

VCE Mathematical Methods (CAS) Units 3 & 4

Written Examination 2

Suggested Solutions

SECTION 1

1	A	B	C	D	E
2	A	B	C	D	E
3	A	B	C	D	E
4	A	B	C	D	E
5	A	B	C	D	E
6	A	B	C	D	E
7	A	B	C	D	E
8	A	B	C	D	E
9	A	B	C	D	E
10	A	B	C	D	E
11	A	B	C	D	E

12	A	B	C	D	E
13	A	B	C	D	E
14	A	B	C	D	E
15	A	B	C	D	E
16	A	B	C	D	E
17	A	B	C	D	E
18	A	B	C	D	E
19	A	B	C	D	E
20	A	B	C	D	E
21	A	B	C	D	E
22	A	B	C	D	E

SECTION 1**Question 1 C**

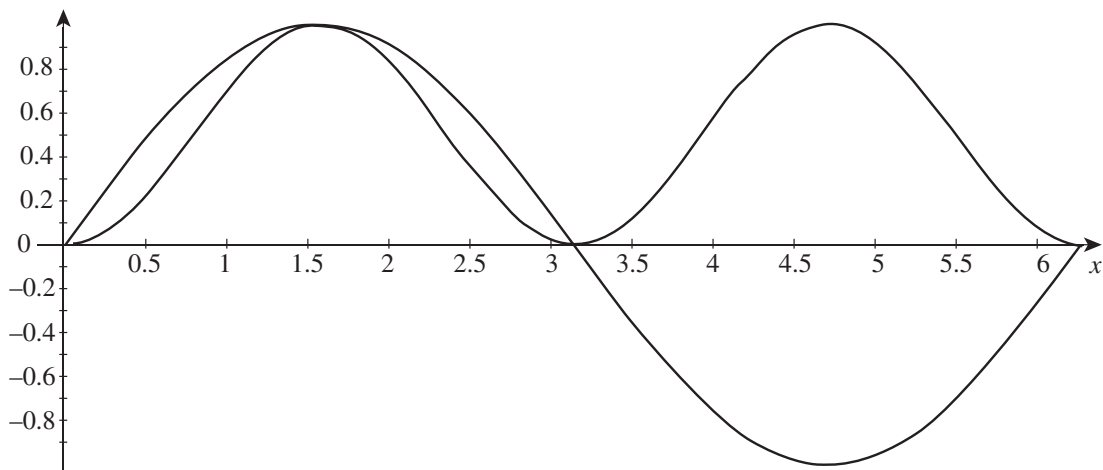
$$\sin(x) = \sin^2(x)$$

$$\sin(x)(1 - \sin(x)) = 0$$

$$\sin(x) = 0 \text{ or } 1$$

$$x = 0, \pi, 2\pi \text{ or } \frac{\pi}{2}, \text{ i.e. a total of 4 solutions.}$$

Alternatively, a graph shows 4 intersections over the domain.

**Question 2 E**

$$\text{We have } \int_{-2}^4 f(x)dx = \int_{-2}^3 f(x)dx + \int_3^4 f(x)dx$$

$$\text{So } a = \int_{-2}^3 f(x)dx + b$$

$$a = -\int_3^{-2} f(x)dx + b$$

$$\int_3^{-2} f(x)dx = b - a$$

Question 3 E

$$f\left(-\frac{1}{3}\right) = \frac{1}{-\frac{1}{3}} = -3$$

$$f(-3) = \frac{1}{-3} + \frac{1}{2} = \frac{1}{6}$$

Question 4 A

If h is differentiable at $x = 1$, then

$$h(1) = 1 + A + B = -1 + 8 + 4$$

$$A + B = 10$$

$$\text{also } h'(x) = \begin{cases} 2x + A & x < 1 \\ -2x + 8 & x \geq 1 \end{cases}$$

$$h'(1) = 2 + A = -2 + 8$$

$$A = 4$$

Thus $B = 6$.

Question 5 C

Checking each function:

Inverse of $f(x) = x$ is clearly $f^{-1}(x) = x$.

$$g(x) = \frac{4}{x} \text{ so inverse is given by } x = \frac{4}{y}, \text{ i.e. } y = \frac{4}{x} \text{ so } g^{-1}(x) = \frac{4}{x}.$$

$$h(x) = \frac{x}{x-1} \text{ so inverse is given by } x = \frac{y}{y-1}. \text{ CAS solve gives } y = \frac{x}{x-1} \text{ so } h^{-1}(x) = \frac{x}{x-1}.$$

$$j(x) = \frac{x-2}{x} \text{ so inverse is given by } x = \frac{y-2}{y}. \text{ CAS solve gives } y = \frac{-2}{x-1} \neq j(x).$$

Question 6 B

The wheel has a diameter of 18 cm so $h_{\max} = 18$ and $h_{\min} = 0$.

The period of the function is 12 seconds so, for a sine or cosine function, $\frac{2\pi}{n} = 12 \Rightarrow n = \frac{\pi}{6}$.

