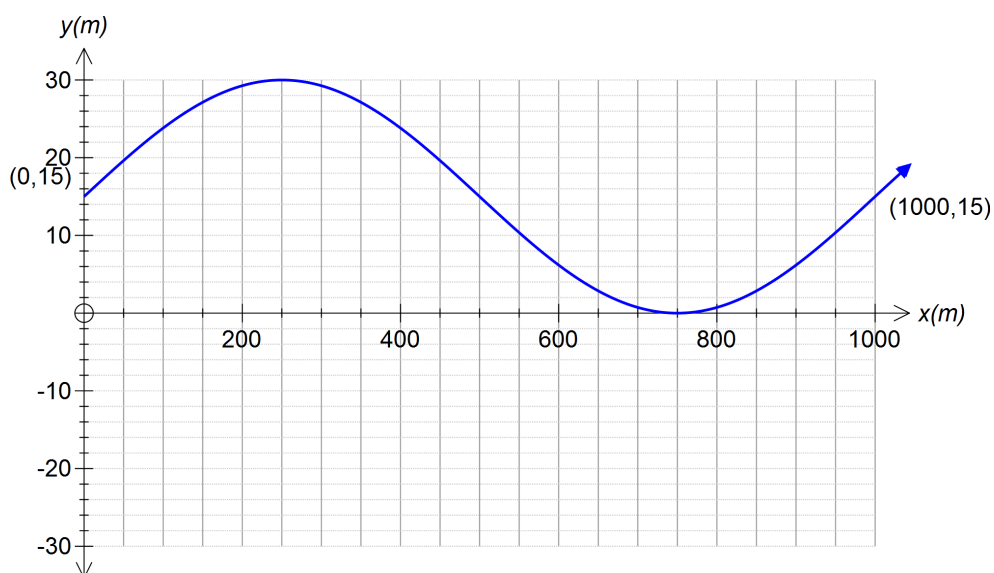
$$\text{Solve } \int_{30}^a (2 \log_e(t+1)) dt = 50 \text{ for } a \quad \mathbf{1M}$$

$$t = 37.0... \text{ min}$$

It takes an extra 7 minutes **1A**



Question 2 (17 marks)



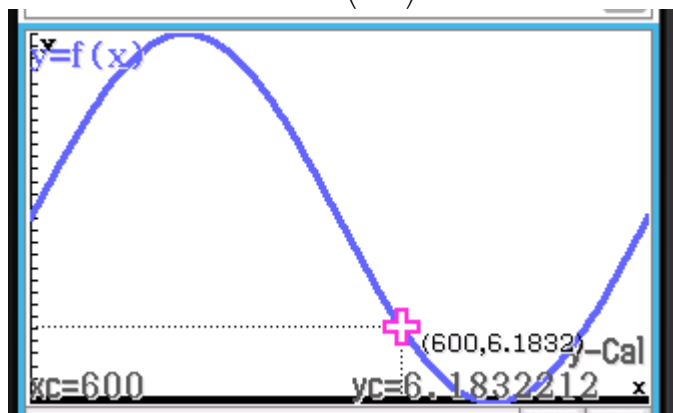
- a. i. a is amplitude so $a = 15$ **1A**
- ii. c is vertical translation so $c = 15$ **1A**
- iii. $\text{period} = 1000 = \frac{2\pi}{\pi/b}$

giving $1000 = 2b$

$b = 500$. **1M**

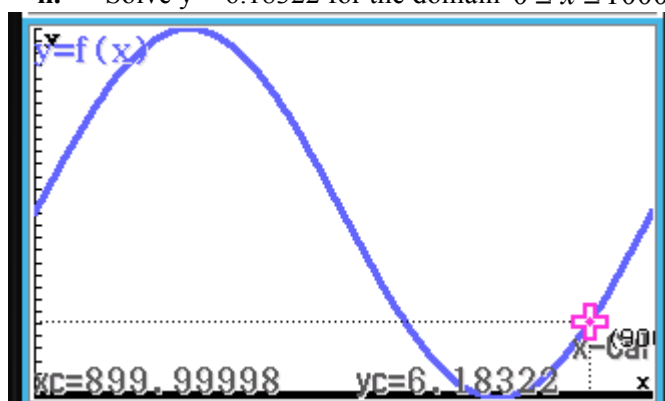
b. $y = 15 + 15 \sin\left(\frac{\pi x}{500}\right)$ 1A

