



Victorian Certificate of Education 2013

SUPERVISOR TO ATTACH PROCESSING LABEL HERE

STUDENT NUMBER

Letter

Figures

Words

MATHEMATICAL METHODS (CAS)

Written examination 2

Thursday 7 November 2013

Reading time: 3.00 pm to 3.15 pm (15 minutes)

Writing time: 3.15 pm to 5.15 pm (2 hours)

QUESTION AND ANSWER BOOK

Structure of book

Section	Number of questions	Number of questions to be answered	Number of marks
1	22	22	22
2	4	4	58
			Total 80

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners, rulers, a protractor, set-squares, aids for curve sketching, one bound reference, one approved CAS calculator (memory DOES NOT need to be cleared) and, if desired, one scientific calculator. For approved computer-based CAS, their full functionality may be used.
- Students are NOT permitted to bring into the examination room: blank sheets of paper and/or white out liquid/tape.

Materials supplied

- Question and answer book of 24 pages with a detachable sheet of miscellaneous formulas in the centrefold.
- Answer sheet for multiple-choice questions.

Instructions

- Detach the formula sheet from the centre of this book during reading time.
- Write your **student number** in the space provided above on this page.
- Check that your **name** and **student number** as printed on your answer sheet for multiple-choice questions are correct, **and** sign your name in the space provided to verify this.
- All written responses must be in English.

At the end of the examination

- Place the answer sheet for multiple-choice questions inside the front cover of this book.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.

SECTION 1**Instructions for Section 1**

Answer **all** questions in pencil on the answer sheet provided for multiple-choice questions.

Choose the response that is **correct** for the question.

A correct answer scores 1, an incorrect answer scores 0.

Marks will **not** be deducted for incorrect answers.

No marks will be given if more than one answer is completed for any question.

Question 1

The function with rule $f(x) = -3 \tan(2\pi x)$ has period

- A. $\frac{2}{\pi}$
- B. 2
- C. $\frac{1}{2}$
- D. $\frac{1}{4}$
- E. 2π

Question 2

The midpoint of the line segment that joins $(1, -5)$ to $(d, 2)$ is

- A. $\left(\frac{d+1}{2}, -\frac{3}{2}\right)$
- B. $\left(\frac{1-d}{2}, -\frac{7}{2}\right)$
- C. $\left(\frac{d-4}{2}, 0\right)$
- D. $\left(0, \frac{1-d}{3}\right)$
- E. $\left(\frac{5+d}{2}, 2\right)$

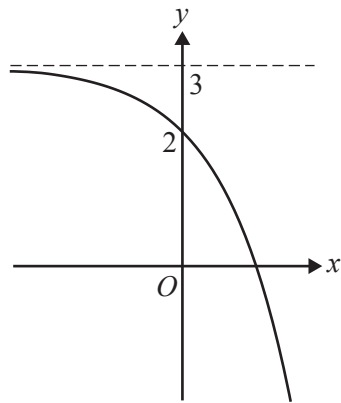
Question 3

If $x + a$ is a factor of $7x^3 + 9x^2 - 5ax$, where $a \in \mathbb{R} \setminus \{0\}$, then the value of a is

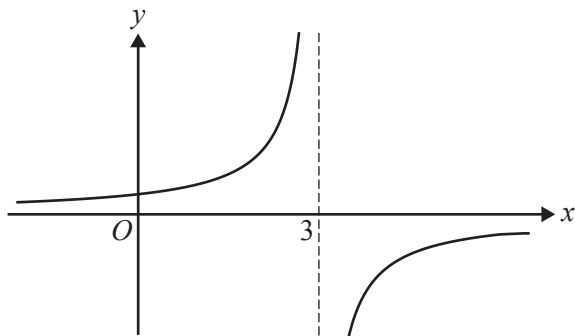
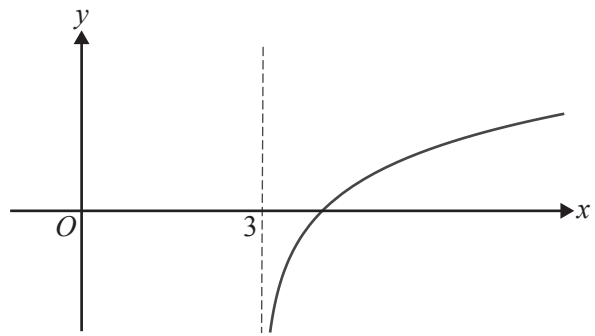
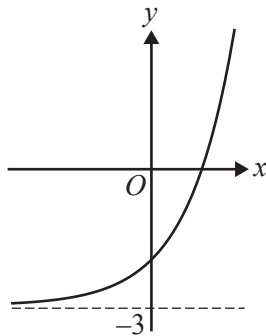
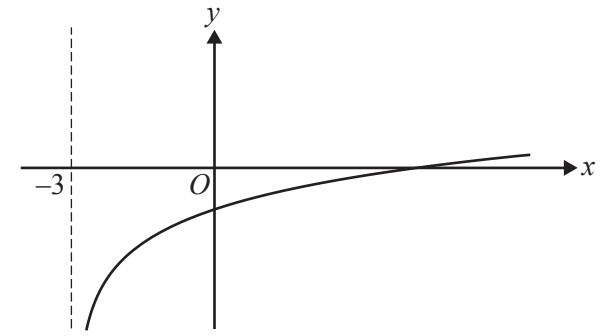
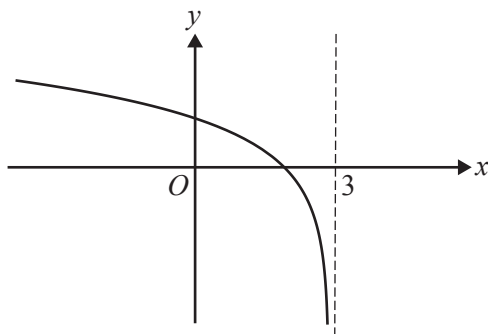
- A. -4
- B. -2
- C. -1
- D. 1
- E. 2

