

Q1a $y = mx + c$ and $y_1 = -4x - x^2$:

$$mx + c = -4x - x^2, x^2 + mx + 4x + c = 0, x^2 + (m+4)x + c = 0$$

One solution only for $y = mx + c$ to be a tangent, $\therefore \Delta = 0$

$$\therefore (m+4)^2 - 4c = 0, m^2 + 8m + 4(4-c) = 0$$

$$y = mx + c \text{ and } y_2 = (x-2)^2 - 2:$$

$$(x-2)^2 - 2 = mx + c, x^2 - 4x + 2 - mx - c = 0,$$

$$x^2 - (m+4)x + (2-c) = 0, \Delta = (m+4)^2 - 4(2-c) = 0$$

$$\therefore m^2 + 8m + 4(2+c) = 0$$

Q1b $m^2 + 8m + 4(4-c) = 0$ (1)

$$m^2 + 8m + 4(2+c) = 0$$
 (2)

(2) - (1): $c = 1$

Substitute in (2), $m^2 + 8m + 12 = 0, (m+6)(m+2) = 0$

$$\therefore m = -6, -2$$

The common tangents are: $y = -6x + 1, y = -2x + 1$

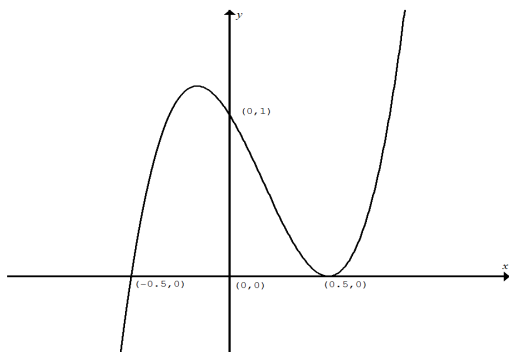
Q2 $f(x+k) = 2 - f(x), a \cos\left(\frac{x+k}{3}\right) + 1 = 2 - \left(a \cos\left(\frac{x}{3}\right) + 1\right)$

$$\therefore \cos\left(\frac{x+k}{3}\right) = -\cos\left(\frac{x}{3}\right) = \cos\left(\frac{x}{3} + n\pi\right)$$

$$\therefore \frac{k}{3} = n\pi \text{ where } n \text{ is an integer}$$

Since $-3\pi \leq k \leq 3\pi, \therefore k = \pm 3\pi$

Q3a $g(x) = 8x^3 - 4x^2 - 2x + 1 = 4x^2(2x-1) - (2x-1)$
 $= (4x^2 - 1)(2x-1) = (2x+1)(2x-1)^2$



Q3b $g(x) = f(1-2x), g\left(\frac{x}{2}\right) = f(1-x), g\left(\frac{-x}{2}\right) = f(1+x)$

$$g\left(\frac{-(x-1)}{2}\right) = f(x), \therefore f(x) = g\left(\frac{1-x}{2}\right)$$

$$= \left(2\left(\frac{1-x}{2}\right) + 1\right) \left(2\left(\frac{1-x}{2}\right) - 1\right)^2 = (2-x)(-x)^2 = -x^3 + 2x^2$$

$$\therefore a = -1, b = 2, c = 0 \text{ and } d = 0$$

Q4a $\Pr(2 \text{ cups of coffee}) = \Pr(\text{ccct}) + \Pr(\text{ctct}) + \Pr(\text{cttc})$

$$= 1 \times \frac{3}{5} \times \frac{2}{5} \times \frac{4}{5} + 1 \times \frac{2}{5} \times \frac{1}{5} \times \frac{2}{5} + 1 \times \frac{2}{5} \times \frac{4}{5} \times \frac{1}{5} = \frac{36}{125}$$

Q4b $\Pr(\text{at least 2 cups of coffee}) = 1 - \Pr(1 \text{ cup of coffee})$

$$= 1 - \Pr(\text{cttt}) = 1 - 1 \times \frac{2}{5} \times \frac{4}{5} \times \frac{4}{5} = \frac{93}{125}$$

