

insightTM

Year 12 *Trial Exam Paper*

2014

MATHEMATICAL METHODS (CAS)

Written examination 1

Worked solutions

This book presents:

- worked solutions, giving you a series of points to show you how to work through the questions
- mark allocations
- tips on how to approach the questions

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Question 1a.**Worked solution**

$$\frac{d}{dx}(x \sin(2x)) = \sin(2x) + 2x \cos(2x)$$

Mark allocation: 2 marks

- 1 mark for evidence of using the product rule.
- 1 mark for the correct answer.

Question 1b.**Worked solution**

$$f(x) = e^{\sqrt{x}}$$

$$f'(x) = \frac{1}{2\sqrt{x}} e^{\sqrt{x}}$$

$$f'(4) = \frac{1}{4} e^2$$

Mark allocation: 2 marks

- 1 mark for the correct derivative $f'(x)$.
- 1 mark for the correct answer.

**Tip**

- *Remember to re-read the question before moving on. Many students differentiate correctly but then forget to evaluate.*

Question 1c.**Worked solution**

Average value of a function is

$$\begin{aligned}
 & \frac{1}{b-a} \int_a^b f(x) dx \\
 &= \frac{1}{k-1} \int_1^k \frac{1}{5-x} dx \\
 &= -\frac{1}{k-1} [\log_e |5-x|]_1^k \\
 &= -\frac{1}{k-1} [\log_e (5-k) - \log_e (4)] \\
 &= -\frac{1}{k-1} \log_e \left(\frac{5-k}{4} \right) \\
 &= \frac{1}{k-1} \log_e \left(\frac{4}{5-k} \right)
 \end{aligned}$$

Setting $\frac{1}{k-1} \log_e \left(\frac{4}{5-k} \right) = \frac{1}{2} \log_e (2)$ gives $k = 3$.

Mark allocation: 3 marks

- 1 mark for setting up $\frac{1}{k-1} \int_1^k \frac{1}{5-x} dx$.
- 1 mark for recognising $\frac{1}{k-1} [\log_e |5-x|]_1^k$ (or equivalent) as the integral.
- 1 mark for answer $k = 3$.

Question 2**Worked solution**

$$\begin{bmatrix} 0 & 4 \\ -3 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

$$\left. \begin{array}{l} 4y = x' \\ -3x = y' \end{array} \right\} \Rightarrow \begin{array}{l} y = \frac{x'}{4} \\ x = -\frac{y'}{3} \end{array}$$

Substituting into the equation $2y - 3x = 5$ gives $2\frac{x'}{4} - 3\frac{-y'}{3} = 5$, which

simplifies to $\frac{x'}{2} + y' = 5 \Rightarrow y' = -\frac{x'}{2} + 5$.

Mark allocation: 2 marks

- 1 mark for expanding the matrix to get equations for x and y in terms of y' and x' .
- 1 mark for the answer $y' = -\frac{x'}{2} + 5$ or equivalent versions.

Question 3a.**Worked solution**

$$\begin{aligned}f(g(x)) &= |x|^2 - 4|x| + 3 \\ &= x^2 - 4|x| + 3\end{aligned}$$

$$\text{Or } f(g(x)) = \begin{cases} x^2 - 4x + 3, & x \geq 0 \\ x^2 + 4x + 3, & x < 0 \end{cases}$$

Mark allocation: 1 mark

- 1 mark for the correct answer.

Question 3b.**Worked solution**

Considering $f(g(x)) = \begin{cases} x^2 - 4x + 3, & x \geq 0 \\ x^2 + 4x + 3, & x < 0 \end{cases}$

For $x \geq 0$, $f(x) = x^2 - 4x + 3$, so x -intercepts occur at

$$x^2 - 4x + 3 = 0$$

$$(x-1)(x-3) = 0$$

So $x = 1$, $x = 3$.

Turning point at $x = \frac{-b}{2a}$, so

$$x = \frac{4}{2} = 2$$

$$y = -1$$

Turning point is $(2, -1)$.