Question 4

Part of the graph of $y = f(x)$, where $f: \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = 3 - e^x$, is shown below.



Which one of the following could be the graph of $y = f^{-1}(x)$, where f^{-1} is the inverse of f ?

A.**B.****C.****D.****E.**

Question 5

If $f: (-\infty, 1) \rightarrow \mathbb{R}$, $f(x) = 2 \log_e(1 - x)$ and $g: [-1, \infty) \rightarrow \mathbb{R}$, $g(x) = 3\sqrt{x+1}$, then the maximal domain of the function $f+g$ is

- A. $[-1, 1)$
- B. $(1, \infty)$
- C. $(-1, 1]$
- D. $(-\infty, -1]$
- E. $\mathbb{R}$

Question 6

For the function $f(x) = \sin(2\pi x) + 2x$, the average rate of change for $f(x)$ with respect to x over the interval

$\left[\frac{1}{4}, 5\right]$ is

- A. 0
- B. $\frac{34}{19}$
- C. $\frac{7}{2}$
- D. $\frac{2\pi + 10}{4}$
- E. $\frac{23}{4}$

Question 7

The function $g: [-a, a] \rightarrow \mathbb{R}$, $g(x) = \sin\left(2\left(x - \frac{\pi}{6}\right)\right)$ has an inverse function. The maximum possible value of a is

- A. $\frac{\pi}{12}$
- B. 1
- C. $\frac{\pi}{6}$
- D. $\frac{\pi}{4}$
- E. $\frac{\pi}{2}$

Question 8

When Xenia travels to work, she either drives or takes the bus.

If she takes the bus to work one day, the probability that she takes the bus to work the next day is $\frac{7}{10}$.

If she drives to work one day, the probability that she drives to work the next day is $\frac{3}{5}$.

(Assume that Xenia will always travel to work according to these conditions only.)

What is the long-term probability that Xenia will take the bus to work?

- A. $\frac{3}{4}$
- B. $\frac{7}{10}$
- C. $\frac{4}{7}$
- D. $\frac{6}{13}$
- E. $\frac{3}{7}$

Question 9

Harry is a soccer player who practises penalty kicks many times each day.

Each time Harry takes a penalty kick, the probability that he scores a goal is 0.7, independent of any other penalty kick.

One day Harry took 20 penalty kicks.

Given that he scored at least 12 goals, the probability that Harry scored exactly 15 goals is closest to

- A. 0.1789
- B. 0.8867
- C. 0.8
- D. 0.6396
- E. 0.2017

Question 10

For events A and B , $\Pr(A \cap B) = p$, $\Pr(A' \cap B) = p - \frac{1}{8}$ and $\Pr(A \cap B') = \frac{3p}{5}$.

If A and B are independent, then the value of p is

- A. 0
- B. $\frac{1}{4}$
- C. $\frac{3}{8}$
- D. $\frac{1}{2}$
- E. $\frac{3}{5}$

Question 11

If the tangent to the graph of $y = e^{ax}$, $a \neq 0$, at $x = c$ passes through the origin, then c is equal to

- A. 0
- B. $\frac{1}{a}$
- C. 1
- D. a
- E. $-\frac{1}{a}$

Question 12

Let $y = 4 \cos(x)$ and x be a function of t such that $\frac{dx}{dt} = 3e^{2t}$ and $x = \frac{3}{2}$ when $t = 0$.