c. i. $y(600) = 15 + 15 \sin\left(\frac{6\pi}{5}\right)$



Depth of the lake 600 metres from the café is 6.18 metres (or 618 **centimetres**) 1A

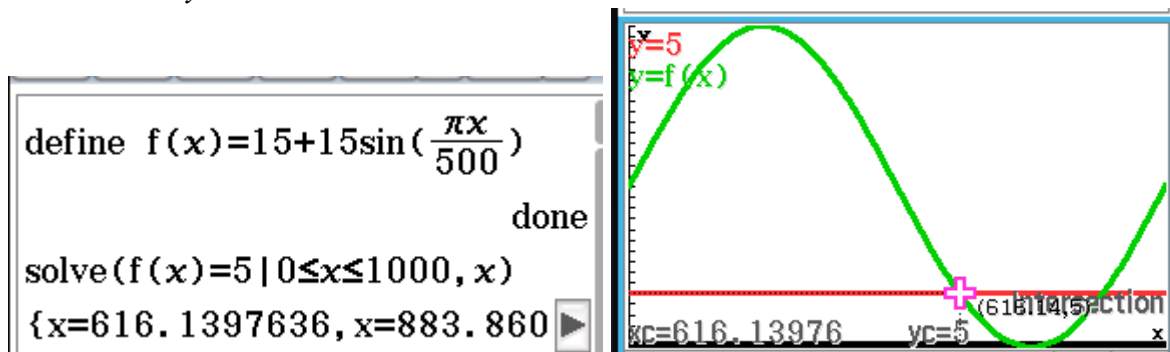
ii. Solve $y = 6.18322$ for the domain $0 \leq x \leq 1000$



Depth of water found at $x = 600$ and $x = 900$ to nearest **metre**. 1A

Complete when $y \geq 5$

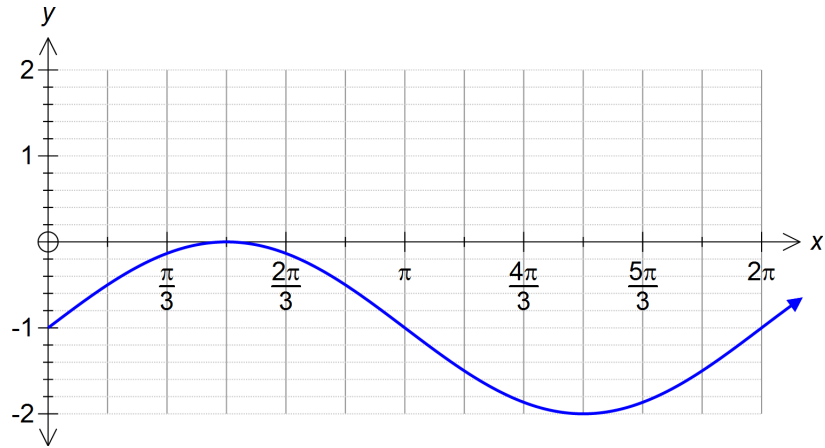
d. i. Solve $y \geq 5$ 1M



Depth of water suitable for $x \in [0, 616] \cup [884, 1000]$ to the nearest **metre** **1A**

