SACRED HEART GIRLS' COLLEGE OAKLEIGH

Mathematical Methods CAS 2013

Unit 3 SAC 1: TEST
Part A

Name: SOLUTIONS

Teacher (please circle): Ms Gates   Mr Smith   Mrs Mak

No CAS and no summary notes permitted

Part A: 5 short answer questions

Writing Time: 25 minutes

Marks: 19

SHORT ANSWER QUESTIONS

Instructions:
Answer **all** questions in the spaces provided.
In all questions where a numerical answer is required an exact value must be given unless otherwise specified.
In questions where more than one mark is available, appropriate working **must** be shown.
Unless otherwise indicated, the diagrams in this test are **not** drawn to scale.

Question 1

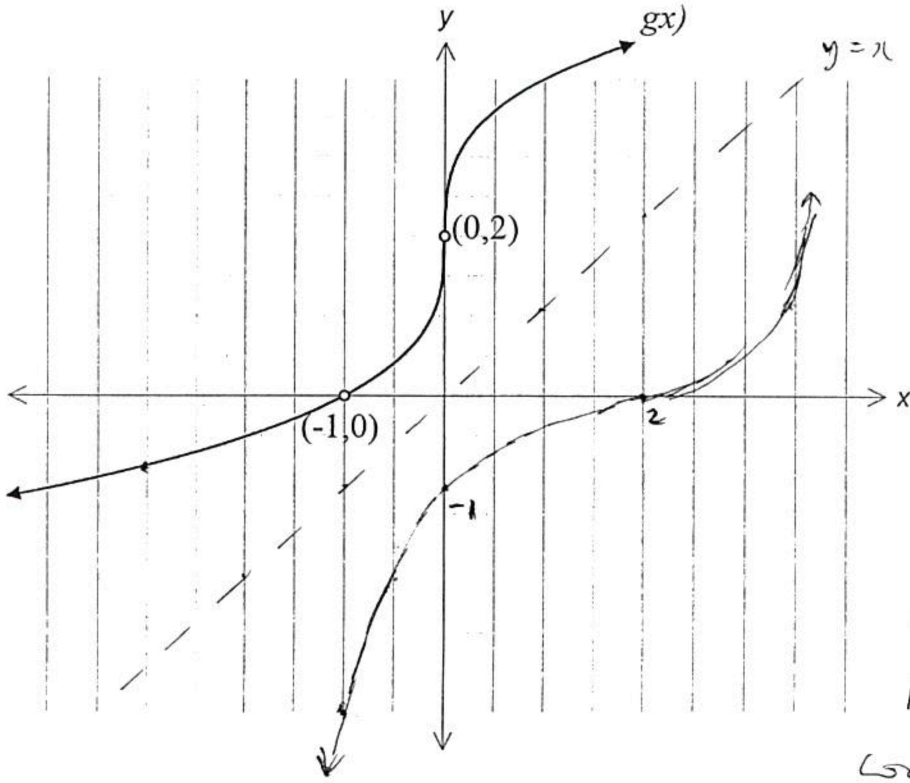
a. State the sequence of transformations required to change $f(x) = x^{\frac{1}{3}}$ into $g(x) = 2x^{\frac{1}{3}} + 3$.

DILATION of factor 2 from x-axis 1

followed by TRANSLATION of 3 units in positive y direction 1

The graph of the curve of $g(x)$ is shown below.

b. Sketch the curve of $g^{-1}(x)$ on the same set of axes below.



1 for x-intercept
1 for y-intercept
1 for shape in correct place

2+3=5 marks

Question 2

The transformation $T: R^2 \to R^2$ is defined by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

The image of the curve $y = 4x^2 - 1$ under the transformation T has equation $y = ax^2 + bx + c$. Find the values of a, b and c.

