

SECTION 1

|   |   |   |   |   |   |   |   |   |    |    |
|---|---|---|---|---|---|---|---|---|----|----|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
| A | D | E | D | E | C | A | D | B | D  | D  |

|    |    |    |    |    |    |    |    |    |    |    |
|----|----|----|----|----|----|----|----|----|----|----|
| 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 |
| A  | D  | C  | A  | B  | A  | A  | D  | C  | C  | A  |

Q1 The inverse of  $y = \frac{x}{x-1}$  is  $x = \frac{y}{y-1}$ ,  $x(y-1) = y$ ,

$$xy - x = y, xy - y = x, y = \frac{x}{x-1}, \therefore f^{-1}(x) = \frac{x}{x-1} \quad A$$

Q2  $\log_a(1-x) = 1 - \log_a x$ ,  $a > 0$  and  $0 < x < 1$   
 $\log_a(1-x) + \log_a x = 1$ ,  $\log_a x(1-x) = 1$ ,  $x(1-x) = a$ ,

$$x^2 - x + a = 0, x = \frac{1 \pm \sqrt{1-4a}}{2}$$

$\therefore 1-4a > 0$  and  $\sqrt{1-4a} < 1$ ,  $\therefore 0 < a < 0.25$  D

Q3  $\frac{x+a}{x-b} \geq 0$  where  $a, b \in \mathbb{R}^+$ ,

either  $x+a \geq 0$  and  $x-b > 0$ , i.e.  $x \geq -a$  and  $x > b$ , i.e.  $x > b$   
or  $x+a \leq 0$  and  $x-b < 0$ , i.e.  $x \leq -a$  and  $x < b$ , i.e.  $x \leq -a$   
 $\therefore$  the maximal domain is  $(-\infty, -a] \cup (b, \infty)$  E

$$Q4 f(x) = \frac{x^2 - a}{x + \sqrt{a}} = \frac{(x + \sqrt{a})(x - \sqrt{a})}{x + \sqrt{a}} = x - \sqrt{a} \text{ for } x \neq -\sqrt{a}$$

$$\therefore f(x) \neq -\sqrt{a} - \sqrt{a}, \text{ i.e. } -2\sqrt{a}$$

The range of  $f(x)$  is  $\mathbb{R} \setminus \{-2\sqrt{a}\}$  D

$$Q5 y = mx - 2m = m(x-2)$$

$$y = x^3 - 6x^2 + 8x = x(x^2 - 6x + 8) = x(x-2)(x-4)$$

The functions have the same  $x$ -intercepts,  $x = 2$

$$\text{For } y = x^3 - 6x^2 + 8x, \frac{dy}{dx} = 3x^2 - 12x + 8 = -4 \text{ at } x = 2$$

$\therefore y = mx - 2m$  intersect  $y = x^3 - 6x^2 + 8x$  once only when  $m \leq -4$  E

$$Q6 f(x) = x(x^2 - 1)$$

$$f(x-1) = (x-1)(x^2 - 2x)$$

$$f(1-x) = (1-x)(-2x + x^2) = -(x-1)(x^2 - 2x)$$

$$\therefore f(1-x) = -f(x-1)$$

$$Q7 \cos A + \sin B = 0$$

$$\sin B = -\cos A = -\sin\left(\frac{\pi}{2} - A\right) = \sin\left(A - \frac{\pi}{2}\right), B = A - \frac{\pi}{2} \quad A$$

$$Q8 f(x) = c - 9x + 6x^2 - x^3$$

When  $c = 0$ ,  $f(x) = -9x + 6x^2 - x^3 = -x(3-x)^2$ ,  $\therefore x$ -intercepts are at  $x = 0$  and  $x = 3$  (a local maximum)

Let  $f'(x) = -9 + 12x - 3x^2 = -3(x-1)(x-3) = 0$ ,  
local minimum point is  $(1, -4)$