Now $t = 0$ corresponds to $h_{\max} = 18$, which suggests a cosine function with amplitude 9 and vertical translation 9.

$$\text{Thus } h(t) = 9 + 9 \cos\left(\frac{\pi t}{6}\right).$$

As this is not an alternative given, use $\cos(x) = \sin\left(\frac{\pi}{2} - x\right)$.

$$\text{Thus } h(t) = 9 + 9 \sin\left(\frac{\pi}{2} - \frac{\pi t}{6}\right).$$

$$h(t) = 9 + 9 \sin\left(\frac{\pi}{6}(3-t)\right) = 9\left(1 + \sin\left(\frac{\pi}{6}(3-t)\right)\right).$$

Question 7 D

$$\frac{f}{g}(x) = \frac{\sqrt{x+9}}{x-6}$$

For f

$$x+9 \geq 0$$

$$x \geq -9$$

We must exclude $x = 6$ because we cannot divide by zero.

Thus $[-9, 6) \cup (6, \infty)$.

Question 8 A

$$\text{Given } h(x) = g(f(x)),$$

$$h'(x) = g'(f(x))f'(x)$$

$$h'(2) = g'(f(2))f'(2)$$

$$h'(2) = g'(1)f'(2) = (-3)(6) = -18$$

Question 9 B

$$\sqrt{x} + \sqrt{y} = 5$$

Solving for y by CAS gives $y = 25 - 10\sqrt{x} + x$.

$$\text{Differentiating, } \frac{dy}{dx} = -\frac{5}{\sqrt{x}} + 1.$$

$$\text{At } x = 16, \text{ gradient of tangent is } -\frac{5}{4} + 1 = -\frac{1}{4}.$$

Equation of tangent:

$$y - 1 = -\frac{1}{4}(x - 16) \text{ which has a } y \text{ intercept of } 5. \text{ Therefore } k = 5.$$

Equation of normal:

$$y - 1 = 4(x - 16) \text{ which has a } y \text{ intercept of } -63. \text{ Therefore } h = -63.$$

$$k - h = 68$$

Question 10 C

The average rate of change of $f(x) = 3x^2 + 2x + k$ over the interval $[0, 2]$ is given by

$$\frac{f(2) - f(0)}{2} = \frac{(12 + 4 + k) - k}{2} = 8$$

$$\text{Thus } \frac{1}{2} \int_0^2 (3x^2 + 2x + k) dx = 8$$

$$\left[x^3 + x^2 + kx \right]_0^2 = 16$$

$$8 + 4 + 2k = 16$$

$$k = 2$$

Question 11 A

Define the events R_i and B_i , such that R_i represents a red ball drawn from urn i and B_i represents a blue ball drawn from urn i , $i = 1, 2$

Let x be the number of blue balls in urn 2.

$$\frac{11}{25} = \Pr(R_1 \cap R_2) + \Pr(B_1 \cap B_2)$$

$$\frac{11}{25} = \Pr(R_1)\Pr(R_2) + \Pr(B_1)\Pr(B_2)$$

OR

$$\begin{array}{l} \frac{4}{10} \text{ } R \text{ } \frac{16}{16+x} \text{ } R \\ \frac{6}{10} \text{ } B \text{ } \frac{x}{16+x} \text{ } B \end{array}$$

$$\frac{11}{25} = \frac{4}{10} \left(\frac{16}{x+16} \right) + \frac{6}{10} \left(\frac{x}{x+16} \right)$$

Solving on CAS gives $x = 4$.

Question 12 A

Let the random variable X represent the number of successful first serves.

$$X \sim Bi(n = 180, p = 0.65)$$

$$\mu = 180 \times 0.65 = 117$$

$$\sigma = \sqrt{180 \times 0.65 \times 0.35} = \frac{3\sqrt{455}}{10}$$

Question 13 C

The initial state matrix is $S_o = \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix}$

The win-lose probabilities can be tabulated:

		Tomorrow	
		Win	Lose
Today	Win	0.80	0.25
	Lose	0.20	0.75

Thus the transition matrix is $T = \begin{bmatrix} 0.8 & 0.25 \\ 0.2 & 0.75 \end{bmatrix}$

The probability that the team will win its fourth match equals $T^3 S_o = \begin{bmatrix} 0.8 & 0.25 \\ 0.2 & 0.75 \end{bmatrix}^3 \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix}$

Question 14 D

Let the increase in unit price per hat be \$ x .

Number of hats sold is $200 - 5x$.

Revenue from selling this number of hats is $(200 - 5x)(90 + x)$.

Cost from manufacturer for this number of hats is $60(200 - 5x)$.