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2x \\ 5y \end{bmatrix} + \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

$x' = 2x + 2$ and $y' = 5y - 3$

$x = \frac{x' - 2}{2}$ and $y = \frac{y' + 3}{5}$ 1

$\frac{y' + 3}{5} = 4\left(\frac{x' - 2}{2}\right)^2 - 1$

$\frac{y' + 3}{5} = (x' - 2)^2 - 1$

$y' + 3 = 5(x' - 2)^2 - 5$

$y' = 5(x' - 2)^2 - 8$ 1

image is $y = 5(x - 2)^2 - 8$

$a =$ $= 5(x^2 - 4x + 4) - 8$

3 marks

$= 5x^2 - 20x + 12$

$a = 5, \ b = -20, \ c = 12$ 1

Question 3

a. Show that $f(x) = \frac{x+3}{x-2} + 1$ is equal to $f(x) = \frac{5}{x-2} + 2$.

$$f(x) = \frac{x-2}{x-2} + \frac{5}{x-2} + 1 \qquad 1$$

$$= 1 + \frac{5}{x-2} + 1$$

$$= \frac{5}{x-2} + 2$$

OR

$$x-2 \,\overline{\big)\, x+3}\ \ 1$$
$$\underline{-(x-2)}$$
$$5$$

$$f(x) = 1 + \frac{5}{x-2} + 1 \qquad 1$$

$$= \frac{5}{x-2} + 1$$

b. Hence, find the rule for the inverse of $f(x)$.

inverse, $x = \frac{5}{y-2} + 2$ 1

$$x - 2 = \frac{5}{y-2}$$

$$y - 2 = \frac{5}{x-2}$$

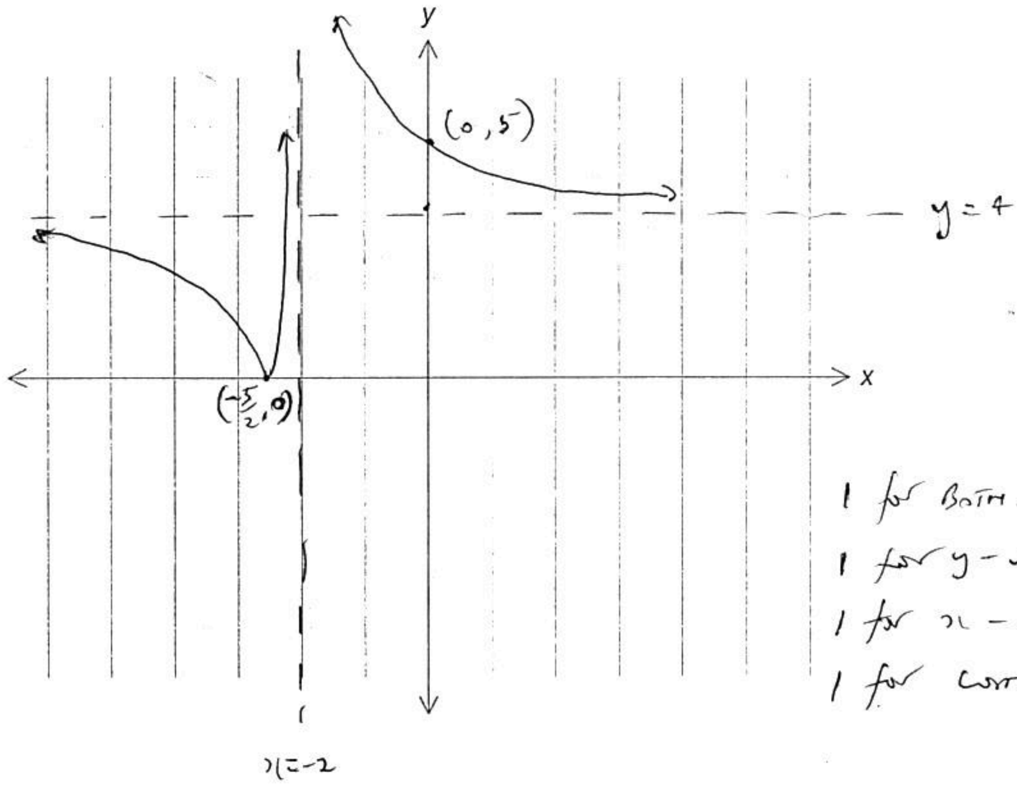
$$y = \frac{5}{x-2} + 2$$

$$f^{-1}(x) = \frac{5}{x-2} + 2 \qquad 1$$

1+2=3 marks

Question 4

On the set of axes below, sketch the graph of the function f with rule $(x) = \left|4 + \frac{2}{x+2}\right|$.
Label axes intercepts as coordinates and asymptotes with their equations.