For three positive  $x$ -intercepts,  $c > 0$  and  $c < 4$  D

$$Q9 y = (1-x)^3 + x - 1 \rightarrow (2-x)^3 - (2-x) \rightarrow (2+x)^3 - (2+x) \\ = (2+x)((2+x)^2 - 1) = (2+x)(1+x)(3+x) \quad B$$

$$Q10 y = g(|x+a|) \quad D$$

For  $x+a > 0$ , i.e.  $x > -a$ ,  $y = g(x+a)$

For  $x+a < 0$ , i.e.  $x < -a$ ,  $y = g(-(x+a))$

The two sections are reflection of each other in the line  $x = -a$ .

$$Q11 \text{ For } x < a, f(x) = 2x - 1, f'(x) = 2$$

$$\text{As } x \rightarrow a, f(x) \rightarrow 2a - 1, f'(x) \rightarrow 2$$

$$\text{For } x \geq a, f(x) = ax^2 + bx, f'(x) = 2ax + b$$

$$\text{As } x \rightarrow a, f(x) \rightarrow a^3 + ab, f'(x) \rightarrow 2a^2 + b$$

Since  $f(x)$  is differentiable at  $x = a$ ,

$$\therefore a^3 + ab = 2a - 1 \text{ and } 2a^2 + b = 2 \therefore a = 1 \quad D$$

Q12 A degree 4 polynomial may not have an inflection point,  
e.g.  $P(x) = x^4 + 4x^3 + 6x^2 + 4x + 1 = (x+1)^4$  A

$$Q13 1 \leq 2 \sin \frac{\pi x}{2} < 2, x \in \left[\frac{1}{3}, 1\right) \cup \left(1, \frac{5}{3}\right] \quad D$$

$$Q14 f(-x) = -f(x), \therefore \int_{-1}^1 f(-x) dx = 0$$

Translate by 1 unit in the positive  $x$ -direction,

$$\int_{-1+1}^{1+1} f(-(x-1)) dx = 0, \text{ i.e. } \int_0^2 f(1-x) dx = 0 \quad C$$

$$Q15 f(x) = \sqrt{x}, f'(x) = \frac{1}{2\sqrt{x}}$$

$$\sqrt{50} \approx \sqrt{49} + 1 \times \frac{1}{2\sqrt{49}} = \frac{99}{14} \quad A$$

$$Q16 \quad B$$

$$Q17 \int_{-a}^a \cos^2 \theta d\theta = 2 \times \int_0^a \cos^2 \theta d\theta = 2 \times \frac{1}{2} \int_0^a (1 + \cos 2\theta) d\theta \\ = \left[ \theta + \frac{\sin 2\theta}{2} \right]_0^a = a + \frac{1}{2} \sin 2a = a + \sin a \cos a \quad A$$

$$Q18 \mu = np = 9, \sigma^2 = npq = 6, \therefore 9q = 6, q = \frac{2}{3}, p = \frac{1}{3} \\ \Pr(X > 12) = \Pr(X \geq 13) \approx 0.079 \quad A$$

$$\begin{array}{cc} & L & O \\ \text{Q19} & L & \begin{bmatrix} 0.15 & 0.10 \\ 0.85 & 0.90 \end{bmatrix} \\ & O & \end{array}$$

Long run probability of on time =  $\frac{0.85}{0.10+0.85} = \frac{17}{19}$

Q20  $\int_{-4}^0 p dx + \int_1^{9p} 4p dx = 1$ , where  $p > 0$ ,  $[px]_{-4}^0 + [4px]_1^{9p} = 1$

$4p + 36p^2 - 4p = 1$ ,  $p = \frac{1}{6}$

Q21  $\Pr(B'|A) = 1 - \Pr(B|A) = 1 - \frac{\Pr(B \cap A)}{\Pr(A)} = 1 - \frac{1}{2\Pr(A)}$

Q22  $2^x = 5^y = 100^z$ ,  $2^x = 5^y = 10^{2z}$ ,  $2^x = 5^y = 2^{2z} 5^{2z}$

$\therefore x = \log_2(2^{2z} 5^{2z}) = 2z + 2z \log_2 5$  and

$y = \log_5(2^{2z} 5^{2z}) = 2z \log_5 2 + 2z$

$\therefore \frac{x-2z}{2z} = \log_2 5$  and  $\frac{y-2z}{2z} = \log_5 2$

$\therefore \frac{x-2z}{2z} = \frac{2z}{y-2z}$ ,  $z = \frac{xy}{2(x+y)}$

## SECTION 2

Q1a Translations of 3 units to the left and 4 units downwards

Q1bi  $y = 2$ ,  $[-4, 4]$

Q1bii  $x = -4$ ,  $[-2, 2]$

Q1c  $WXYZ$  is similar to  $ABCD$  because the corresponding sides are dilated by the same factor, 2, e.g.  $WX = 2AB$  and  $WZ = 2AD$

