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GROUP**

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**MATHS METHODS (CAS) 3 & 4
TRIAL EXAMINATION 1
SOLUTIONS
2014**

Question 1 (4 marks)

a. $y = 3x^2 e^{4x}$
 $\frac{dy}{dx} = 6xe^{4x} + 3x^2 \times 4e^{4x}$ (product rule)
 $= 6xe^{4x} + 12x^2 e^{4x}$

(1 mark) – correct first term
(1 mark) – correct second term

b. $g(x) = \frac{\sin(x)}{2x}$
 $g'(x) = \frac{2x \times \cos(x) - 2 \times \sin(x)}{(2x)^2}$ (quotient rule) **(1 mark)**

$$g'\left(\frac{\pi}{2}\right) = \frac{2 \times \frac{\pi}{2} \times 0 - 2 \times 1}{\left(2 \times \frac{\pi}{2}\right)^2}$$

since $\cos\left(\frac{\pi}{2}\right) = 0$
and $\sin\left(\frac{\pi}{2}\right) = 1$

$$= \frac{0 - 2}{\pi^2}$$

$$= \frac{-2}{\pi^2}$$

(1 mark)

Question 2 (2 marks)

$$\int \frac{1}{2x-3} dx = \frac{1}{2} \log_e |2x-3|$$

Note, because we were asked for ‘an’ antiderivative c is not required (because c could, in this case, equal zero)

(1 mark) – recognition that a logarithm was required
(1 mark) – correct answer

Question 3 (2 marks)

$$f : R \setminus \{0\} \rightarrow R, f(x) = \frac{5}{x}$$

To show, $f(f(x)) = f(f^{-1}(x))$

$$LS = f(f(x))$$

$$= f\left(\frac{5}{x}\right)$$

$$= \frac{5}{\frac{5}{x}}$$

$$= 5 \div \frac{5}{x}$$

$$= 5 \times \frac{x}{5}$$

$$= x$$

(1 mark)

Now find $f^{-1}(x)$.

$$f(x) = \frac{5}{x}$$

$$\text{Let } y = \frac{5}{x}$$

Swap x and y for inverse.

$$x = \frac{5}{y}$$

$$y = \frac{5}{x}$$

$$f^{-1}(x) = \frac{5}{x}$$

$$RS = f(f^{-1}(x))$$

$$= f\left(\frac{5}{x}\right)$$

$$= x$$

$$= LS$$

Have shown.

(1 mark)

Question 4 (5 marks)

a. $\log_6(3) - 2\log_6(x) + \log_6(2) = 1$

$$\log_6(3) - \log_6(x^2) + \log_6(2) = 1$$

$$\log_6\left(\frac{3 \times 2}{x^2}\right) = 1 \quad (1 \text{ mark})$$

$$6^1 = \frac{6}{x^2}$$

$$6x^2 = 6$$

$$x^2 = 1$$

$$x = \pm 1$$

(1 mark)

but $x > 0$ (for the term $-2\log_6(x)$ to be defined)

So $x = 1$.

(1 mark)

b. $8^{1-2x} = 2^{4+x}$

$$(2^3)^{1-2x} = 2^{4+x}$$

$$2^{3-6x} = 2^{4+x}$$

(1 mark)

So $3 - 6x = 4 + x$

$$-1 = 7x$$

$$x = -\frac{1}{7}$$

(1 mark)

Question 5 (2 marks)

$$\sin\left(\frac{x}{3}\right) = \frac{1}{\sqrt{2}}$$

$$x \in [0, 6\pi]$$

$$\frac{x}{3} = \frac{\pi}{4}, \frac{3\pi}{4}$$

$$\frac{x}{3} \in [0, 2\pi]$$

$$x = \frac{3\pi}{4}, \frac{9\pi}{4}$$



sin is positive in the 1st and 2nd quadrants

(1 mark) – one correct answer

(1 mark) – second correct answer

Question 6 (5 marks)

- a. $\Pr(X=0) + \Pr(X=1) + \Pr(X=2) = 0.5$
and, $\Pr(X=3) + \Pr(X=4) = 0.5$.

The median of X is therefore halfway between 2 and 3 so the median of X is 2.5

(1 mark)

- b. Method 1 – using intuition and the table.

$$\begin{aligned}\Pr(X \leq 2 \mid X > 0) &= \frac{0.3}{0.8} \\ &= \frac{3}{8}\end{aligned}$$

(1 mark)

(1 mark)

Since we're told $X > 0$, then we eliminate $\Pr(X=0)$ which gives us a denominator of 0.8.