1 for BOTH ~~intercepts~~ ASYMPTOTES
1 for y - intercept
1 for x - intercept
1 for correct curve

4 marks

x - intercept, $0 = 4 + \frac{2}{x+2}$

$$-4(x+2) = 2$$

$$x+2 = -\frac{1}{2}$$

$$x = -\frac{5}{2}$$

Question 5

Let $f(x) = 2x + 1$ and $g(x) = 2\sqrt{x}$.

a. Write down the rule of $f(g(x))$.

$f(g(x)) = 2 \times 2\sqrt{x} + 1$

$= 4\sqrt{x} + 1$ 1

b. State the maximal domain for $f(g(x))$.

$x \geq 0$ 1

c. Evaluate $f(g(75))$.

$f(g(75)) = 4\sqrt{75} + 1$ 1

$= 4\sqrt{25 \times 3} + 1$

1+1+2=4 marks

$= 20\sqrt{3} + 1$ 1

PART B

MULTIPLE CHOICE

Instructions:
Answer **all** questions in pencil on the answer sheet provided for multiple-choice questions.
Choose the response that is **correct** for that question.
A correct answer scores 1, an incorrect answer scores 0.
Marks will **not** be deducted for incorrect answers.
No marks will be given if more than one answer is completed for any question.
Only the answers on the Answer Sheet will be marked.

Question 1

M (1,1) is the midpoint between the points A (−2,3) and B (x,y). The coordinates of the point B are

A. $\left(-\frac{1}{2},2\right)$
B. $\left(\frac{1}{2},-2\right)$
C. $(-5,5)$
D. $(3,5)$
E. $(4,-1)$

A $(-2,3)$ B (x,y)

$\left(\frac{x-2}{2}, \frac{y+3}{2}\right)$

$(1, 1)$

$\frac{x-2}{2}=1$

$x-2=2$

$x=4$

$\frac{y+3}{2}=1$

$y+3=2$

$y=-1$

Question 2

The function $f(x) = x^4 - 3x^3 + kx^2 + 4$ has one real solution when

CAS TRIAL AND ERROR USING SOLVE

A. $k = 2$
B. $k = 1$
C. $k < 1$
D. $k > 2$
E. $1 < k < 2$

Question 3

Considering the two functions $f(x) = \frac{1}{\sqrt{x-2}}$ and $g(x) = \sqrt{4-x}$. If $h(x) = g(f(x))$ then the domain of $h(x)$ is

$\text{dom } g(f(x)) = \text{dom } f(x) \quad x>2 \rightarrow (2,\infty)$

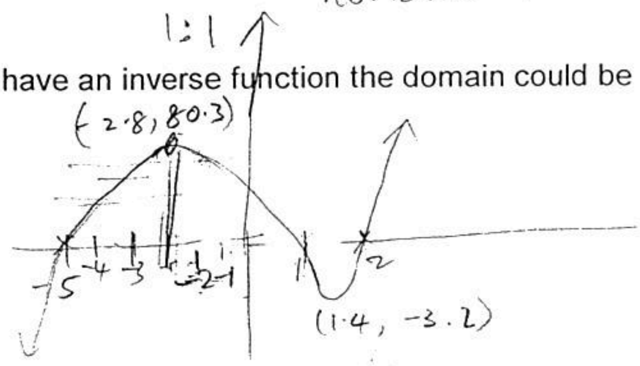
A. $[2,\infty)$
B. $(-\infty,0)$
C. $(-\infty,2)$
D. $(-\infty,4]$
E. $(2,\infty)$