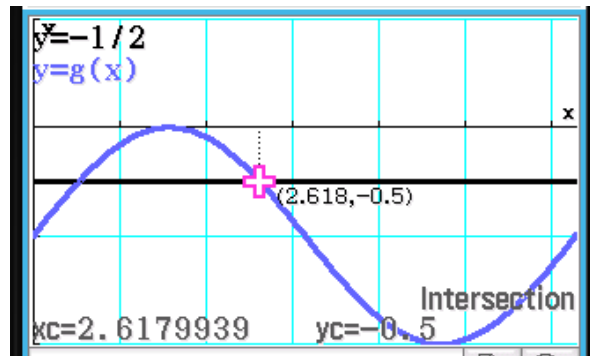
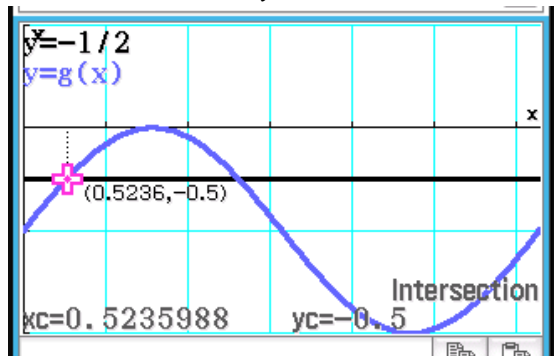
ii. Length of race, to nearest **metre**, before it is interrupted by shallow water is 616 metres. **1A**

$$f(x) = \sin(x) - 1 \text{ for } 0 \leq x \leq 2\pi$$



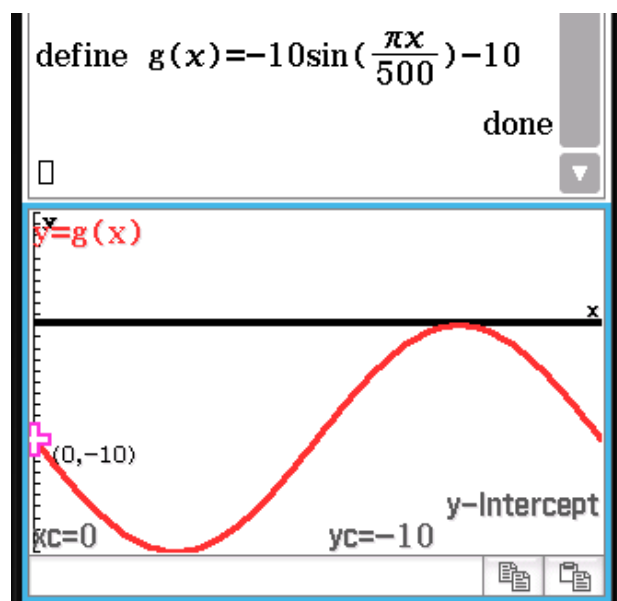
e. zero depth at $\left(\frac{\pi}{2}, 0\right)$ **1A**

Stand in water when $y > -0.5$ **1M**



f. Tourists can walk at distance from the café, to nearest **centimetre**, for $x \in (0.52, 2.62)$ **1A**

$$g(x) = -10 \sin\left(\frac{\pi x}{500}\right) - 10 \text{ for } 0 \leq x \leq 1000$$



g. Straight line, between $x = 750$ and $x = 875$.

Using the equation of a straight line $\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$

where $\frac{y - y(750)}{x - 750} = \frac{y(875) - y(750)}{875 - 750}$ 1M

$\left(\frac{y - g(750)}{x - 750} = \frac{g(875) - g(750)}{875 - 750} \right)$

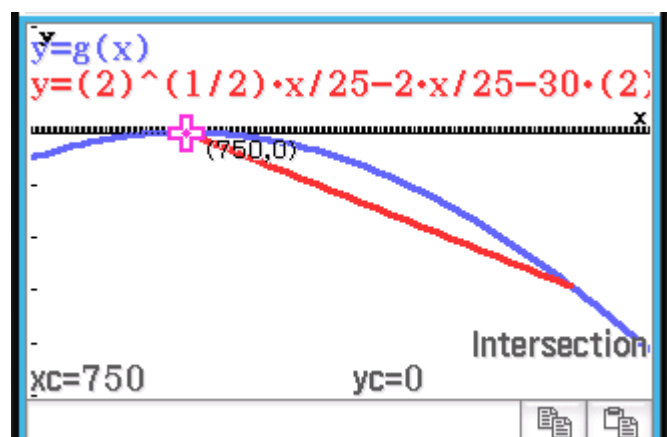
$\left\{ y = \frac{-(x - 750) \cdot (-\sqrt{2} + 2)}{25} \right\}$