We want $\Pr(X \leq 2)$, so we want $\Pr(X=1)$ and $\Pr(X=2)$. Remember that $\Pr(X=0)$ is eliminated.

So the numerator is $0.2 + 0.1 = 0.3$.

Method 2 – using the conditional probability formula

$$\Pr(X \leq 2 \mid X > 0) = \frac{\Pr(X \leq 2 \cap X > 0)}{\Pr(X > 0)} \text{ (from the formula sheet)} \quad \textbf{(1 mark)}$$

$$\begin{aligned}&= \frac{\Pr(X=1) + \Pr(X=2)}{\Pr(X=1) + \Pr(X=2) + \Pr(X=3) + \Pr(X=4)} \\ &= \frac{0.2 + 0.1}{0.2 + 0.1 + 0.4 + 0.1} \\ &= \frac{0.3}{0.8} \\ &= \frac{3}{8}\end{aligned}$$

(1 mark)

- c. $\text{Var}(X) = E(X^2) - [E(X)]^2$

$$= 0^2 \times 0.2 + 1^2 \times 0.2 + 2^2 \times 0.1 + 3^2 \times 0.4 + 4^2 \times 0.1 - 2^2$$

(1 mark)

$$= 0.2 + 0.4 + 3.6 + 1.6 - 4$$

$$= 1.8$$

(1 mark)

This formula is not given on the formula sheet. It is easier to use than the one on the formula sheet so therefore worth remembering.

Question 7 (4 marks)

- a. Since f is a probability density function,

$$\int_0^1 a \cos\left(\frac{\pi x}{2}\right) dx = 1$$

(1 mark)

$$a \left[\frac{1}{\pi/2} \sin\left(\frac{\pi x}{2}\right) \right]_0^1 = 1 \quad (a \text{ is a constant})$$

(1 mark)

$$a \left\{ \frac{2}{\pi} \sin\left(\frac{\pi}{2}\right) - \frac{2}{\pi} \sin(0) \right\} = 1$$

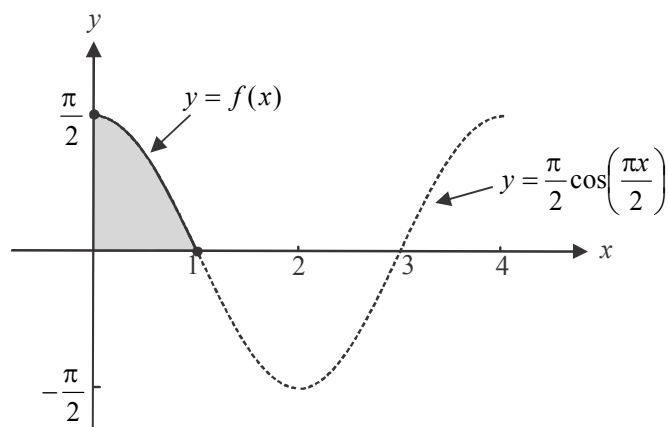
$$a \left(\frac{2}{\pi} \times 1 - 0 \right) = 1$$

$$\frac{2a}{\pi} = 1$$

$$a = \frac{\pi}{2}$$

(1 mark)

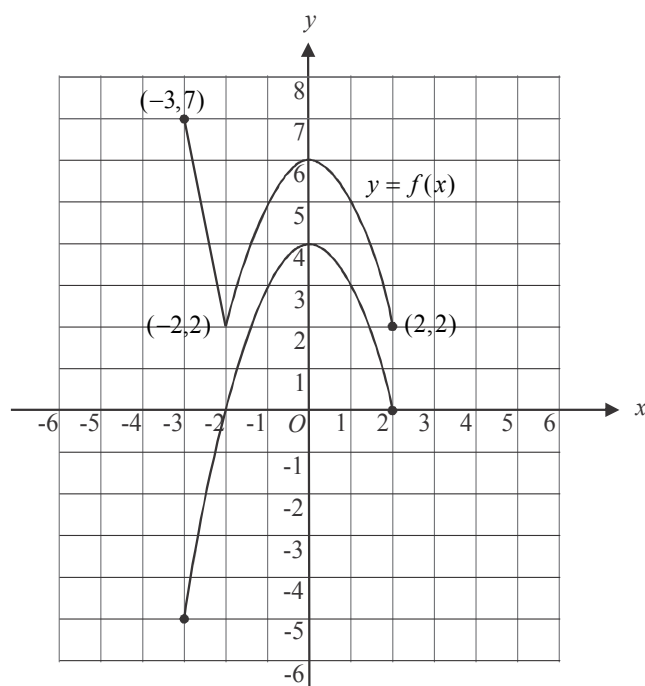
- b. The mode of X is the value of x for which $f(x)$ is a maximum.
Do a quick sketch.