	dom	ran
$f(x)$	$(2,\infty)$	$(0,\infty)$
$g(x)$	$(-\infty,4]$	$[0,\infty)$

horizontal line test

Question 4

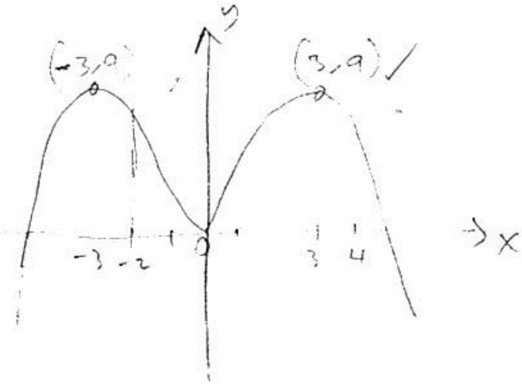
In order for $f(x) = 2(x-2)(x-1)(x+5)$ to have an inverse function the domain could be



A. $[-5,-2]$
B. $[-2,-1]$
C. $[-5,-1]$
D. $[1,5]$
E. R

Question 5

The maximum value of $f: [-2,4) \to R, f(x) = 9 - (|x| - 3)^2$ is



A. 8
B. 4
C. 3
D. 9
E. ∞

Question 6

For the system of simultaneous linear equations

$$x + 2y - z = 2$$
$$2x + 5y - (a+2)z = 3$$
$$-x + (a-5)y + z = 1$$

CAS

The values of a for which there is a unique solution are

A. 0 and 3
B. [0,3]
C. $R\backslash$ [0,3]
D. $R \backslash$ {0,3}
E. $R \backslash$ (0,3)

$$\det \begin{bmatrix} 1 & 2 & -1 \\ 2 & 5 & -(a+2) \\ -1 & (a-5) & 1 \end{bmatrix} \qquad \det = a.(a-3) \neq 0$$

$R \backslash \{0,3\}$

EXTENDED RESPONSE

Instructions:
Answer **all** questions in the spaces provided.
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Question 1

Two straight lines meet at right angles. The equation of the line y_1 is $y_1 = \frac{x}{2}$
The line y_2 passes through the point $(10,0)$.

a. Show that the equation of the line y_2 is $y_2 = -2x + 20$.