The value of $\frac{dy}{dt}$ when $x = \frac{\pi}{2}$ is

- A. 0
- B. $3\pi \log_e\left(\frac{\pi}{2}\right)$
- C. -4π
- D. -2π
- E. $-12e$

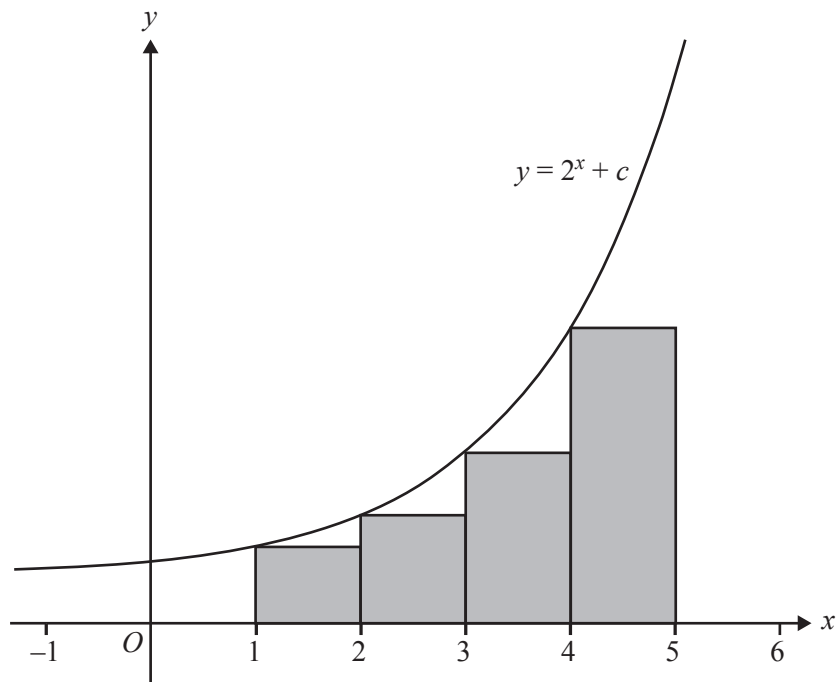
Question 13

If the equation $f(2x) - 2f(x) = 0$ is true for all real values of x , then the rule for f could be

- A. $\frac{x^2}{2}$
- B. $\sqrt{2x}$
- C. $2x$
- D. $\log_e\left(\frac{|x|}{2}\right)$
- E. $x - 2$

Question 14

Consider the graph of $y = 2^x + c$, where c is a real number. The area of the shaded rectangles is used to find an approximation to the area of the region that is bounded by the graph, the x -axis and the lines $x = 1$ and $x = 5$.



If the total area of the shaded rectangles is 44, then the value of c is

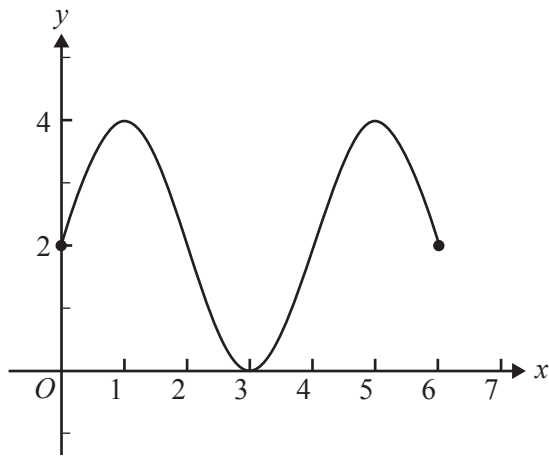
- A. 14
- B. -4
- C. $\frac{14}{5}$
- D. $\frac{7}{2}$
- E. $-\frac{16}{5}$

Question 15

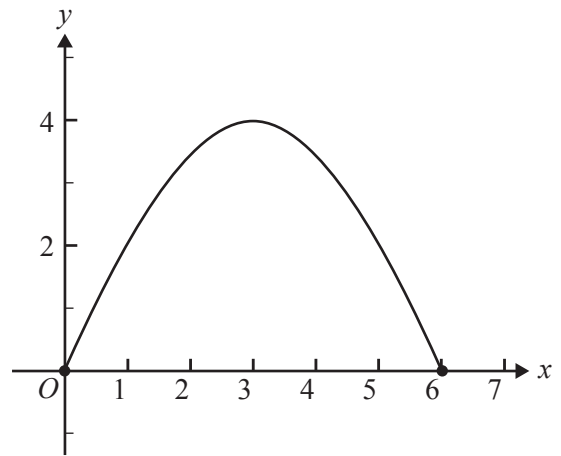
Let h be a function with an average value of 2 over the interval $[0, 6]$.

The graph of h over this interval could be

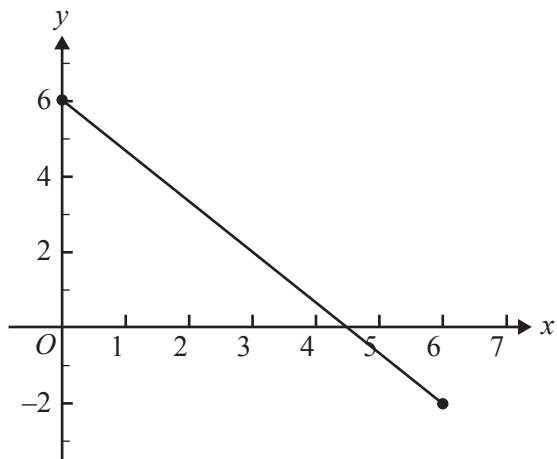
A.