1M

The equation of the line that represents the slide is,:

expand $\left(y = \frac{-(x - 750) \cdot (-\sqrt{2} + 2)}{25} \right)$

$y = \frac{\sqrt{2} \cdot x}{25} - \frac{2 \cdot x}{25} - 30 \cdot \sqrt{2} + 60$



graph of the water slide

Simplified equation of water slide $y = -\frac{x}{25}(2 - \sqrt{2}) - 30\sqrt{2} + 60$ **1A**

h. Transform $y = 15 + 15 \sin\left(\frac{\pi x}{500}\right)$ to the image graph $g(x) = -10 \sin\left(\frac{\pi x}{500}\right) - 10$

Step 1: Dilate from the x -axis by a factor of $\frac{2}{3}$ units **1A**

Step 2: Reflect in the x -axis **1A**

Question 3 (16 marks)

Rowing Ergo (R), Cycle (C), Step (S).

60 staff members, all take part in at least one class.

$$\Pr(R) = 0.25$$

$$\Pr(C) = 0.55$$

$$\Pr(S) = 0.75$$

R, C and S are independent.

a. i. Independent $\therefore \Pr(R \cap S) = \Pr(R) \times \Pr(S)$

$$\Pr(R \cap S) = 0.25 \times 0.75 = 0.1875 \quad \text{or} \quad \frac{3}{16} \quad \mathbf{1A}$$