For $x < 0$, $f(x) = x^2 + 4x + 3$, so x -intercepts occur at

$$x^2 + 4x + 3 = 0$$

$$(x+1)(x+3) = 0$$

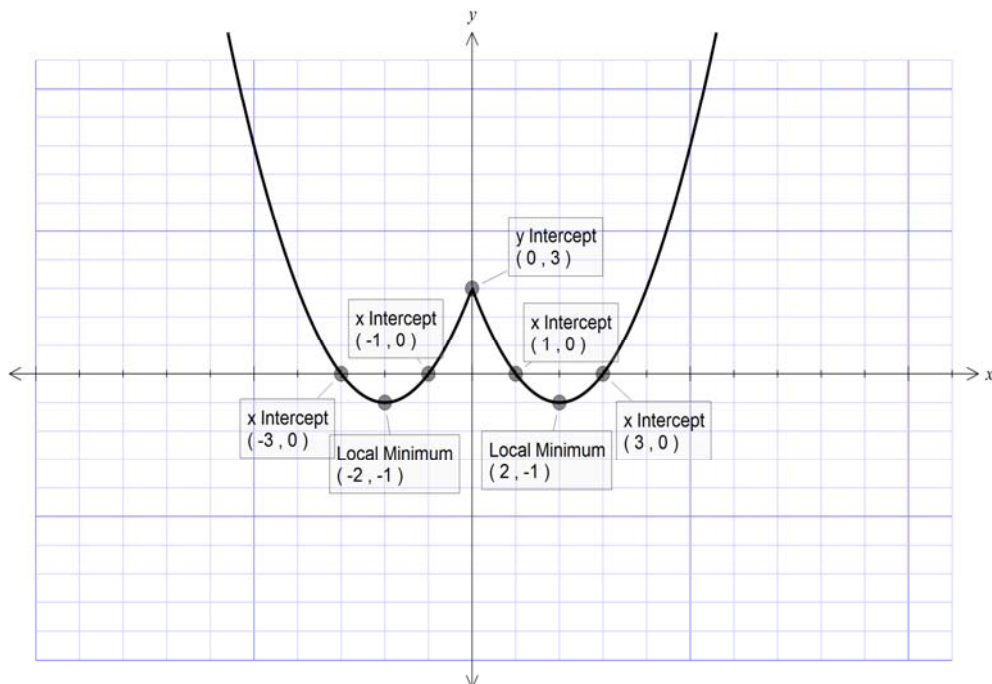
So $x = -1$, $x = -3$.

Turning point at $x = \frac{-b}{2a}$, so

$$x = \frac{-4}{2} = -2$$

$$y = -1$$

Turning point is $(-2, -1)$.



Mark allocation: 3 marks

- 1 mark for shape of graph showing two parabolic sections and a cusp at the y-axis.
- 1 mark for all intercepts labelled correctly.
- 1 mark for labelling the turning points correctly.

**Tips**

- To sketch graphs of the form $y = f(|x|)$, first sketch the graph for $x \geq 0$, then reflect the graph in the y-axis and this 'mirror image' becomes the graph for $x < 0$.
- Be careful to draw cusps as pointy sections, not as curves.

Question 3c.**Worked solution**

Domain of the derivative is $R \setminus \{0\}$.

Mark allocation: 1 mark

- 1 mark for the correct answer.

**Tips**

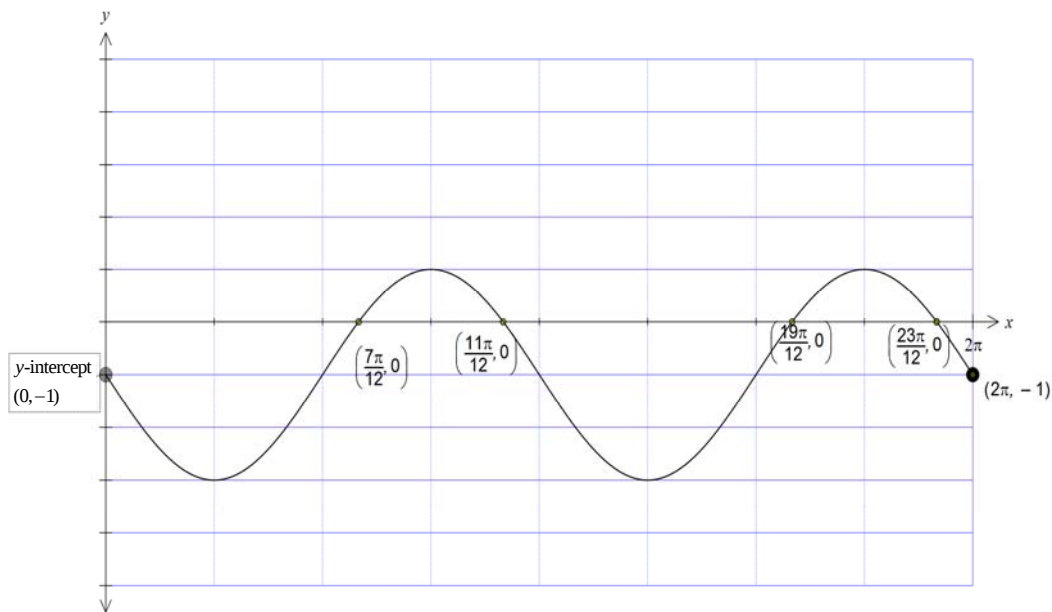
- Graphs are never differentiable at cusps.
- Remember to use a 'back slash' or reverse solidus; symbol: \.