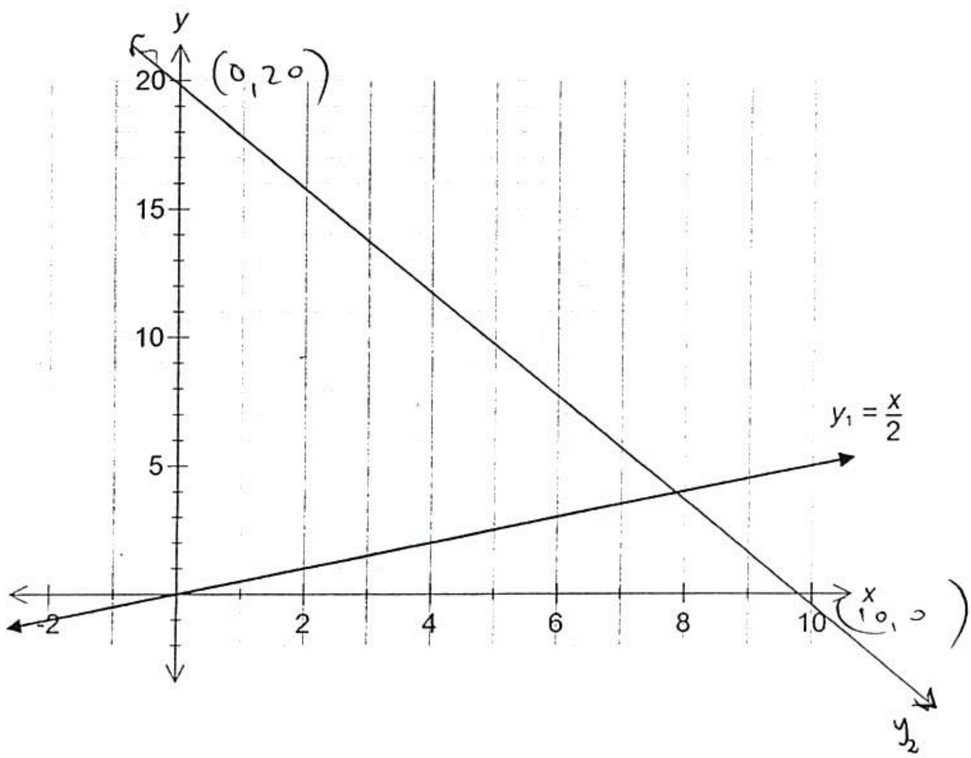
$m_2 = -2$ 1

$y - 0 = -2(x - 10)$

$y_2 = -2x + 20$ 1

2 marks

b. Sketch the line y_2 on the axes below.



1 mark

c. Find the coordinates of the point of intersection of the two lines.

(8, 4)

1 mark

Now consider the more general case where the first line has an equation of

$y_1 = ax$ where a is a positive constant.

The second line has an equation in the form of

$y_2 = bx + c$ where b and c are constants.

The lines are still perpendicular and the second line still passes through the point $(10,0)$.

d. Find an expression for b in terms of a.

$a \times b = -1$

$b = -\frac{1}{a}$

1 mark

e. Find an expression for c in terms of a.

$10b + c = 0$

$c = -10b$

$c = -10 \times -\frac{1}{a}$

$c = \frac{10}{a}$

1 mark

f. Show that the value of the x coordinate of the point of intersection of these two lines is

$$x = \frac{10}{a^2 + 1}$$

$ax = -\frac{1}{a}x + \frac{10}{a}$ 1

$ax + \frac{1}{a}x = \frac{10}{a}$

$x\left(a + \frac{1}{a}\right) = \frac{10}{a}$

$x\left(\frac{a^2+1}{a}\right) = \frac{10}{a}$

$x = \frac{10a}{a(a^2+1)}$

$x = \frac{10}{a^2+1}$ 1

2 marks

g. Find an expression for the value of the y coordinate.

$y = \frac{10a}{a^2+1}$

1 mark

Total 9 marks

Question 2

Dorothy Smart, the environmentalist, has received an emergency call about an oil spill in Bass Strait. The oil is forming a circular oil slick and the area of the oil slick is given by the function

$$A: [0, b) \to R, A(r) = \pi r^2$$

where r is the radius in km and A is the area in km^2.

Dorothy and her team need to contain the spill before it covers an area of 1200π km^2 or it will reach environmentally sensitive areas of the region.

a. Find the value of b.

$b = 20\sqrt{3}$

1 mark

In order to determine the spread of the oil slick over time Dorothy defines the function

$$r: R^+ \to R, r(t) = 4t^{\frac{2}{3}}$$

where r is the radius in km and t is time in days.

She attempts to perform the composition $A(r(t))$ only to find it does not exist.

b. Explain why the composite function $A(r(t))$ does not exist.

for $A(r(t))$ to exist $ran_r \subseteq dom\, A$

$\to [0, \infty) \not\subseteq [0, 20\sqrt{3})$ 1

$\therefore A(r(t))$ does not exist

1 mark

c. Define a restriction r^* such that $A(r^*(t))$ is defined.

$\text{ran } r = [0, 20\sqrt{3})$

$20\sqrt{3} = 4t^{2/3}$ 1

$t = 5\sqrt{5} \times 3^{3/4} \quad t > 0$

$r^* : [0, 5\sqrt{5} \times 3^{3/4}) \to R, \ r^*(t) = 4t^{2/3}$ 1

2 marks

d. Find $A(r^*(t))$.

$A(r^*(t)) : [0, 5\sqrt{5} \times 3^{3/4}) \to R, \ A(r^*(t)) = 16\pi t^{4/3}$

1 mark

e. Find the maximum number of days, to the nearest day, Dorothy and her team have to clean up the spill before serious and irreversible damage occurs to the environment.

$1200\pi = 16\pi t^{4/3}$

$t = 25$ days

1 mark

Total 6 marks

END OF SAC