As profit = revenue - cost,

$$\text{profit} = P(x) = (200 - 5x)(90 + x) - 60(200 - 5x)$$

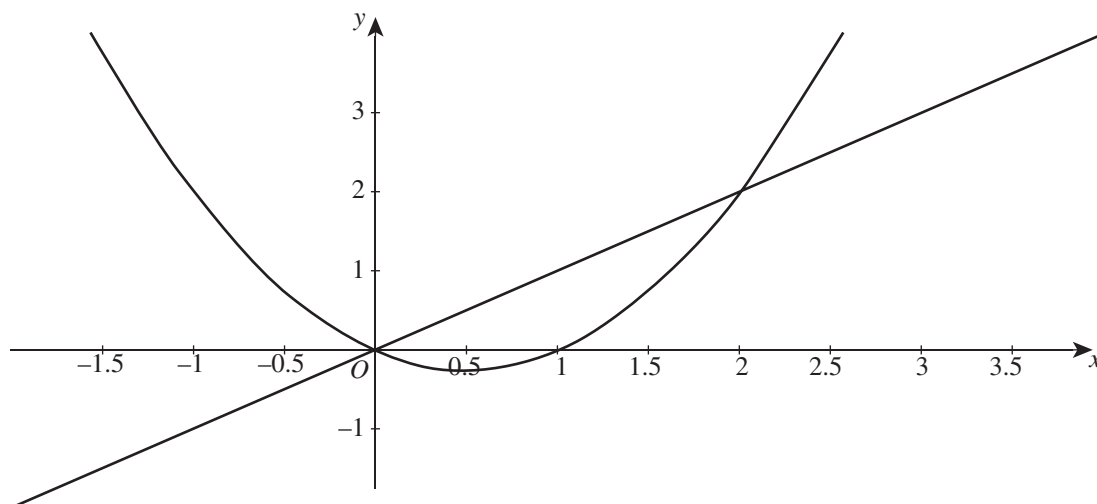
Simplifying on CAS gives $P(x) = -5x^2 + 50x + 6000$.

Maximum of when $P(x)$ when $P'(x) = 0$,

$$-10x + 50 = 0$$

$$x = 5$$

Number of hats sold is $200 - 5x = 200 - 25 = 175$

Question 15 C

The graphs meet when $x = x^2 - x$ $x = 0, 2$

Area bounded by the graphs equals $\int_0^2 x - (x^2 - x) dx = \int_0^2 (2x - x^2) dx = \frac{4}{3}$.

As $x = k$ divides the region in half, $\int_0^k (2x - x^2) dx = \frac{2}{3}$

$$\left[x^2 - \frac{x^3}{3} \right]_0^k = \frac{2}{3}$$

$$k^2 - \frac{k^3}{3} = \frac{2}{3}$$

Solving gives $k = 1$.

Question 16 A

The graph of the derivative needs to change from positive to negative within the domain.

This only occurs for the graph of f .

Question 17 D

Range of $f(x) - 2$ will be $[-9, 3]$.

So the range of $|f(x) - 2|$ will be $[0, 9]$ since the absolute value turns the negative results positive.

Finally, the range of $2|f(x) - 2| + 1$ equals $[1, 19]$, by doubling the range and adding 1.

Question 18 C

$$x^2 + kx + k = 0$$

As $x = -\frac{1}{2}$ is a root, the equation can be written in factored form as $\left(x + \frac{1}{2}\right)(x + 2k) = 0$

Expanding gives $x^2 + 2kx + \frac{1}{2}x + k = 0$

Equating coefficients of the x term

$$2k + \frac{1}{2} = k \quad k = -\frac{1}{2}$$

Now the other root is $x = -2k = -2 \times -\frac{1}{2} = 1$

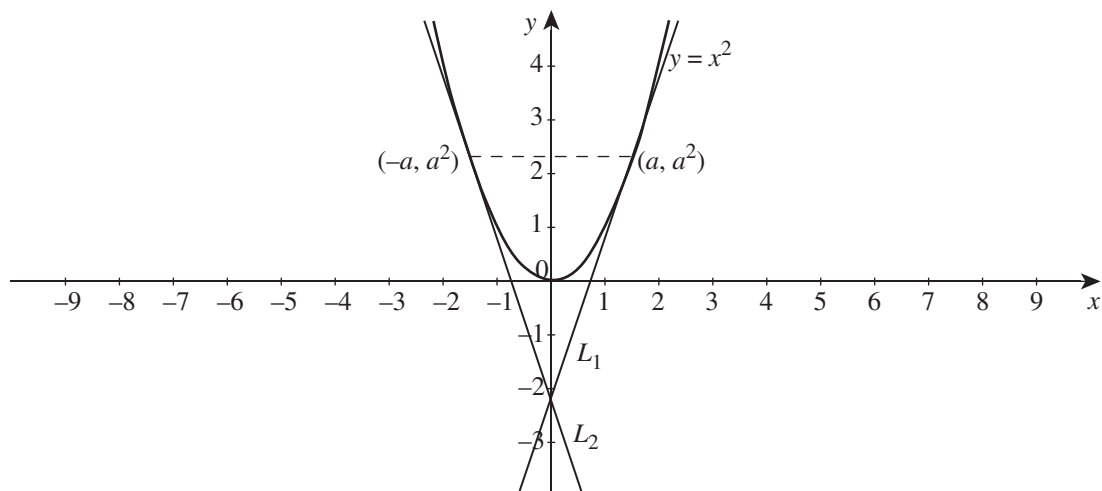
Alternatively, solving on CAS:

$$\left(-\frac{1}{2}\right)^2 + k\left(-\frac{1}{2}\right) + k = 0$$

$$\text{gives } k = -\frac{1}{2}$$

So $x^2 - \frac{1}{2}x - \frac{1}{2} = 0$.