ii. $\Pr(R | S) = \frac{\Pr(R \cap S)}{\Pr(S)}$

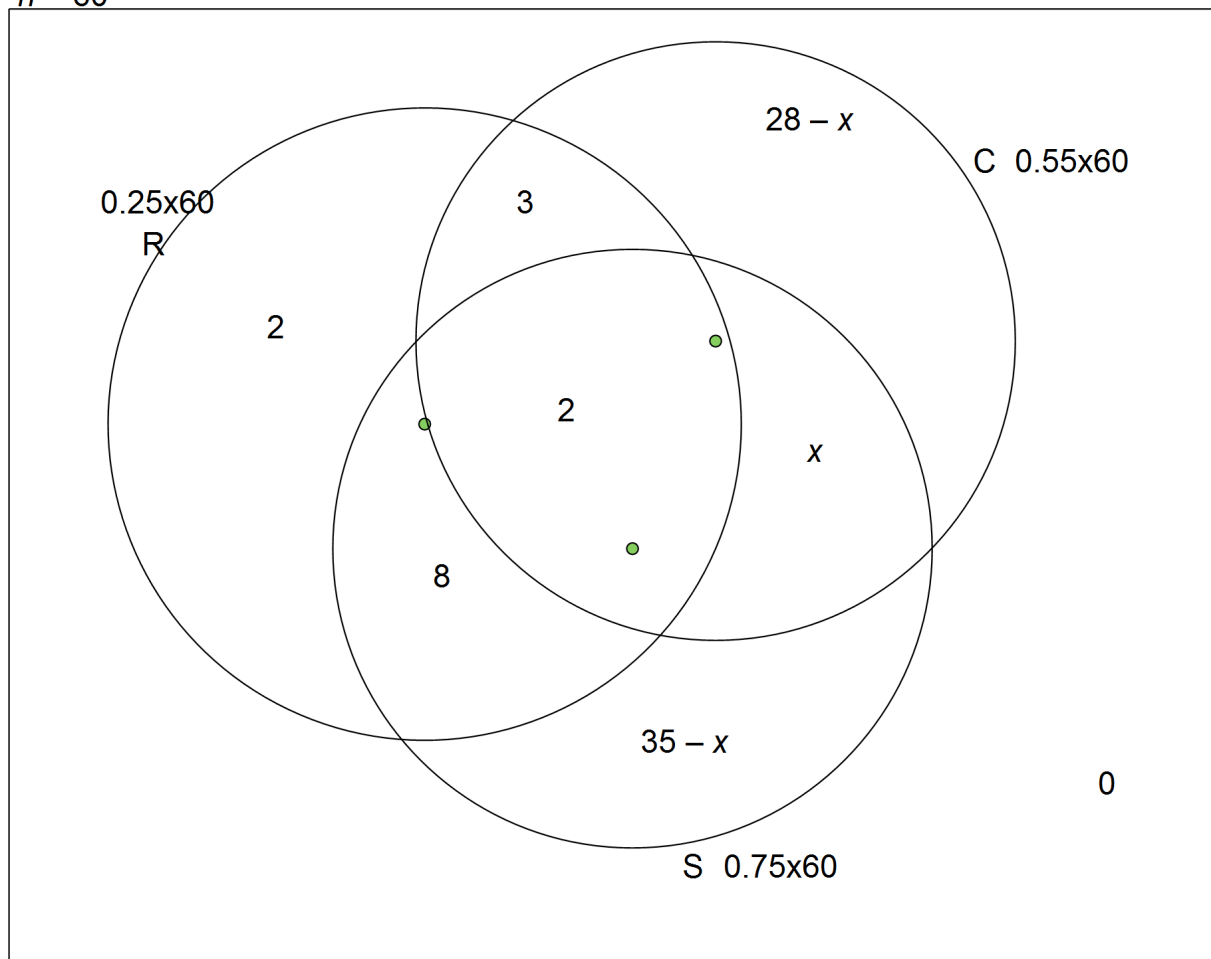
$$\Pr(R | S) = \frac{0.1875}{0.75} = 0.25 \quad \text{or} \quad \frac{1}{4} \quad \mathbf{1A}$$

b. i. $\Pr(R \cap S \cap C') = 0.25 \times 0.75 \times 0.45$ **1M**

$$\Pr(R \cap S \cap C') = 0.084375 \quad \text{or} \quad \frac{27}{320} \quad \mathbf{1A}$$

ii. Venn diagram.

$n = 60$



1M

From Venn diagram we get $2 + 3 + 2 + 8 + 28 - x + x + 35 - x = 60$

$x = 18$

10 Staff members take C only **1A**

60 staff members

If a staff member takes Step there is a probability of 20% of them taking Step the next day.

If a staff member takes Cycle there is a probability 35% of them taking Step Class the next day.

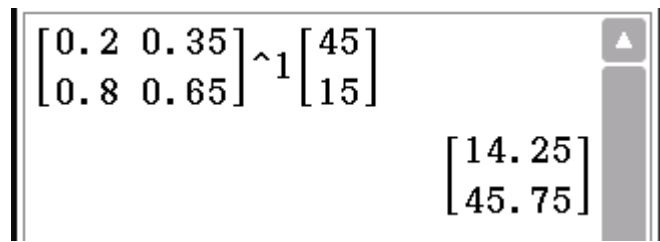
No R classes are offered.

On Monday 15 take Cycle Class.

$$T = \begin{bmatrix} 0.2 & 0.35 \\ 0.8 & 0.65 \end{bmatrix}$$

$$S_0 = \begin{bmatrix} 45 \\ 15 \end{bmatrix}$$

c. i. Cycle class on Tuesday using $S_1 = T^1 S_0$

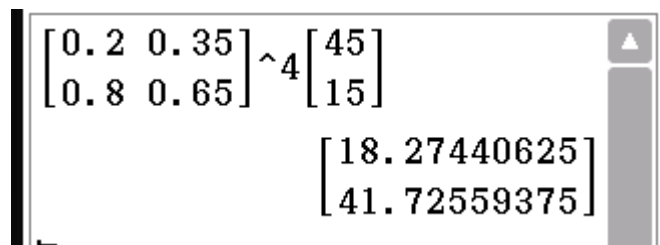


$$\begin{bmatrix} 0.2 & 0.35 \end{bmatrix}^1 \begin{bmatrix} 45 \\ 15 \end{bmatrix} = \begin{bmatrix} 14.25 \\ 45.75 \end{bmatrix}$$