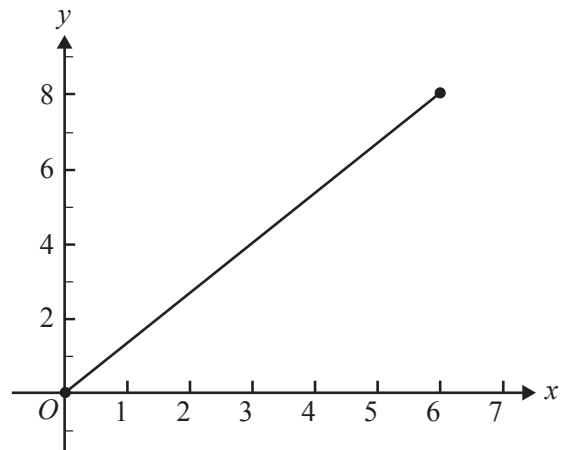
B.



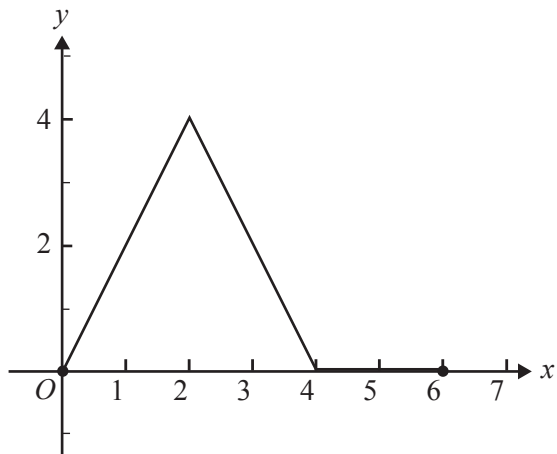
C.



D.

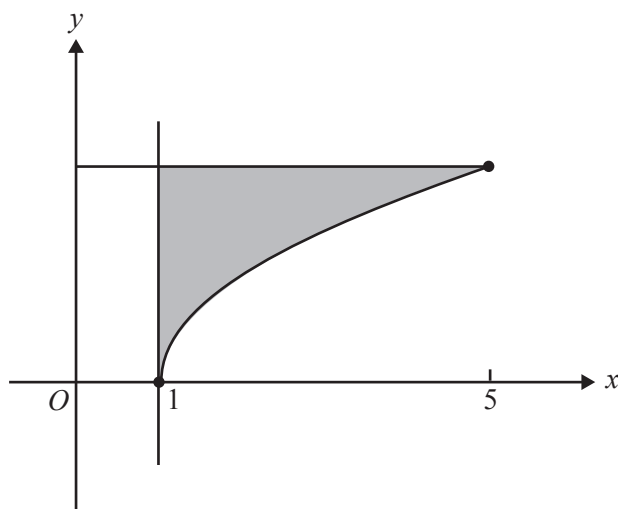


E.



Question 16

The graph of $f: [1, 5] \rightarrow \mathbb{R}, f(x) = \sqrt{x-1}$ is shown below.



Which one of the following definite integrals could be used to find the area of the shaded region?

- A. $\int_1^5 (\sqrt{x-1}) dx$
- B. $\int_0^2 (\sqrt{x-1}) dx$
- C. $\int_0^5 (2 - \sqrt{x-1}) dx$
- D. $\int_0^2 (x^2 + 1) dx$
- E. $\int_0^2 (x^2) dx$

Question 17

A and B are events of a sample space.

Given that $\Pr(A|B) = p$, $\Pr(B) = p^2$ and $\Pr(A) = p^{\frac{1}{3}}$, $\Pr(B|A)$ is equal to

- A. p
- B. $p^{\frac{4}{3}}$
- C. $p^{\frac{7}{3}}$
- D. $p^{\frac{8}{3}}$
- E. p^3

Question 18

Let $g(x) = \log_2(x)$, $x > 0$.

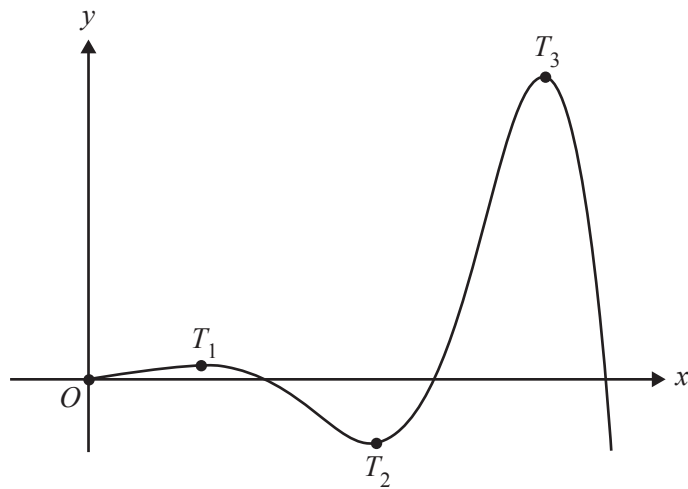
Which one of the following equations is true for all positive real values of x ?