Q1d  $2y = x^2 - 6x + 17$ ,  $2y = x^2 - 6x + 9 - 9 + 17$ ,

$\therefore 2y = (x-3)^2 + 8$ ,  $2(y-4) = (x-3)^2$

$\therefore$  Translations of 3 units to the left and 4 units downwards will take the vertex of parabola **II** to the origin.

Q1e  $2(y-4) = (x-3)^2 \rightarrow 2((y+4)-4) = ((x+3)-3)^2$

The new equation is  $2y = x^2$ .

Q1f  $2y = x^2$ ,  $\frac{2y}{4} = \frac{x^2}{4}$ ,  $\left(\frac{y}{2}\right) = \left(\frac{x}{2}\right)^2$

$\therefore 2y = x^2$  is the dilations of  $y = x^2$  by the same factor of 2 vertically and horizontally,  $\therefore 2y = x^2$  and  $y = x^2$  are similar.

Q1g Consider a general parabola  $y = a(x-h)^2 + k$ . Two translations will move its vertex to the origin and its equation becomes  $y = ax^2$ ,  $\therefore ay = a^2x^2$ ,  $(ay) = (ax)^2$ .  $\therefore y = ax^2$  is the

dilations of  $y = x^2$  by the same factor of  $\frac{1}{a}$  vertically and

horizontally.  $\therefore$  all parabolas are similar to  $y = ax^2$ . Hence they are similar to each other.

Q2a  $y = \frac{1}{a-b}(e^{-bt} - e^{-at})$

When  $t = \log_e 2$ ,  $e^t = 2$

D  $y = \frac{1}{a-b}(e^{-bt} - e^{-at}) = \frac{1}{a-b}((e^t)^{-b} - (e^t)^{-a}) = \frac{1}{a-b}(2^{-b} - 2^{-a})$   
 $= \frac{1}{(a-b)}\left(\frac{1}{2^b} - \frac{1}{2^a}\right) = \frac{1}{(a-b)}\left(\frac{2^a - 2^b}{2^a 2^b}\right) = \frac{2^a - 2^b}{(a-b)2^{a+b}}$

C Q2b  $y = \frac{1}{a-b}(e^{-bt} - e^{-at})$

When  $t = 1$ ,  $y = \frac{e-1}{e^2}$ ,  $\therefore \frac{e-1}{e^2} = \frac{1}{a-b}(e^{-b} - e^{-a})$

C  $\therefore \frac{e-1}{e^2} = \frac{e^a - e^b}{(a-b)e^{a+b}}$  ..... (1)

When  $t = 2$ ,  $y = \frac{e^2-1}{e^4}$ ,  $\therefore \frac{e^2-1}{e^4} = \frac{1}{a-b}(e^{-2b} - e^{-2a})$

$\therefore \frac{e^2-1}{e^4} = \frac{e^{2a} - e^{2b}}{(a-b)e^{2(a+b)}} \dots\dots\dots (2)$

A Q2c Factorise both sides of (2):

$\frac{(e-1)(e+1)}{e^2 e^2} = \frac{(e^a - e^b)(e^a + e^b)}{(a-b)e^{a+b}e^{a+b}} \dots\dots\dots (3)$

(3)/(1)  $\frac{e+1}{e^2} = \frac{e^a + e^b}{e^{a+b}}$

$\therefore e^{-1} + e^{-2} = e^{-b} + e^{-a}$ ,  $\therefore a = 2$  and  $b = 1$

Q2di  $y = \frac{1}{a-b}(e^{-bt} - e^{-at})$ ,  $\frac{dy}{dt} = \frac{1}{a-b}(-be^{-bt} + ae^{-at})$