$$\begin{bmatrix} 0.2 & 0.35 \\ 0.8 & 0.65 \end{bmatrix} \begin{bmatrix} 45 \\ 15 \end{bmatrix} = \begin{bmatrix} 14.25 \\ 45.75 \end{bmatrix} \quad \mathbf{1M}$$

46 staff take Cycle class on Tuesday **1A**

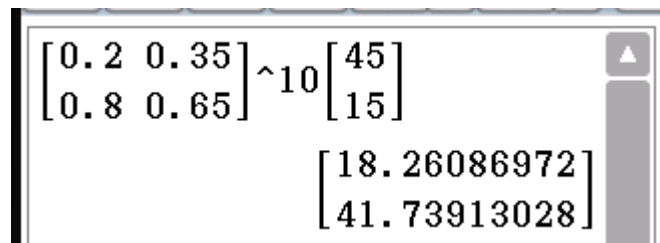
ii. Step class on Friday using $S_4 = T^4 S_0$



$$\begin{bmatrix} 0.2 & 0.35 \\ 0.8 & 0.65 \end{bmatrix}^4 \begin{bmatrix} 45 \\ 15 \end{bmatrix} = \begin{bmatrix} 18.27440625 \\ 41.72559375 \end{bmatrix}$$

18 staff take Step class on Friday **1A**

iii. Cycle and Step using $S_\infty = T^\infty S_0$



$$\begin{bmatrix} 0.2 & 0.35 \\ 0.8 & 0.65 \end{bmatrix}^{10} \begin{bmatrix} 45 \\ 15 \end{bmatrix} = \begin{bmatrix} 18.26086972 \\ 41.73913028 \end{bmatrix}$$

18 staff take Step class and 42 take Cycle class long term **2A**

10 Mathematics staff members. Independent events

$$\Pr(C) = 0.2$$

$$\Pr(S) = 0.6$$

$$\mathbf{d.} \quad \Pr(S \cap S \cap C \cap C \cap C) = 0.6^2 \times 0.2^3$$

$$\Pr(S \cap S \cap C \cap C \cap C) = 0.00288 \quad \text{or} \quad \frac{9}{3125} \quad \mathbf{1A}$$

e. $\text{Bi}(n, p) = \text{Bi}(10, 0.6)$ 1M

$\text{binomialCDF}(7, 10, 10, 0.6)$ 0.3822806016
--

$$\Pr(X > 6) = \Pr(X \geq 7) = 0.3823 \quad 1A$$

f. Now $\Pr(C) = 0.3$

$$\text{Bi}(n, p) = \text{Bi}(n, 0.3)$$

$$\Pr(X \geq 2) > 0.65$$

Let $\Pr(X \geq 2) = 0.65$

$$1 - [\Pr(X = 0) + \Pr(X = 1)] = 0.65$$

$$\therefore \Pr(X = 0) + \Pr(X = 1) = 0.35 \quad 1M$$

$${}^nC_0 q^n p^0 + {}^nC_1 q^{n-1} p^1 = 0.35$$

Where $q = 0.7, p = 0.3$.

$${}^nC_0 0.7^n 0.3^0 + {}^nC_1 0.7^{n-1} 0.3^1 = 0.35$$

Giving

$$0.7^n + n \times 0.7^{n-1} \times 0.3 = 0.35. \quad \text{Solve for } n.$$

We need 7 Science staff members.