- A. $2g(8x) = g(x^2) + 8$
- B. $2g(8x) = g(x^2) + 6$
- C. $2g(8x) = (g(x) + 8)^2$
- D. $2g(8x) = g(2x) + 6$
- E. $2g(8x) = g(2x) + 64$

Question 19

Part of the graph of a function $f: [0, \infty) \rightarrow \mathbb{R}$, $f(x) = e^{x\sqrt{3}} \sin(x)$ is shown below.

The first three turning points are labelled T_1 , T_2 and T_3 .



The x -coordinate of T_3 is

- A. $\frac{8\pi}{3}$
- B. $\frac{16\pi}{3}$
- C. $\frac{13\pi}{6}$
- D. $\frac{17\pi}{6}$
- E. $\frac{29\pi}{6}$

Question 20

A transformation $T: R^2 \rightarrow R^2$, $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 5 \\ 0 \end{bmatrix}$ maps the graph of a function f to the graph of $y = x^2$, $x \in R$.

The rule of f is

- A. $f(x) = -(x + 5)^2$
- B. $f(x) = (5 - x)^2$
- C. $f(x) = -(x - 5)^2$
- D. $f(x) = -x^2 + 5$
- E. $f(x) = x^2 - 5$

Question 21

The cubic function $f: R \rightarrow R$, $f(x) = ax^3 - bx^2 + cx$, where a , b and c are positive constants, has no stationary points when

- A. $c > \frac{b^2}{4a}$
- B. $c < \frac{b^2}{4a}$
- C. $c < 4b^2a$
- D. $c > \frac{b^2}{3a}$
- E. $c < \frac{b^2}{3a}$

Question 22

Butterflies of a particular species die T days after hatching, where T is a normally distributed random variable with a mean of 120 days and a standard deviation of σ days.

If, from a population of 2000 newly hatched butterflies, 150 are expected to die in the first 90 days, then the value of σ is closest to

- A. 7 days
- B. 13 days
- C. 17 days
- D. 21 days
- E. 37 days

SECTION 2**Instructions for Section 2**

Answer **all** questions in the spaces provided.

In all questions where a numerical answer is required, an exact value must be given unless otherwise specified.

In questions where more than one mark is available, appropriate working **must** be shown.

Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Question 1 (12 marks)

Trigg the gardener is working in a temperature-controlled greenhouse. During a particular 24-hour time interval, the temperature (T °C) is given by $T(t) = 25 + 2\cos\left(\frac{\pi t}{8}\right)$, $0 \leq t \leq 24$, where t is the time in hours from the beginning of the 24-hour time interval.

- a.** State the maximum temperature in the greenhouse and the values of t when this occurs. 2 marks

- b.** State the period of the function T . 1 mark

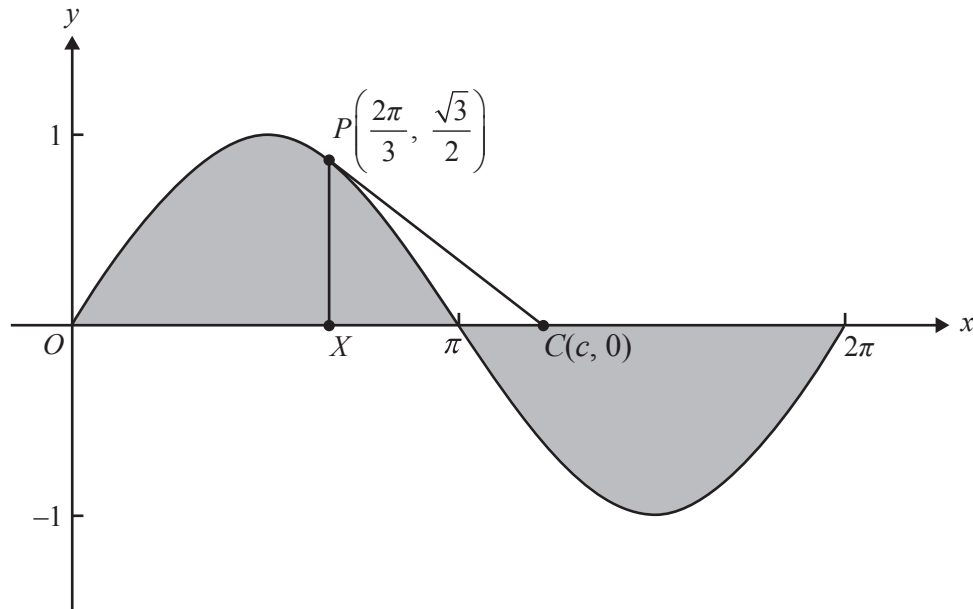
- c.** Find the smallest value of t for which $T = 26$. 2 marks

- d. For how many hours during the 24-hour time interval is $T \geq 26$?