Q2dii For maximum  $y$ , let  $\frac{dy}{dt} = \frac{1}{a-b}(-be^{-bt} + ae^{-at}) = 0$

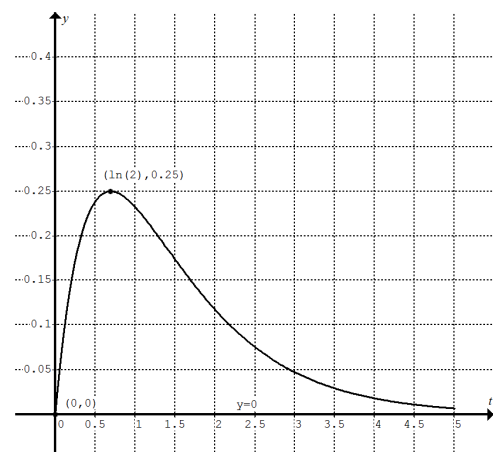
$\therefore ae^{-at} = be^{-bt}$ ,  $e^{(a-b)t} = \frac{a}{b}$ ,  $(a-b)t = \log_e \frac{a}{b}$

$\therefore t = \frac{1}{a-b} \log_e \frac{a}{b}$

Q2diii If  $a = 2$  and  $b = 1$ ,  $y_{\max}$  occurs when  $t = \log_e 2$

From part a,  $y_{\max} = \frac{2^2 - 2^1}{(2-1)2^{2+1}} = \frac{1}{4}$

Q2e

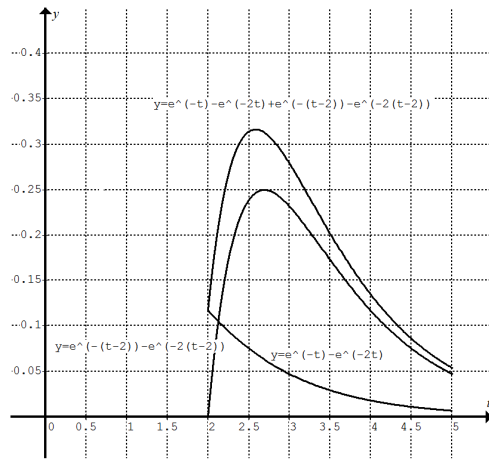


Q2f  $y_{\text{first}} = e^{-t} - e^{-2t}$  for  $t \geq 0$

$y_{\text{second}} = e^{-(t-2)} - 2^{-2(t-2)}$  for  $t \geq 2$

$y = y_{\text{first}} + y_{\text{second}} = e^{-t} - e^{-2t} + e^{-(t-2)} - 2^{-2(t-2)}$  for  $t \geq 2$

Q2g



Q2h  $t \approx 3.85$  by CAS

Q3ai  $\int_0^{1000} (1 - \cos nx) dx = \frac{1000^2}{3}$ ,  $\int_0^{1000} (1 - \cos nx) dx = \frac{1000}{3}$

$\left[ x - \frac{\sin nx}{n} \right]_0^{1000} = \frac{1000}{3}$ ,  $1000 - \frac{\sin 1000n}{n} = \frac{1000}{3}$

$\frac{\sin 1000n}{n} = \frac{2000}{3}$ ,  $\therefore 3 \sin 1000n = 2000n$

Q3aii Use CAS to solve  $3 \sin 1000n = 2000n$  for  $n$ ,  
 $n \approx 0.001496$

$y \approx 1000(1 - \cos(0.001496 x))$

Q3bi Before reflection:  $y = A \log_e \left( 1 + \frac{x}{150} \right)$

After reflection:  $x = A \log_e \left( 1 + \frac{y}{150} \right)$ ,  $\log_e \left( 1 + \frac{y}{150} \right) = \frac{x}{A}$

$1 + \frac{y}{150} = e^{\frac{x}{A}}$ ,  $y = 150 \left( e^{\frac{x}{A}} - 1 \right)$

Q3bii  $\int_0^{1000} 150 \left( e^{\frac{x}{A}} - 1 \right) dx = \frac{1000^2}{3}$ ,  $\int_0^{1000} \left( e^{\frac{x}{A}} - 1 \right) dx = \frac{1000^2}{450}$