Solving on CAS gives $x = -\frac{1}{2}$ or $x = 1$

Question 19 D

Let the vertex of the triangle at the point of contact for L_1 have coordinates (a, a^2) . In quadrant 2 the corresponding coordinates of the point of contact for L_2 will have coordinates $(-a, a^2)$.

Consider L_1 : $\frac{dy}{dx} = 2x = 2a$

But we know the triangle is equilateral so $m_1 = \tan(60^\circ) = \sqrt{3}$.

$$2a = \sqrt{3}$$

$$\text{Thus } a = \frac{\sqrt{3}}{2}$$

The length of each side of the triangle is $2a$.

Using the Sine rule for area formula (on formula sheet):

$$A = \frac{1}{2} \times (2a)(2a) \sin(60^\circ)$$

$$A = 2a^2 \frac{\sqrt{3}}{2} = \sqrt{3} \times \left(\frac{\sqrt{3}}{2}\right)^2 = \frac{3\sqrt{3}}{4}$$

Question 20 A

We require $\Pr(V < 2 \mid V \geq 1.5) = \frac{\Pr(1.5 \leq V \leq 2)}{\Pr(V \geq 1.5)}$.

Using CAS, compute $\frac{\int_{1.5}^2 \frac{3}{v^4} dv}{1 - \int_1^{1.5} \frac{3}{v^4} dv}$ which gives $= 0.5781$.

Question 21 A

The gradient function $y = f'(x)$ has 4 x intercepts symmetrically placed either side of the y -axis. The function $f(x)$ has stationary points at those locations.

Both **A** and **C** satisfy this condition completely.

Also notice $f'(0)$ is undefined corresponding to the cusp on each of the graphs in **A** and **C**.

Notice that $f'(x) < 0$ for positive x values up to approximately 0.7. The gradient of a tangent to graph **A** is negative for these x values, but graph **C** has a positive gradient for these x values

Question 22 C

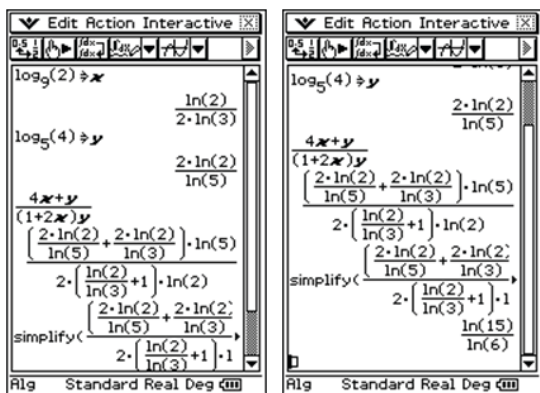
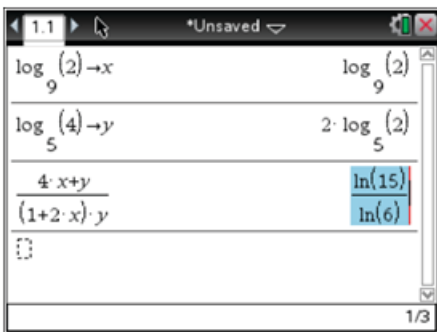
On CAS, define $x = \log_9(2)$ and $y = \log_5(4)$

Check each alternative systematically.

$$\frac{4x + y}{(1 + 2x)y} = \frac{\log_e(15)}{\log_e(6)}$$

By the change of base rule,

$$\frac{\log_e(15)}{\log_e(6)} = \log_6(15)$$



SECTION 2

Question 1 (15 marks)

- a. $x = -\frac{3}{4}$ and $y = \frac{1}{2}$ represent the vertical and horizontal asymptotes respectively.

$$\text{Thus } \text{dom}(f) = \mathbb{R} \setminus \left\{ -\frac{3}{4} \right\}. \quad \text{A1}$$

The graph touches the x axis and is otherwise above it. We do not exclude $y = \frac{1}{2}$.

$$\text{Thus } \text{ran}(f) = [0, \infty). \quad \text{A1}$$

b. Given $\frac{1}{2} \left| 1 - \frac{5}{4x+3} \right| = \left| \frac{ax+b}{cx+d} \right|,$

$$\text{LHS} = \frac{1}{2} \left| \frac{4x+3-5}{4x+3} \right| = \frac{1}{2} \left| \frac{4x-2}{4x+3} \right| = \left| \frac{2x-1}{4x+3} \right|. \quad \text{M1}$$

$$\text{This gives } a = 2, b = -1, c = 4 \text{ and } d = 3. \quad \text{A1}$$

- c. i. g must be a one-to-one function with range $[0, \infty)$.