2 marks

Trigg is designing a garden that is to be built on flat ground. In his initial plans, he draws the graph of $y = \sin(x)$ for $0 \leq x \leq 2\pi$ and decides that the garden beds will have the shape of the shaded regions shown in the diagram below. He includes a garden path, which is shown as line segment PC .

The line through points $P\left(\frac{2\pi}{3}, \frac{\sqrt{3}}{2}\right)$ and $C(c, 0)$ is a tangent to the graph of $y = \sin(x)$ at point P .



- e. i. Find $\frac{dy}{dx}$ when $x = \frac{2\pi}{3}$.

1 mark

- ii. Show that the value of c is $\sqrt{3} + \frac{2\pi}{3}$.

1 mark

In further planning for the garden, Trigg uses a transformation of the plane defined as a dilation of factor k from the x -axis and a dilation of factor m from the y -axis, where k and m are positive real numbers.

f. Let X' , P' and C' be the image, under this transformation, of the points X , P and C respectively.

i. Find the values of k and m if $X'P' = 10$ and $X'C' = 30$.

2 marks

ii. Find the coordinates of the point P' .

1 mark

Question 2 (11 marks)

FullyFit is an international company that owns and operates many fitness centres (gyms) in several countries. At every one of FullyFit's gyms, each member agrees to have his or her fitness assessed every month by undertaking a set of exercises called **S**. There is a five-minute time limit on any attempt to complete **S** and if someone completes **S** in less than three minutes, they are considered fit.

- a.** At FullyFit's Melbourne gym, it has been found that the probability that any member will complete **S** in less than three minutes is $\frac{5}{8}$. This is independent of any other member.

In a particular week, 20 members of this gym attempt **S**.

- i.** Find the probability, correct to four decimal places, that at least 10 of these 20 members will complete **S** in less than three minutes.

2 marks

- ii.** Given that at least 10 of these 20 members complete **S** in less than three minutes, what is the probability, correct to three decimal places, that more than 15 of them complete **S** in less than three minutes?

3 marks

- b. Paula is a member of FullyFit's gym in San Francisco. She completes **S** every month as required, but otherwise does not attend regularly and so her fitness level varies over many months. Paula finds that if she is fit one month, the probability that she is fit the next month is $\frac{3}{4}$, and if she is not fit one month, the probability that she is not fit the next month is $\frac{1}{2}$.

If Paula is not fit in one particular month, what is the probability that she is fit in exactly two of the next three months?

2 marks

- c. When FullyFit surveyed all its gyms throughout the world, it was found that the time taken by members to complete **S** is a continuous random variable X , with a probability density function g , as defined below.

$$g(x) = \begin{cases} \frac{(x-3)^3 + 64}{256} & 1 \leq x \leq 3 \\ \frac{x+29}{128} & 3 < x \leq 5 \\ 0 & \text{elsewhere} \end{cases}$$

- i. Find $E(X)$, correct to four decimal places.

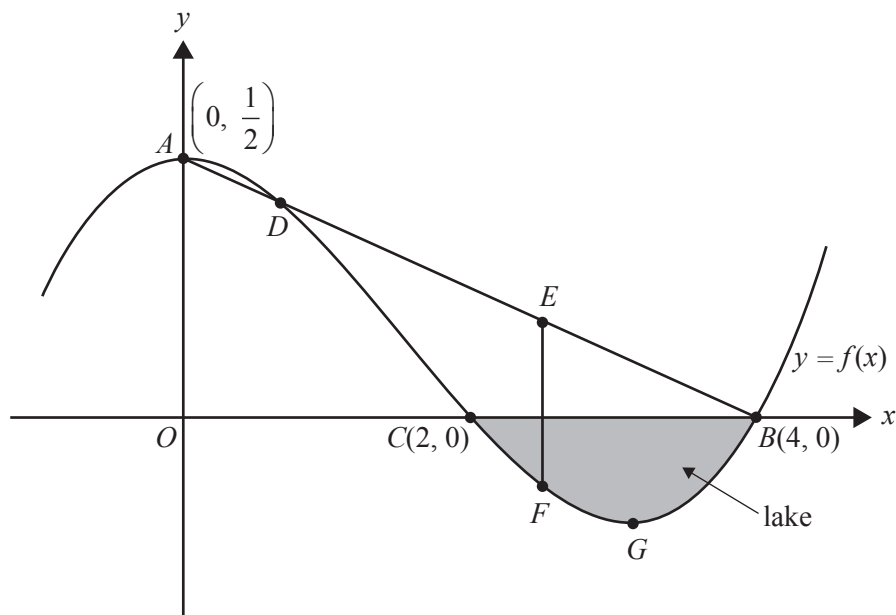
2 marks

- ii. In a random sample of 200 FullyFit members, how many members would be expected to take more than four minutes to complete **S**? Give your answer to the nearest integer.