The maximum value of $f(x)$ is $\frac{\pi}{2}$ and it occurs when $x = 0$.

The mode is zero.

(1 mark)

Question 8 (6 marks)**a.**

(1 mark) – correct shape
(1 mark) – correct position
 including endpoints

b. i.

$$f(x) = |4 - x^2| + 2$$

When the graph of f is translated 2 units in the negative direction of the y -axis, its rule becomes

$$y = |4 - x^2| + 2 - 2, \text{ that is, } y = |4 - x^2|$$

When this graph is then translated 2 units in the positive direction of the x -axis, its rule becomes $y = |4 - (x - 2)^2|$

When this graph is then dilated from the y -axis by a factor of $\frac{1}{2}$, its rule

$$\text{becomes } y = \left| 4 - \left(\frac{x}{\frac{1}{2}} - 2 \right)^2 \right|, \text{ that is, } h(x) = |4 - (2x - 2)^2|.$$

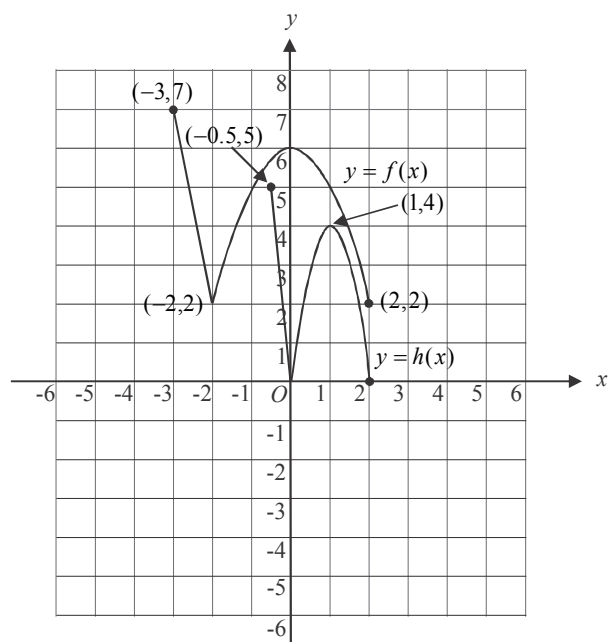
(1 mark) – for $y = |4 - (x - 2)^2|$

(1 mark) – for $h(x) = |4 - (2x - 2)^2|$

ii. Method 1 – drawing a graph

To find the domain, do a quick sketch of $y = f(x)$ after it has undergone the two translations and the dilation to become $y = h(x)$.

$$d_h = [-0.5, 2] \quad \textbf{(1 mark)}$$



Method 2 – using the rule

$$f(x) = |4 - x^2| + 2 \quad d_f = [-3, 2] \quad r_f = [2, 7] \text{ from part a.}$$

After a translation 2 units in the negative direction of the y -axis, we have:

$$y = |4 - x^2| \quad d = [-3, 2] \quad r = [2 - 2, 7 - 2] = [0, 5]$$

After a translation 2 units in the positive direction of the x -axis, we have:

$$y = |4 - (x - 2)^2| \quad d = [-3 + 2, 2 + 2] = [-1, 4] \quad r = [0, 5]$$

After a dilation from the y -axis by a factor of $\frac{1}{2}$, we have:

$$h(x) = |4 - (2x - 2)^2| \quad d_h = [-1 \times \frac{1}{2}, 4 \times \frac{1}{2}] = [-\frac{1}{2}, 2] \quad r_h = [0, 5]$$

$$\text{So } d_h = \left[-\frac{1}{2}, 2\right] \quad (1 \text{ mark})$$

iii. Using either method in part ii., $r_h = [0, 5]$ (1 mark)

Question 9 (4 marks)

a. $\frac{d}{dx}(1-x)\log_e(x) = -1 \times \log_e(x) + (1-x) \times \frac{1}{x}$ (product rule)

$$= -\log_e(x) + \frac{1}{x} - \frac{x}{x}$$

$$= -\log_e(x) + x^{-1} - 1$$

(1 mark)

b. The area of the shaded region is given by the integral $\int_{0.5}^1 -\log_e(x) dx$.