1A

Question 4 (15 marks)

a. $A = -1$

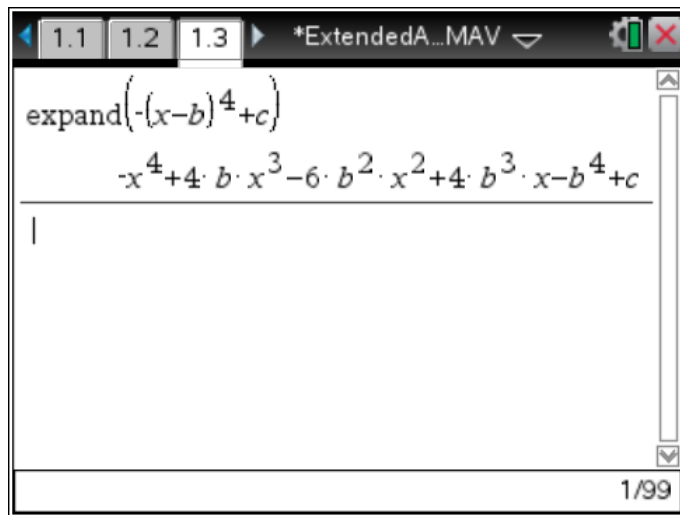
Expand $-(x - B)^4 + C = -x^4 + 4Bx^3 - 6B^2x^2 + 4B^3x - B^4 + C$ **1M**

Equate coefficients

$4B = 8, B = 2$

$-B^4 + C = -15, -16 + C = -15, C = 1$

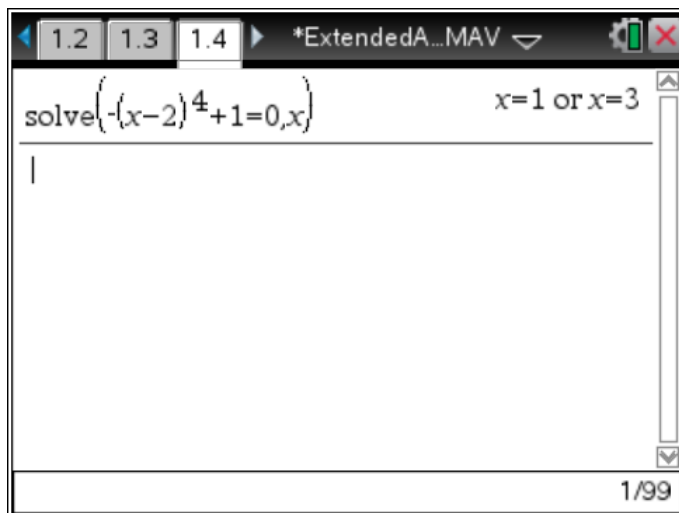
$f(x) = -(x - 2)^4 + 1$ **1A**



b. $f(1) = -(1 - 2)^4 + 1 = 0$ **1M**

Could use the turning point (2, 1)

Solve $f(x) = -(x - 2)^4 + 1 = 0$ for $x, x = 3, b = 3$ **1A**



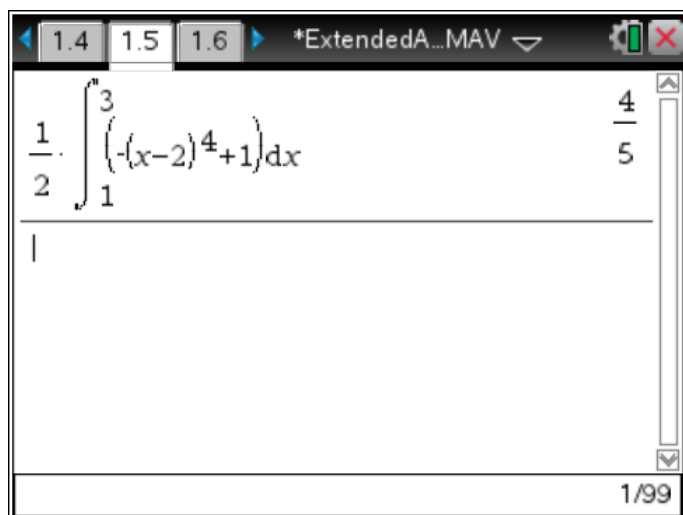
c. i. $[1, 3]$ **1A**

ii $(1, 3)$ **1A**

d. $c = 2$ **1A**

e. g is not differentiable at $x = 3$ and therefore is not differentiable for all values of x . **1A**