Question 4a.**Worked solution**

Range is $[-3, 1]$ and period is π .

Mark allocation: 2 marks

- 1 mark for the range.
- 1 mark for the period.

Question 4b.**Worked solution**

To find the x -intercepts, first solve $-2 \sin(2x) - 1 = 0$:

$$-2 \sin(2x) - 1 = 0$$

$$\sin(2x) = -\frac{1}{2}$$

$$2x = \pi + \frac{\pi}{6}, 2\pi - \frac{\pi}{6}$$

$$2x = \frac{7\pi}{6}, \frac{11\pi}{6}$$

$$x = \frac{7\pi}{12}, \frac{11\pi}{12}$$

Additional intercepts are found by adding the period of π to both answers.

This gives x -intercepts of $\frac{7\pi}{12}, \frac{11\pi}{12}, \frac{19\pi}{12}, \frac{23\pi}{12}$.

Mark allocation: 3 marks

- 1 mark for showing two cycles.
- 1 mark for all x -intercepts labelled correctly.
- 1 mark for both end points labelled correctly.

Question 4c.**Worked solution**

This is best done graphically.

Look to place a horizontal line through the graph and have this line intersect the graph in *four* places.

It can be observed that this happens when $-3 < p < -1 \cup -1 < p < 1$.

Mark allocation: 1 mark

- 1 mark for the correct answer.

Question 5a.**Worked solution**

Interchange x and y :

$$x = 3\log_e(4 - y)$$

$$\frac{x}{3} = \log_e(4 - y)$$

$$e^{\frac{x}{3}} = 4 - y$$

$$y = 4 - e^{\frac{x}{3}}$$

So $f^{-1}(x) = 4 - e^{\frac{x}{3}}$.

Mark allocation: 2 marks

- 1 mark for swapping x and y .
- 1 mark for the correct rule.

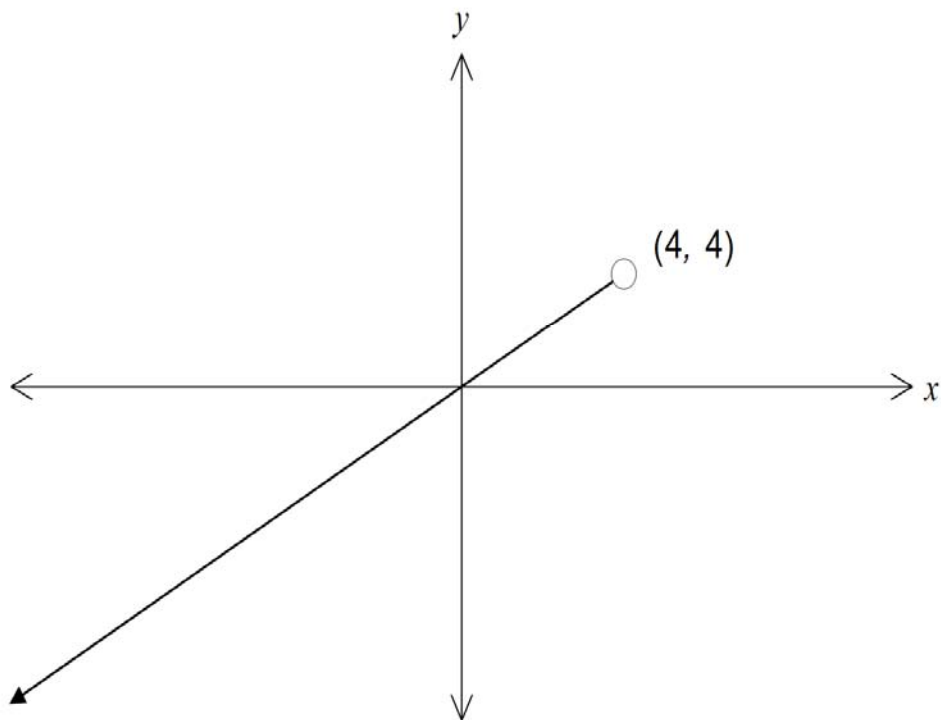
**Tip**

- *You must use the correct notation. In this case ' $y =$ ' is not acceptable; the answer must be written with f^{-1} .*