$$m = -\frac{3}{4}, n = \frac{1}{2} \quad \text{A1 A1}$$

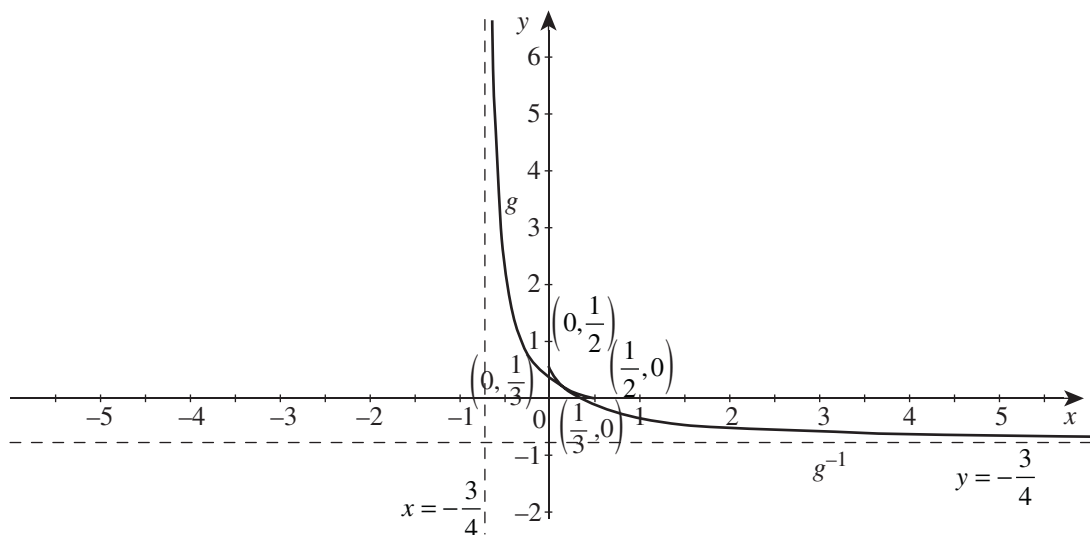
ii. For $x \in \left(-\frac{3}{4}, \frac{1}{2} \right], g(x) = -\frac{2x-1}{4x+3} = \frac{1-2x}{4x+3}.$

$$\text{The inverse is given by solving } x = \frac{1-2y}{4y+3}. \quad \text{M1}$$

$$\text{Use CAS : } y = \frac{1-3x}{4x+2}$$

$$\text{Thus } g^{-1} : [0, \infty) \rightarrow \mathbb{R}, g^{-1}(x) = \frac{1-3x}{4x+2}. \quad \text{A1}$$

iii.



Graph of g and g^{-1} A1
 Correct intercepts A1
 Correct asymptotes A1

iv. $g^{-1}(x) - g(x) = 0$

Solve on CAS the equation $\frac{1-3x}{4x+2} = \frac{1-2x}{4x+3}$, giving $x = \frac{\sqrt{41}-5}{8}$. A1

d. For $x \leq 0.5$, $f(x) = \frac{1-2x}{4x+3}$.

Using CAS, $f'(x) = \frac{-10}{(4x+3)^2}$. M1

Thus $f'(0) = -\frac{10}{9}$ and the equation of the tangent here is $y = -\frac{10}{9}x + \frac{1}{3}$. A1

Solving simultaneously on CAS:

$$y = -\frac{10}{9}x + \frac{1}{3} \text{ and } y = \left| \frac{2x-1}{4x+3} \right| \text{ and gives intersection at } (-1.25777, 1.73086)$$

This gives $p = -1.258$, $q = 1.731$. A1

Question 2 (15 marks)

a. Using CAS, $y = \frac{x^3}{e^x}$ gives $\frac{dy}{dx} = \frac{(3x^2 - x^3)}{e^x}$. A1

b. Stationary points occur when $\frac{dy}{dx} = \frac{(3x^2 - x^3)}{e^x} = 0$
 $x^2(3-x) = 0 \Rightarrow x = 0, 3$ M1

Thus a maximum at $\left(3, \frac{27}{e^3}\right)$ and a stationary point of inflection at $(0, 0)$. A1 A1

c. $\frac{x^3}{e^x} \leq \frac{27}{e^3}$

As $e^3 > 0$, we rewrite the in-equation:

$$x^3 e^{3-x} \leq 27$$
 M1

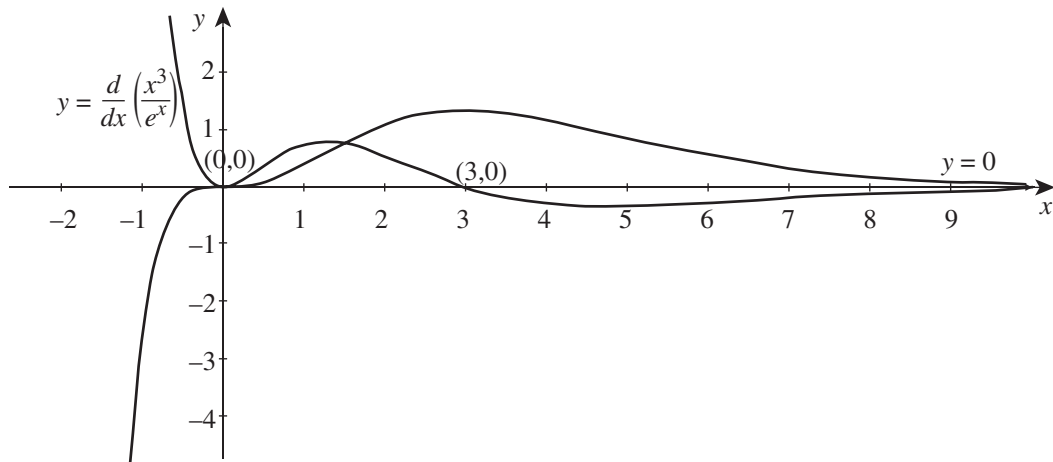
Taking logs of both sides:

$$\log_e(x^3) + \log_e(e^{3-x}) \leq \log_e(27)$$
 M1

$$3\log_e(x) + 3 - x \leq 3\log_e(3)$$

Thus $3\log_e(x) \leq x + 3\log_e(3) - 3$. A1

d. i.



Intercept coordinates and asymptote
Shape

A1
A1

ii. $\frac{dy}{dx} = \frac{(3x^2 - x^3)}{e^x}$

The maximum and minimum will occur when $\frac{d}{dx} \left(\frac{3x^2 - x^3}{e^x} \right) = 0$.

Thus $\frac{x(x^2 - 6x + 6)}{e^x} = 0$

$x = 3 \pm \sqrt{3}$ ($x = 0$ is also a solution.)

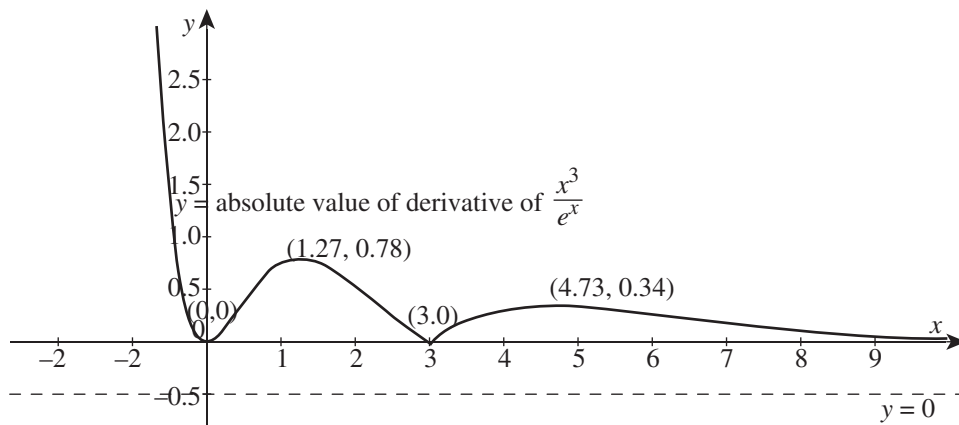
M1

From graph, maximum occurs at $x = 3 - \sqrt{3}$, giving $\frac{dy}{dx} = 6(2\sqrt{3} - 3)e^{\sqrt{3}-3}$.

From graph, minimum occurs at $x = 3 + \sqrt{3}$, giving $\frac{dy}{dx} = -6(2\sqrt{3} + 3)e^{-\sqrt{3}-3}$.

A1

iii. A function is strictly decreasing if for all, $a < b$, $f(a) > f(b)$.



From graph, $[1.27, 3] \cup [4.73, \infty)$.

A1 M1

- e. Solving the equation $\frac{x^6}{e^{2x}} = k(x-3)$ is equivalent to solving $\left(\frac{x^3}{e^x}\right)^2 = k(x-3)$

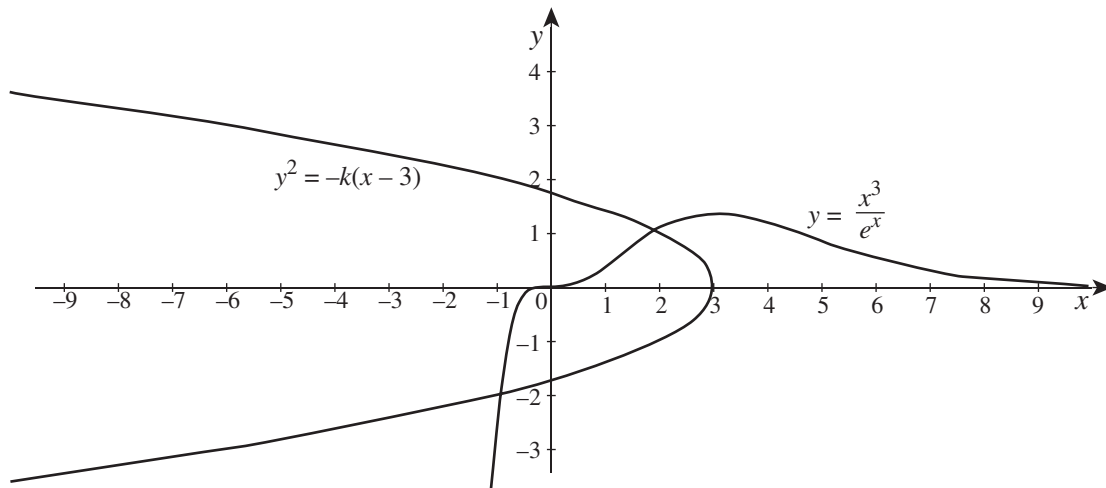
i.e. $\left(\frac{x^3}{e^x}\right) = \pm\sqrt{k(x-3)}$.