Q4c $\Pr(3 \text{ cups of coffee}) = \Pr(\text{ccct}) + \Pr(\text{cctc}) + \Pr(\text{ctcc})$

$$= 1 \times \frac{3}{5} \times \frac{3}{5} \times \frac{2}{5} + 1 \times \frac{3}{5} \times \frac{2}{5} \times \frac{1}{5} + 1 \times \frac{2}{5} \times \frac{1}{5} \times \frac{3}{5} = \frac{30}{125}$$

$$\Pr(4 \text{ cups of coffee}) = \Pr(\text{cccc}) = 1 \times \frac{3}{5} \times \frac{3}{5} \times \frac{3}{5} = \frac{27}{125}$$

x cups of coffee	1	2	3	4
$\Pr(X = x)$	$\frac{32}{125}$	$\frac{36}{125}$	$\frac{30}{125}$	$\frac{27}{125}$

$$E(X) = 1 \times \frac{32}{125} + 2 \times \frac{36}{125} + 3 \times \frac{30}{125} + 4 \times \frac{27}{125} = \frac{302}{125}$$

Q5a $f(x) = \log_{10}(9-x^2), 9-x^2 > 0, \therefore -3 < x < 3$

The domain is $(-3, 3)$.

Q5b $f(x) = \log_{10}(9-x^2) = \frac{\log_e(9-x^2)}{\log_e 10}$

$$f'(x) = \frac{-2x}{(\log_e 10)(9-x^2)}$$

$$f'(-1) = \frac{-2(-1)}{(\log_e 10)(9-(-1)^2)} = \frac{1}{4\log_e 10}$$

Q5c $\text{Area} \approx 1 \times \log_{10} 5 + 1 \times \log_{10} 8 + 1 \times \log_{10} 9 + 1 \times \log_{10} 8$

$$= \log_{10}(5 \times 8 \times 9 \times 8) = \log_{10} 2880 \text{ or } 1 + \log_{10} 288$$

Q6 $-12 \sin \frac{\pi x}{3}$ completes 2 cycles and $\frac{1}{12} \cos \frac{\pi x}{6}$ completes a cycle in $[-6, 6]$, $\therefore$ the average value of each one in the interval is zero.

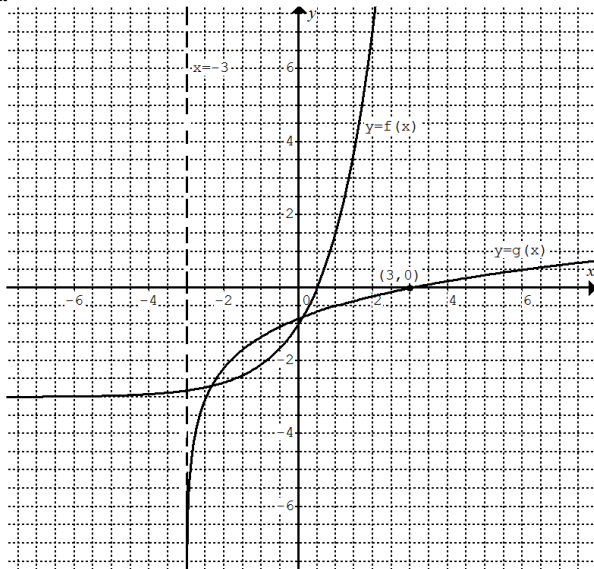
Average value of $f(x)$

$$\frac{\int_{-6}^6 f(x) dx}{6 - (-6)} = \frac{\int_{-6}^6 \left(1 - 12 \sin \frac{\pi x}{3} + \frac{1}{12} \cos \frac{\pi x}{6}\right) dx}{12}$$

$$= \frac{\int_{-6}^6 1 dx}{12} + \frac{\int_{-6}^6 -12 \sin \frac{\pi x}{3} dx}{12} + \frac{\int_{-6}^6 \frac{1}{12} \cos \frac{\pi x}{6} dx}{12}$$

$$= \frac{\int_{-6}^6 1 dx}{12} = 1$$

Q7a

Q7b Sequence of transformations of $f(x)$:

$$f(x) \rightarrow f^{-1}(x) \rightarrow f^{-1}(x-2) \rightarrow f^{-1}\left(\frac{x}{3}-2\right)$$

$$\therefore g(x) = f^{-1}\left(\frac{x}{3}-2\right)$$

$$\text{Q8a } y = |x|e^{-|x|} = \begin{cases} xe^{-x}, & x \geq 0 \\ -xe^x, & x \leq 0 \end{cases}$$

Note: The two sections in the hybrid function are reflections of each other in the y-axis, i.e. the y-axis is the axis of symmetry of the function.

$$\frac{dy}{dx} = \begin{cases} e^{-x} - xe^{-x}, & x > 0 \\ -e^x - xe^x, & x < 0 \end{cases}$$