$\left[ A e^{\frac{x}{A}} - x \right]_0^{1000} = \frac{1000^2}{450}$ ,  $A e^{\frac{1000}{A}} - 1000 - A = \frac{1000^2}{450}$

$A \left( e^{\frac{1000}{A}} - 1 \right) = \frac{29000}{9}$ ,  $e^{\frac{1000}{A}} = 1 + \frac{29000}{9A}$

$\therefore A \log_e \left( 1 + \frac{29000}{9A} \right) = 1000$

Q3biii Use CAS to solve  $A \log_e \left( 1 + \frac{29000}{9A} \right) = 1000$  for  $A$ ,

$A \approx 496.7$

$y \approx 496.7 \log_e \left( 1 + \frac{x}{150} \right)$

Q3c Let  $D$  metres be the northerly distance between curve  $I$  and curve  $II$ .

$D \approx 496.7 \log_e \left( 1 + \frac{x}{150} \right) - 1000(1 - \cos(0.001496 x))$

$\frac{dD}{dx} = \frac{496.7 \times \frac{1}{150}}{1 + \frac{x}{150}} - 1000(0.001496 \sin(0.001496 x))$

For longest distance, let  $\frac{dD}{dx} = 0$

$\frac{496.7}{150 + x} - 1.496 \sin(0.001496 x) = 0$

By CAS,  $x \approx 417.633$ ,  $D_{\text{longest}} \approx 472$  m

Q4a  $\mu = 588.4$ ,  $\sigma = 77.9$ ,  $\Pr(X > 583.6) \approx 0.525$ , i.e. 52.5%

Q4b  $\mu \pm \sigma = 588.4 \pm 77.9 = 510.5$ ,  $666.3 \approx 511$ , 666

Q4c  $\mu = 583.6$ ,  $\sigma = 82.2$ ,  $\Pr(510.5 < X < 666.3) \approx 0.656$ , i.e. 65.6%

Q4d Binomial distribution:  $n = 10$ ,  $p = 0.525$ ,  
 $\Pr(X > 6) \approx 0.22$

Q4e  $\mu = 588.4$ ,  $\sigma = 77.9$

$\Pr(X > 583.6 | X < 788) = \frac{\Pr(583.6 < X < 788)}{\Pr(X < 788)} \approx 0.522$ , i.e.

52.2%

Q4f  $\Pr(X < 668) = 0.80$  and  $\Pr(X > 544) = 0.70$

$\Pr\left(Z < \frac{668 - \mu}{\sigma}\right) = 0.80$  and  $\Pr\left(Z > \frac{544 - \mu}{\sigma}\right) = 0.70$

$\frac{668 - \mu}{\sigma} = 0.8416$ ,  $\frac{544 - \mu}{\sigma} = -0.5244$

$\therefore \mu \approx 591.6$  and  $\sigma \approx 90.8$

Q4g In Vic.,  $\Pr(X < a) = 0.95$ ,  $a \approx 720.47$

In NSW,  $\Pr(X > 720.47) \approx 0.08$ , i.e. the top 8%,  $x = 8$

Q4h  $\Pr(baaa) = 1 \times 0.4 \times 0.9 \times 0.9 \approx 0.32$

Q4i  $\Pr(aaab) + \Pr(aaba) + \Pr(aaaa)$

$= 1 \times 1 \times 0.9 \times 0.1 + 1 \times 1 \times 0.1 \times 0.4 + 1 \times 1 \times 0.9 \times 0.9 \approx 0.94$

Q4j  $\Pr(aaaa) = 1 \times 0.9 \times 1 \times 0.9 = 0.81$

$\Pr(aaab) + \Pr(abaa) = 1 \times 0.9 \times 1 \times 0.1 + 1 \times 0.1 \times 1 \times 0.9 = 0.18$

$\Pr(abab) = 1 \times 0.1 \times 1 \times 0.1 = 0.01$

$E(X) = 4 \times 0.81 + 3 \times 0.18 + 2 \times 0.01 = 3.8$

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