M1

The graph below illustrates, that for a negative k , 2 solutions are obtained.

Thus $k < 0$.

A1



Question 3 (18 marks)

- a. i. Let X represent the number of these enquiries which came through the phone.

$$X \sim \text{Bi}(n=100, p=0.4)$$

$$E(X) = np = 100 \times 0.4 = 40$$

A1

$$\sigma_X = \sqrt{np(1-p)} = \sqrt{100 \times 0.4 \times 0.6} = 4.90$$

A1

- ii. Require $\Pr(X \geq 30) = 0.9852$. Using CAS the answer is directly obtained from binomialcdf:

$$\text{binomCdf}(100, 0.4, 30, 100)$$

A1

- b. i. For the eighth phone call to result in the first booking from phone enquiries on that day we need to have no bookings from the first seven phone calls, then a booking on the eighth call.

$$\text{Thus required probability} = (1-k)^7 \times k \text{ or } k(1-k)^7.$$

A1

- ii. $\Pr(\text{eighth phone call results in fourth booking}) = k \times \Pr(\text{three bookings from seven phone calls})$

M1

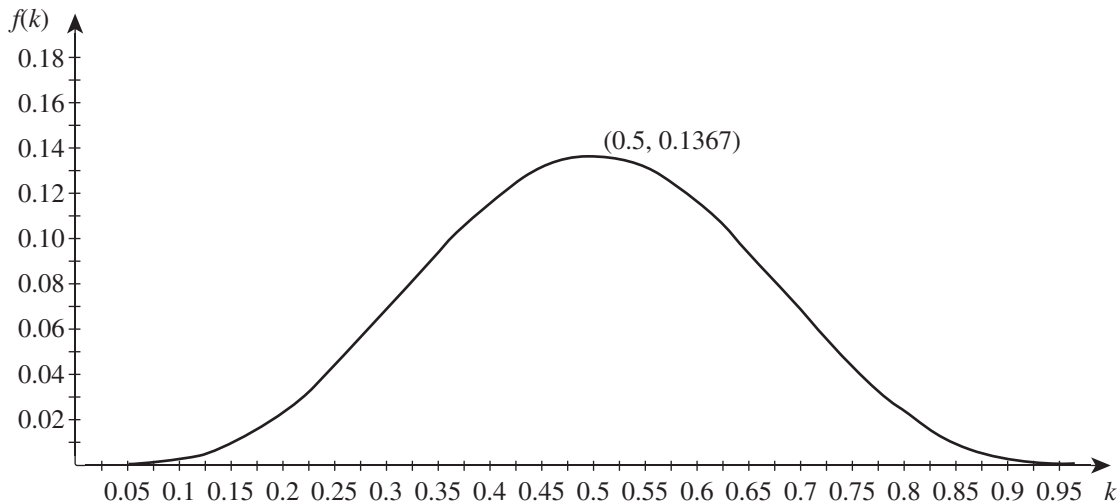
$$k \times {}^7C_3 k^3 (1-k)^4 = 35k^4 (1-k)^4$$

A1

- iii. We need to locate the maximum of the function $f(k) = 35k^4(1-k)^4$

A graph sketch from a CAS shows the maximum turning point at (0.5, 0.1367)

M1



The maximum probability occurs when $k = 0.5$ and equals 0.1367.

A1

- c. $\Pr(\text{no booking}) = 0.42$

$$\text{Thus } 0.4(1-k) + 0.5(1-k^2) + 0.1(1-k^3) = 0.42.$$

M1

Solving on CAS and noting $0 < k < 1$, $k = 0.72$

A1

- d. $\Pr(\text{email via internet booking agency} \mid \text{a booking is made}) = \frac{\Pr(\text{internet} \cap \text{booking})}{\Pr(\text{booking})}$

$$\text{Thus } 0.25 = \frac{0.5k^2}{0.1k^3 + 0.4k + 0.5k^2}.$$

M1

Solving on CAS and noting $0 < k < 1$, $k = 0.27$.