Question 5b.**Worked solution**

$$f^{-1}(f(x)) = x \text{ for } x \in \text{dom}(f(x)).$$

So in this case $f^{-1}(f(x)) = x$ for $x \in (-\infty, 4)$.

**Mark allocation: 1 mark**

- 1 mark for correctly drawn graph with correct domain.

**Tip**

- Always consider the domain of a function.

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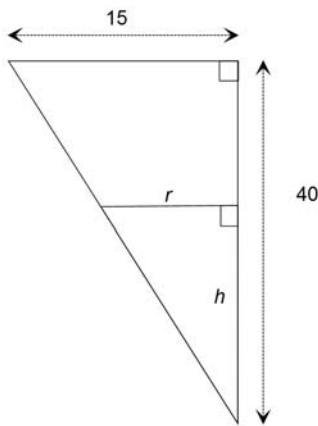
Question 6**Worked solution**

$$\frac{dh}{dt} = \frac{dV}{dt} \times \frac{dh}{dV}, \text{ where } \frac{dV}{dt} = 6 \text{ cm}^3/\text{min}.$$

$\frac{dh}{dV}$ will need to be found by developing a relationship between h and V .

For a cone, $V = \frac{1}{3}\pi r^2 h$.

For this cone, the following pair of similar triangles apply:



This gives $\frac{r}{15} = \frac{h}{40}$, so $r = \frac{3h}{8}$.

The formula for a cone is $V = \frac{1}{3}\pi r^2 h$, so for this cone $V = \frac{1}{3}\pi \left(\frac{3h}{8}\right)^2 h = \frac{3\pi h^3}{64}$.

Therefore, a volume of 24π has a height of

$$24\pi = \frac{3\pi h^3}{64}$$

$$8 \times 64 = h^3$$

$$h = 8 \text{ cm}$$

So differentiating $V = \frac{3\pi h^3}{64}$ gives $\frac{dV}{dh} = \frac{9\pi h^2}{64}$ and $\frac{dh}{dV} = \frac{64}{9\pi h^2}$.

Therefore, $\frac{dh}{dt} = \frac{dV}{dt} \times \frac{dh}{dV}$.

Substituting, this gives $\frac{dh}{dt} = 6 \times \frac{64}{9\pi h^2}$

So, at a height of 8 cm, $\frac{dh}{dt} = 6 \times \frac{64}{9 \times 64\pi}$
 $= \frac{6}{9\pi} = \frac{2}{3\pi} \text{ cm/min}$

Mark allocation: 3 marks

- 1 mark for setting up the rate equation $\frac{dh}{dt} = \frac{dV}{dt} \times \frac{dh}{dV}$.
- 1 mark for obtaining $\frac{dh}{dV} = \frac{64}{9\pi h^2}$.
- 1 mark for the answer $\frac{2}{3\pi}$ cm/min.

Question 7**Worked solution**

$$\begin{array}{cc} A_i & B_i \\ A_{i+1} & \begin{bmatrix} 0.3 & 0.6 \\ 0.7 & 0.4 \end{bmatrix} \\ B_{i+1} & \end{array}$$

$$\Pr(B, A, A, A, B) = 1 \times 0.6 \times 0.3 \times 0.3 \times 0.7 = \frac{3}{5} \times \frac{3}{10} \times \frac{3}{10} \times \frac{7}{10} = \frac{189}{5000}$$

Mark allocation: 2 marks

- 1 mark for either writing the transition matrix or for identifying the chain of probabilities.
- 1 mark for the correct answer.

Question 8a.**Worked solution**

Note that c will be a value that is one standard deviation above the mean of X .
So $c = 36 + 8 = 44$.

Alternatively, using symmetry, $\frac{c-36}{8} = 1$, so $c = 44$.