Note: $\frac{dy}{dx}$ is undefined at $x = 0$

Q8b From Q8a, $\frac{dy}{dx} = e^{-x} - xe^{-x}$ for $x > 0$

$$\therefore \int_a^{\log_e 2} \frac{dy}{dx} dx = \int_a^{\log_e 2} e^{-x} dx - \int_a^{\log_e 2} xe^{-x} dx \text{ where } 0 < a < \log_e 2$$

$$\therefore \int_a^{\log_e 2} xe^{-x} dx = \int_a^{\log_e 2} e^{-x} dx - \int_a^{\log_e 2} \frac{dy}{dx} dx = [-e^{-x}]_a^{\log_e 2} - [xe^{-x}]_a^{\log_e 2}$$

$$= -e^{-\log_e 2} + e^{-a} - (\log_e 2)e^{-\log_e 2} + ae^{-a}$$

$$= -\frac{1}{2} + e^{-a} - \frac{1}{2}\log_e 2 + ae^{-a}$$

$$\int_{-\log_e 2}^{\log_e 2} |x|e^{-|x|} dx = 2 \times \lim_{a \rightarrow 0^+} \int_a^{\log_e 2} xe^{-x} dx$$

$$\therefore \int_{-\log_e 2}^{\log_e 2} |x|e^{-|x|} dx = 2 \times \lim_{a \rightarrow 0^+} \left(-\frac{1}{2} + e^{-a} - \frac{1}{2}\log_e 2 + ae^{-a} \right)$$

$$= 1 - \log_e 2$$

$$\begin{aligned} \text{Q9 } \frac{2^{2x} + 2^{-2x} - 2}{2^x - 2^{-x}} &= 2 \text{ where } 2^x - 2^{-x} \neq 0 \\ \frac{(2^x - 2^{-x})(2^x - 2^{-x})}{2^x - 2^{-x}} &= 2, \therefore 2^x - 2^{-x} = 2 \\ (2^x - 2^{-x}) \times 2^x &= 2 \times 2^x, \therefore (2^x)^2 - 2 \times 2^x - 1 = 0 \\ \therefore 2^x &= 1 + \sqrt{2} \text{ by the quadratic formula} \\ \therefore x &= \log_2(1 + \sqrt{2}) \end{aligned}$$

$$\text{Q10a } \int_0^2 (0.75 - k|x-1|) dx = 1, \int_0^2 0.75 dx - \int_0^2 k|x-1| dx = 1$$

$$\int_0^2 0.75 dx - 2 \times \int_1^2 k(x-1) dx = 1$$

$$[0.75x]_0^2 - 2k \left[\frac{(x-1)^2}{2} \right]_1^2 = 1, 1.5 - k = 1, k = 0.5$$

$$\text{Q10b } f(x) = \begin{cases} 0.75 - 0.5(-x+1), & 0 \leq x \leq 1 \\ 0.75 - 0.5(x-1), & 1 \leq x \leq 2 \\ 0, & \text{otherwise} \end{cases}$$

$$= \begin{cases} 0.5x + 0.25, & 0 \leq x \leq 1 \\ -0.5x + 1.25, & 1 \leq x \leq 2 \\ 0, & \text{otherwise} \end{cases}$$

$$E(X) = \int_0^1 x(0.5x + 0.25) dx + \int_1^2 x(-0.5x + 1.25) dx$$

$$= \int_0^1 (0.5x^2 + 0.25x) dx + \int_1^2 (-0.5x^2 + 1.25x) dx$$

$$= \left[\frac{x^3}{6} + \frac{x^2}{8} \right]_0^1 + \left[-\frac{x^3}{6} + \frac{5x^2}{8} \right]_1^2 = 1$$

$$\text{Q10c } \Pr(X < b) = 1 - \Pr(X > b) \text{ where } 0 < b < 2$$

$$\therefore \Pr(X > b) = 1 - \Pr(X < b) = \frac{3}{16}$$

$$\int_b^2 (-0.5x + 1.25) dx = \frac{3}{16}, \therefore \left[-\frac{x^2}{4} + \frac{5x}{4} \right]_b^2 = \frac{3}{16}$$

$$(-1 + 5) - \left(-\frac{b^2}{4} + \frac{5b}{4} \right) = \frac{3}{16}, 4b^2 - 20b + 21 = 0, \therefore b = \frac{3}{2}$$

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