2 marks

Question 3 (19 marks)

Tasmania Jones is in Switzerland. He is working as a construction engineer and he is developing a thrilling train ride in the mountains. He chooses a region of a mountain landscape, the cross-section of which is shown in the diagram below.



The cross-section of the mountain and the valley shown in the diagram (including a lake bed) is modelled by the function with rule

$$f(x) = \frac{3x^3}{64} - \frac{7x^2}{32} + \frac{1}{2}.$$

Tasmania knows that $A\left(0, \frac{1}{2}\right)$ is the highest point on the mountain and that $C(2, 0)$ and $B(4, 0)$ are the points at the edge of the lake, situated in the valley. All distances are measured in kilometres.

- a. Find the coordinates of G , the deepest point in the lake.

3 marks

Tasmania's train ride is made by constructing a straight railway line AB from the top of the mountain, A , to the edge of the lake, B . The section of the railway line from A to D passes through a tunnel in the mountain.

- b.** Write down the equation of the line that passes through A and B . 2 marks

- c. i.** Show that the x -coordinate of D , the end point of the tunnel, is $\frac{2}{3}$. 1 mark

- ii.** Find the length of the tunnel AD . 2 marks

In order to ensure that the section of the railway line from D to B remains stable, Tasmania constructs vertical columns from the lake bed to the railway line. The column EF is the longest of all possible columns. (Refer to the diagram on page 18.)

- d. i.** Find the x -coordinate of E .

2 marks

- ii.** Find the length of the column EF in metres, correct to the nearest metre.

2 marks

Tasmania's train travels down the railway line from A to B . The speed, in km/h, of the train as it moves down the railway line is described by the function

$$V: [0, 4] \rightarrow R, V(x) = k\sqrt{x} - mx^2,$$

where x is the x -coordinate of a point on the front of the train as it moves down the railway line, and k and m are positive real constants.

The train begins its journey at $A\left(0, \frac{1}{2}\right)$. It increases its speed as it travels down the railway line.

The train then slows to a stop at $B(4, 0)$, that is $V(4) = 0$.

- e.** Find k in terms of m .

1 mark

- f. Find the value of x for which the speed, V , is a maximum.

2 marks

Tasmania is able to change the value of m on any particular day. As m changes, the relationship between k and m remains the same.

- g. If, on one particular day, $m = 10$, find the maximum speed of the train, correct to one decimal place.

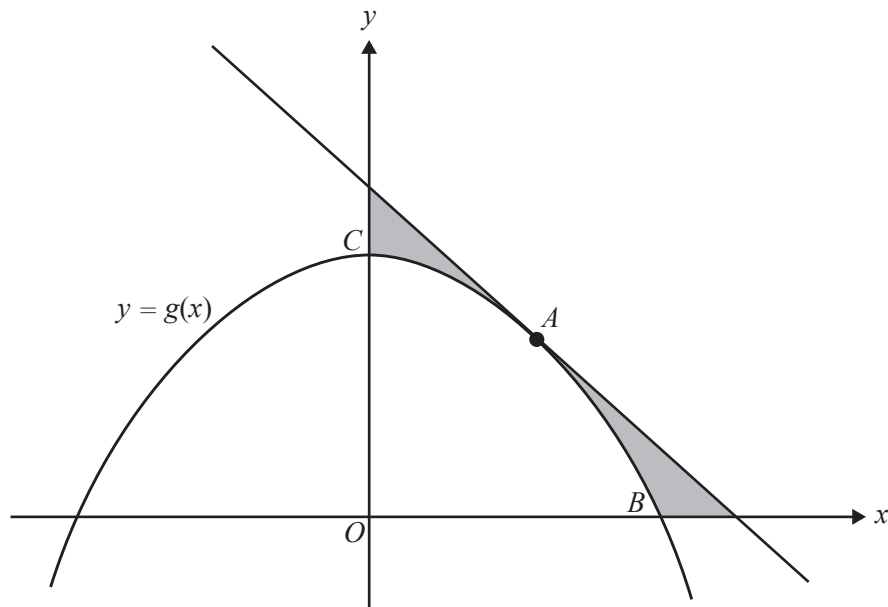
2 marks

- h. If, on another day, the maximum value of V is 120, find the value of m .

2 marks

Question 4 (16 marks)

Part of the graph of a function $g: \mathbb{R} \rightarrow \mathbb{R}$, $g(x) = \frac{16-x^2}{4}$ is shown below.



- a. Points B and C are the positive x -intercept and y -intercept of the graph of g , respectively, as shown in the diagram above. The tangent to the graph of g at the point A is parallel to the line segment BC .

i. Find the equation of the tangent to the graph of g at the point A .

2 marks

- ii. The shaded region shown in the diagram above is bounded by the graph of g , the tangent at the point A , and the x -axis and y -axis.