From part a., we have

$$\frac{d}{dx}(1-x)\log_e(x) = -\log_e(x) + x^{-1} - 1$$

Rearrange this.

$$-\log_e(x) = \frac{d}{dx}(1-x)\log_e(x) - x^{-1} + 1$$

Finding the antiderivative of each and every term on both sides of the equation gives:

$$\int_{0.5}^1 -\log_e(x) dx = \int_{0.5}^1 \frac{d}{dx}(1-x)\log_e(x) dx - \int_{0.5}^1 x^{-1} dx + \int_{0.5}^1 1 dx \quad (1 \text{ mark})$$

$$= [(1-x)\log_e(x)]_{0.5}^1 - [\log_e|x|]_{0.5}^1 + [x]_{0.5}^1 \quad (1 \text{ mark})$$

$$= \{0 - (1-0.5)\log_e(0.5)\} - \{\log_e(1) - \log_e(0.5)\} + \{1 - 0.5\}$$

$$= -0.5\log_e(0.5) + \log_e(0.5) + 0.5$$

$$= 0.5\log_e(0.5) + 0.5$$

So area = $0.5(\log_e(0.5) + 1)$ square units as required.

(1 mark)

Note that $\log_e(1) = 0$ and that $\int_{0.5}^1 \frac{d}{dx}(1-x)\log_e(x) dx$

$$= [(1-x)\log_e(x)]_{0.5}^1$$

because the antiderivative ‘undoes’ the derivative.

Question 10 (6 marks)

a. i. In $\triangle CDE$, $\sin(\theta) = \frac{DE}{CD}$
 $= \frac{DE}{p}$ since $AB = DC = p$ cm
 So $DE = p \sin(\theta)$ (1 mark)

ii. In $\triangle CDE$, $\cos(\theta) = \frac{CE}{CD}$
 $= \frac{CE}{p}$
 $CE = p \cos(\theta)$ (1 mark)

b. $A = \text{area of rectangle } ABCD - \text{area of } \triangle CDE$
 $= 4 \times p - \frac{1}{2} \times DE \times CE$
 $= 4p - \frac{1}{2} \times p \sin(\theta) \times p \cos(\theta)$
 $A = 4p - \frac{p^2}{2} \sin(\theta) \cos(\theta)$ (1 mark)

c. $\frac{dA}{d\theta} = -\frac{p^2}{2} (\cos(\theta) \times \cos(\theta) + \sin(\theta) \times -\sin(\theta))$ (product rule)
 $= -\frac{p^2}{2} (\cos^2(\theta) - \sin^2(\theta))$ (1 mark)

Min occurs when $\frac{dA}{d\theta} = 0$ so

$$-\frac{p^2}{2} (\cos^2(\theta) - \sin^2(\theta)) = 0$$

$$\cos^2(\theta) - \sin^2(\theta) = 0 \quad \text{since } p > 0$$

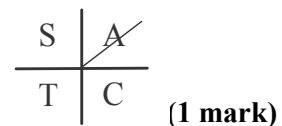
$$\cos^2(\theta) = \sin^2(\theta)$$

$$1 = \frac{\sin^2(\theta)}{\cos^2(\theta)}$$

$$1 = \tan^2(\theta)$$

$$\tan(\theta) = 1 \quad \text{or} \quad \tan(\theta) = -1$$

$$\theta = \frac{\pi}{4} \quad \text{not possible since } 0 < \theta < \frac{\pi}{2}$$



d. From part c., the minimum area occurs when $\theta = \frac{\pi}{4}$. From part b.,

$A = 4p - \frac{p^2}{2} \sin(\theta) \cos(\theta)$ $16 = 4p - \frac{p^2}{2} \sin\left(\frac{\pi}{4}\right) \cos\left(\frac{\pi}{4}\right)$ $16 = 4p - \frac{p^2}{2} \times \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}}$ $16 = 4p - \frac{p^2}{4}$	$64 = 16p - p^2$ $p^2 - 16p + 64 = 0$ $(p - 8)^2 = 0$ $p = 8$
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(1 mark)