f. i. $k = \frac{1}{2} \int_1^3 (f(x)) dx$ **1M**
 $= \frac{4}{5}$ **1A**

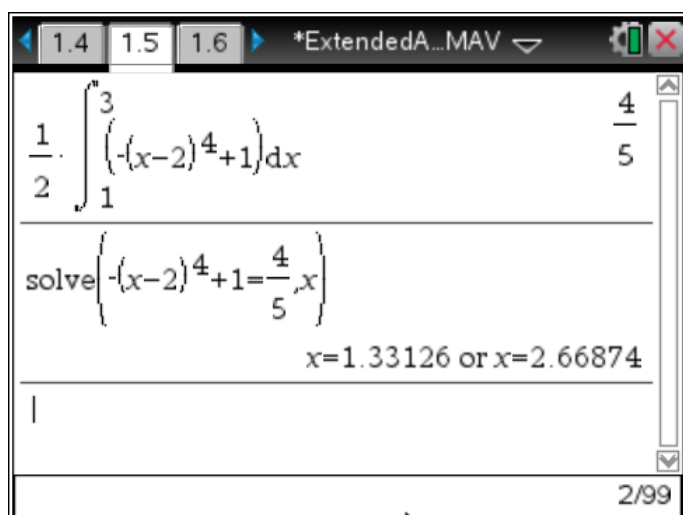


A screenshot of a CAS interface. The top bar shows navigation buttons for 1.4, 1.5, and 1.6, and a dropdown menu set to '*ExtendedA...MAV'. The main display area shows the integral $\frac{1}{2} \cdot \int_1^3 (-(x-2)^4 + 1) dx$ on the left and the result $\frac{4}{5}$ on the right. Below the integral, there is a vertical bar and a large empty space for further input. At the bottom right, a status bar shows '1/99'.

ii. $\int_m^n (f(x) - k) dx = 2 \int_1^m (k - f(x)) dx$

Solve $f(x) = \frac{4}{5}$ for x **1M**

$m = 1.3313$ and $n = 2.6687$ correct to four decimal places **1A**



A screenshot of a CAS interface. The top bar is identical to the previous one. The main display area shows the same integral $\frac{1}{2} \cdot \int_1^3 (-(x-2)^4 + 1) dx$ and result $\frac{4}{5}$. Below this, the input $\text{solve}\left(-(x-2)^4 + 1 = \frac{4}{5}, x\right)$ is shown, followed by the output $x=1.33126 \text{ or } x=2.66874$. At the bottom right, a status bar shows '2/99'.

g. $f_1: [1, 3] \rightarrow \mathbb{R}, f_1(x) = -x^4 + 8x^3 + ax^2 + bx + d$

Find a and b in terms of d .

Solve $f_1(1) = 0$ and $f_1(3) = 0$ for a and b

$$a = \frac{d-57}{3} \text{ and } b = -\frac{4(d-9)}{3} \quad \mathbf{1M}$$

Solve $\frac{d}{dx} \left(-x^4 + 8x^3 + \frac{d-57}{3}x^2 - \frac{4(d-9)}{3}x + d \right) = 0$ for x $\mathbf{1M}$

$$x = 2 \text{ or } x = \frac{12 \pm \sqrt{6d+90}}{6}$$

For three solutions $6d+90 > 0$, $d > -15$ and $\frac{\sqrt{6d+90}}{6} < 1$, $d < -9$ as $1 < x < 3$

$$-15 < d < -9$$

$\mathbf{1A}$