Mark allocation: 1 mark

- 1 mark for the correct answer.

Question 8b.**Worked solution**

Since 20 is 2 standard deviations below the mean of X , d will be an equivalent value that is 2 standard deviations above the mean of Z ; so $d = 2$.

$$\text{Alternatively, } z = \frac{\mu - X}{\sigma} = \frac{36 - 20}{8} = 2$$

Mark allocation: 1 mark

- 1 mark for the correct answer.

Question 9a.**Worked solution**

$$\text{Let } \int_0^4 k(2x + 3) dx = 1.$$

$$\begin{aligned} \text{LHS} &= k \int_0^4 (2x + 3) dx \\ &= k \left[x^2 + 3x \right]_0^4 \\ &= k [28 - 0] \\ &= 28k \end{aligned}$$

$$\text{So } 28k = 1 \Rightarrow k = \frac{1}{28}$$

Mark allocation: 2 marks

- 1 mark for setting up the integral equal to 1 or for using the area of a triangle.
- 1 mark for the correct antiderivative, leading to the correct result of k .

Question 9b.**Worked solution**

$$\begin{aligned} \Pr(X \leq 2 \mid X < 3) &= \frac{\Pr(X \leq 2 \cap X < 3)}{\Pr(X < 3)} \\ &= \frac{\Pr(X \leq 2)}{\Pr(X < 3)} \\ &= \frac{\frac{1}{28} \int_0^2 (2x + 3) dx}{\frac{1}{28} \int_0^3 (2x + 3) dx} \\ &= \frac{\left[x^2 + 3x \right]_0^2}{\left[x^2 + 3x \right]_0^3} = \frac{10}{18} = \frac{5}{9} \end{aligned}$$

Mark allocation: 2 marks

- 1 mark for setting up a conditional probability.
- 1 mark for the correct answer.

Question 10a.**Worked solution**

$$m_{AB} = \frac{f(0) - f(-p)}{0 - (-p)} = \frac{f(0) - f(-p)}{p}$$

$$m_{BC} = \frac{f(p) - f(0)}{p - 0} = \frac{f(p) - f(0)}{p}$$

Let $m_{AB} = m_{BC}$, giving:

$$\frac{f(p) - f(0)}{p} = \frac{f(0) - f(-p)}{p}$$

$$f(p) - f(0) = f(0) - f(-p)$$

$$f(p) + f(-p) = 2f(0)$$

$$\frac{f(p) + f(-p)}{2} = f(0)$$

Mark allocation: 1 mark

- 1 mark for the correct working, leading to the required answer.

Question 10bi.**Worked solution**

$$f(3) = -27 + 9b + 3c + d$$

$$f(-3) = 27 + 9b - 3c + d$$

$$f(0) = d$$

Using the result from part **a** gives:

$$\frac{f(p) + f(-p)}{2} = f(0)$$

$$\frac{-27 + 9b + 3c + d + 27 + 9b - 3c + d}{2} = d$$

$$\frac{18b + 2d}{2} = d$$

$$18b + 2d = 2d$$

$$18b = 0$$

$$b = 0$$

Mark allocation: 2 marks

- 1 mark for $f(3)$, $f(-3)$ and $f(0)$.
- 1 mark for correct working, leading to the required result.

Question 10bii.**Worked solution**

When $b = 0$, $f(x) = -x^3 + cx + d$, so $f'(x) = -3x^2 + c$.

$$m_{AC} = \frac{f(3) - f(0)}{3} = \frac{-27 + 3c + d - d}{3} = -9 + c$$

Let $f'(x) = m_{AC}$.

So, $-9 + c = -3x^2 + c$

$$3 = x^2$$

$$x = \pm\sqrt{3}$$

$x = \sqrt{3}$ gives $f(\sqrt{3}) = -3\sqrt{3} + \sqrt{3}c + d$.

$x = -\sqrt{3}$ gives $f(-\sqrt{3}) = 3\sqrt{3} - \sqrt{3}c + d$.

So, the coordinates are $(\sqrt{3}, -3\sqrt{3} + \sqrt{3}c + d)$ and $(-\sqrt{3}, 3\sqrt{3} - \sqrt{3}c + d)$.

Mark allocation: 3 marks

- 1 mark for finding $f'(x)$.
- 1 mark for setting $f'(x) = m_{AC}$.
- 1 mark for the correct coordinates.

END OF WORKED SOLUTIONS BOOK