Evaluate the area of this shaded region.

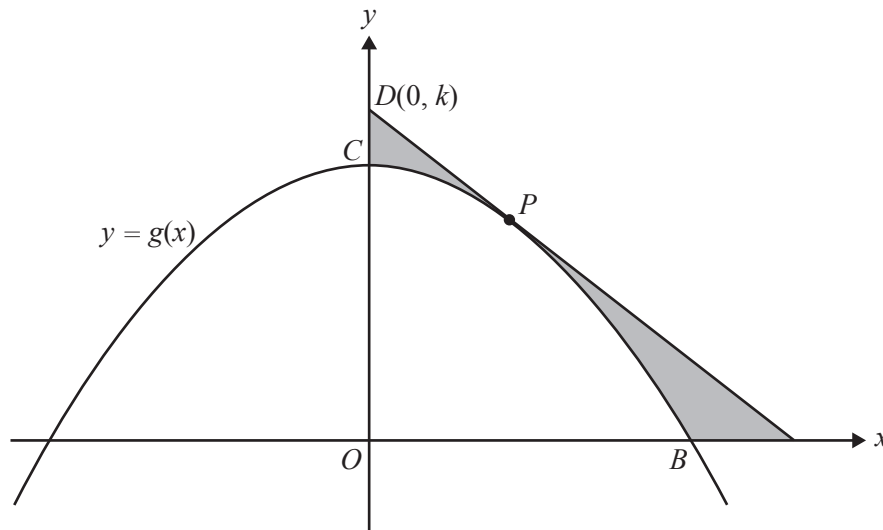
3 marks

- b. Let Q be a point on the graph of $y = g(x)$.

Find the positive value of the x -coordinate of Q , for which the distance OQ is a minimum and find the minimum distance.

3 marks

The tangent to the graph of g at a point P has a **negative** gradient and intersects the y -axis at point $D(0, k)$, where $5 \leq k \leq 8$.



- c. Find the gradient of the tangent in terms of k .

2 marks

- d. i.** Find the rule $A(k)$ for the function of k that gives the area of the shaded region. 2 marks

- ii.** Find the **maximum** area of the shaded region and the value of k for which this occurs. 2 marks

- iii.** Find the **minimum** area of the shaded region and the value of k for which this occurs. 2 marks

MATHEMATICAL METHODS (CAS)

Written examinations 1 and 2

FORMULA SHEET

Directions to students

Detach this formula sheet during reading time.

This formula sheet is provided for your reference.

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Mathematical Methods (CAS)

Formulas

Mensuration

area of a trapezium:	$\frac{1}{2}(a+b)h$	volume of a pyramid:	$\frac{1}{3}Ah$
curved surface area of a cylinder:	$2\pi rh$	volume of a sphere:	$\frac{4}{3}\pi r^3$
volume of a cylinder:	$\pi r^2 h$	area of a triangle:	$\frac{1}{2}bc \sin A$
volume of a cone:	$\frac{1}{3}\pi r^2 h$		

Calculus

$\frac{d}{dx}(x^n) = nx^{n-1}$	$\int x^n dx = \frac{1}{n+1} x^{n+1} + c, n \neq -1$
$\frac{d}{dx}(e^{ax}) = ae^{ax}$	$\int e^{ax} dx = \frac{1}{a} e^{ax} + c$
$\frac{d}{dx}(\log_e(x)) = \frac{1}{x}$	$\int \frac{1}{x} dx = \log_e x + c$
$\frac{d}{dx}(\sin(ax)) = a \cos(ax)$	$\int \sin(ax) dx = -\frac{1}{a} \cos(ax) + c$
$\frac{d}{dx}(\cos(ax)) = -a \sin(ax)$	$\int \cos(ax) dx = \frac{1}{a} \sin(ax) + c$
$\frac{d}{dx}(\tan(ax)) = \frac{a}{\cos^2(ax)} = a \sec^2(ax)$	

product rule: $\frac{d}{dx}(uv) = u \frac{dv}{dx} + v \frac{du}{dx}$	quotient rule: $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$
chain rule: $\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$	approximation: $f(x+h) \approx f(x) + hf'(x)$

Probability

$\Pr(A) = 1 - \Pr(A')$	$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$
$\Pr(A B) = \frac{\Pr(A \cap B)}{\Pr(B)}$	transition matrices: $S_n = T^n \times S_0$
mean: $\mu = E(X)$	variance: $\text{var}(X) = \sigma^2 = E((X - \mu)^2) = E(X^2) - \mu^2$

Probability distribution		Mean	Variance
discrete	$\Pr(X=x) = p(x)$	$\mu = \sum x p(x)$	$\sigma^2 = \sum (x - \mu)^2 p(x)$
continuous	$\Pr(a < X < b) = \int_a^b f(x) dx$	$\mu = \int_{-\infty}^{\infty} x f(x) dx$	$\sigma^2 = \int_{-\infty}^{\infty} (x - \mu)^2 f(x) dx$