A1

- e. i. We require 3 transitions to go from Sunday to Wednesday, i.e. $T^3 \times \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

$$\text{Thus } \begin{bmatrix} \frac{3}{5} & \frac{1}{3} \\ \frac{2}{5} & \frac{2}{3} \end{bmatrix}^3 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{523}{1125} \\ \frac{602}{1125} \end{bmatrix}.$$

M1

So the required probability of dining in the restaurant on Wednesday night equals $\frac{523}{1125}$.

A1

- ii. Consider T^n for large n . For example $\begin{bmatrix} \frac{3}{5} & \frac{1}{3} \\ \frac{2}{5} & \frac{2}{3} \end{bmatrix}^{50} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.4545 \\ 0.5454 \end{bmatrix}$.

The percentage of nights the hotel can assume guests will dine in the hotel restaurant is 45%.

A1

- f. Define Y as the random variable “weight of lobster”.

Then $Y \sim N(\mu, \sigma^2)$.

We are given that $\Pr(|Y - \mu| \leq m) = 0.25$

$$\Pr(-m \leq Y - \mu \leq m) = 0.25 \quad \text{A1}$$

Applying the Z transformation, $Z = \frac{Y - \mu}{\sigma}$ gives $\Pr(-\frac{m}{\sigma} \leq Z \leq \frac{m}{\sigma}) = 0.25$

Thus $\Pr(Z \leq -\frac{m}{\sigma}) = 0.375$.

$$-\frac{m}{\sigma} = \text{invnorm}(0.375) = -0.3186 \text{ Thus } \frac{m}{\sigma} = 0.3186 \quad \text{M1}$$

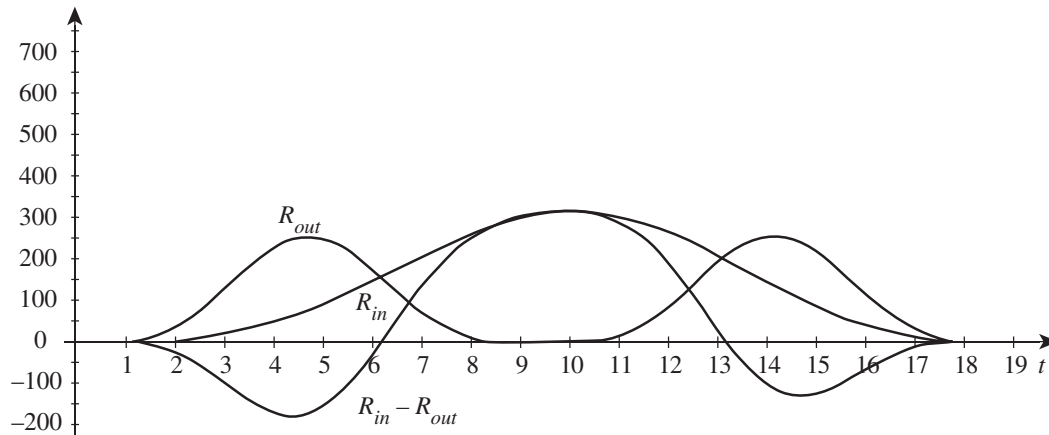
We require $\Pr(Y - \mu) \leq 3m$.

This is equivalent to finding $\Pr(Z) \leq \frac{3m}{\sigma}$, i.e. $\Pr(Z) \leq 3 \times 0.3186$.

$$= \Pr(Z \leq 0.9559) = 0.8304 \quad \text{A1}$$

Question 4 (10 marks)

a.



Shape of graph A1
Critical points correctly located A1

- b. Using the graph and CAS, we require

$y = R_{in} - R_{out}$ to be above axis. Intersection points occur at $t = 6.15095, 13.1152$,
so between $t = 6.151$ and 13.115

A1

- c. The tank contains the initial quantity plus an increase or decrease according to

$$1200 + \int_0^{15} (R_{in} - R_{out}) dt = 1200 + 804.71 = 2004.71. \quad \text{M1}$$

So 2005 litres.

A1

- d. The inflow rate and outflow rate are equal at $t = 6.15095, 13.1152$

At $t = 6.15095$,

$$\text{volume} = 12000 + \int_0^{6.15095} (R_{in} - R_{out}) dt = 1200 - 498.97 = 701.024 . \quad \text{A1}$$

At $t = 13.1152$,

$$\text{volume} = 1200 + \int_0^{13.1152} (R_{in} - R_{out}) dt = 1200 + 984.516 = 2184.516 .$$

At $t = 18$

$$\text{volume} = 1200 + \int_0^{18} (R_{in} - R_{out}) dt = 1200 + 655.2641 = 1855.2639$$

Hence the absolute minimum quantity of liquid occurs at $t = 6.15$. A1

Using appropriate integrals to compute volume M1

- e. At $t = 18$, there is a volume of 1855.26 litres remaining in the tank.

$$\text{Thus } \int_{18}^T 250 \sin^4 \left(\frac{t}{6} \right) dt = 1855.26 . \quad \text{A1}$$

Solving on CAS gives $T = 42.72$.

So tank is empty after 42 hours 43 minutes. A1