



Trial Examination 2013

VCE Mathematical Methods (CAS) Units 3 & 4

Written Examination 1

Suggested Solutions

Question 1 (3 marks)

a. $y = \frac{2}{1-5x} = 2(1-5x)^{-1} \Rightarrow \frac{dy}{dx} = -2(1-5x)^{-2} \times -5$

$$\frac{dy}{dx} = \frac{10}{(1-5x)^2}$$

A1

b. $f(x) = x \cos^2\left(\frac{\pi x}{2}\right)$

$$= x \cdot 2 \times \frac{\pi}{2} \cos\left(\frac{\pi x}{2}\right) \left[-\sin\left(\frac{\pi x}{2}\right)\right] + \cos^2\left(\frac{\pi x}{2}\right)$$

M1

$$= -\pi x \cos\left(\frac{\pi x}{2}\right) \sin\left(\frac{\pi x}{2}\right) + \cos^2\left(\frac{\pi x}{2}\right)$$

$$f'\left(\frac{1}{2}\right) = -\frac{\pi}{2} \cos\left(\frac{\pi}{4}\right) \sin\left(\frac{\pi}{4}\right) + \cos^2\left(\frac{\pi}{4}\right)$$

$$= -\frac{\pi}{2} \cdot \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} + \left(\frac{1}{\sqrt{2}}\right)^2$$

$$= \frac{1}{2} - \frac{\pi}{4}$$

A1

Question 2 (4 marks)

As $\frac{dy}{dx} = \frac{6}{\sqrt{x^3}}$, we have $y = \int \frac{6}{\sqrt{x^3}} dx = \int 6x^{-\frac{3}{2}} dx = -\frac{12}{\sqrt{x}} + c$

M1

$$4 = \frac{-12}{\sqrt{\frac{1}{4}}} + c \Rightarrow c = 28$$

Equation of the curve is $y = -\frac{12}{\sqrt{x}} + 28$

A1

Question 3 (2 marks)

$$f(g(x)) = f\left(1 + e^{\frac{x}{a}}\right) = a \log_e\left(1 + e^{\frac{x}{a}} - 1\right) = a \log_e\left(e^{\frac{x}{a}}\right) = a \times \frac{x}{a} = x$$

M1

As $f(g(x)) = x = f(f^{-1}(x))$, and both f and g are one to one functions, they are inverses.

A1

Question 4 (5 marks)

a. Define the random variable X as “the number of passengers who show up for the flight”.

$$\Pr(\text{everyone gets a seat}) = \Pr(X \leq 40) = 0.1 + 0.45 + 0.35 = 0.9$$

A1

b. Define Y as the number of passengers who do not show up

Then $Y = 43 - X$ so that

$$E(Y) = 43 - E(X)$$

M1

$$E(Y) = 43 - (38 \times 0.1 + 39 \times 0.45 + 40 \times 0.35 + 41 \times 0.05 + 42 \times 0.04 + 43 \times 0.01)$$

$$= 43 - (3.8 + 17.55 + 14 + 2.05 + 1.68 + 0.43)$$

$$= 43 - 39.51$$

$$= 3.49$$

A1

- c. $\Pr(X = 38 \mid \text{not all seats filled}) =$

$$\Pr(X = 38 \mid X < 40) = \frac{\Pr(X = 38)}{\Pr(X < 40)} \quad \text{M1}$$

$$= \frac{0.1}{0.1 + 0.45} = \frac{10}{55} = \frac{2}{11} \quad \text{A1}$$

Question 5 (7 marks)

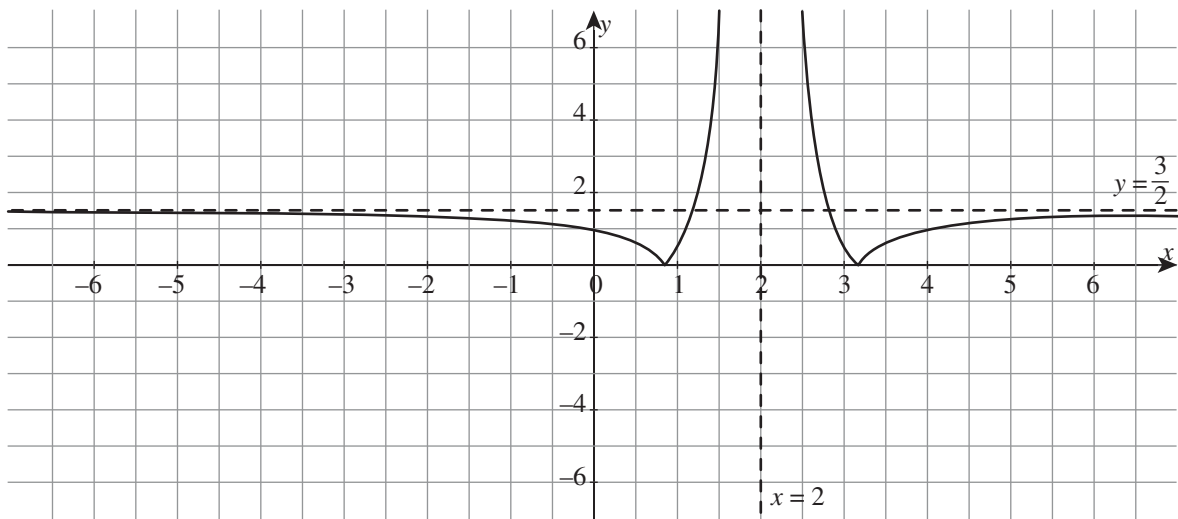
- a. A vertical asymptote with equation $x = 2$ means $b = -2$. A1

$$f(x) = \frac{a}{(x-2)^2} \text{ and } f(0) = -\frac{1}{2} \Rightarrow \frac{a}{4} = -\frac{1}{2} \Rightarrow a = -2 \quad \text{A1}$$

- b. x -intercepts by solving $\frac{3}{2} - \frac{2}{(x-2)^2} = 0$

$$(x-2)^2 = \frac{4}{3}$$

$$x = 2 \pm \frac{2}{\sqrt{3}} \approx 2 \pm 2(0.6) \approx 0.8 \text{ and } 3.2$$



Vertical asymptote $x = 2$ and horizontal asymptote $y = \frac{3}{2}$ A1

x -intercepts correct and shown on graph A1

correct shape A1

- c. Starting with $f(x) = \frac{-2}{(x-2)^2}$, under a dilation of scale factor 2 from the y -axis we have

$$x \rightarrow \frac{x}{2}, \text{ so } y = \frac{-2}{\left(\frac{x}{2} - 2\right)^2} \quad \text{M1}$$

A translation of 3 units in the negative direction of the x -axis, $x \rightarrow x + 3$

$$\text{So } y = \frac{-2}{\left(\frac{x+3}{2} - 2\right)^2} = \frac{-2}{\left(\frac{x-1}{2}\right)^2} = \frac{-8}{(x-1)^2} \quad \text{A1}$$

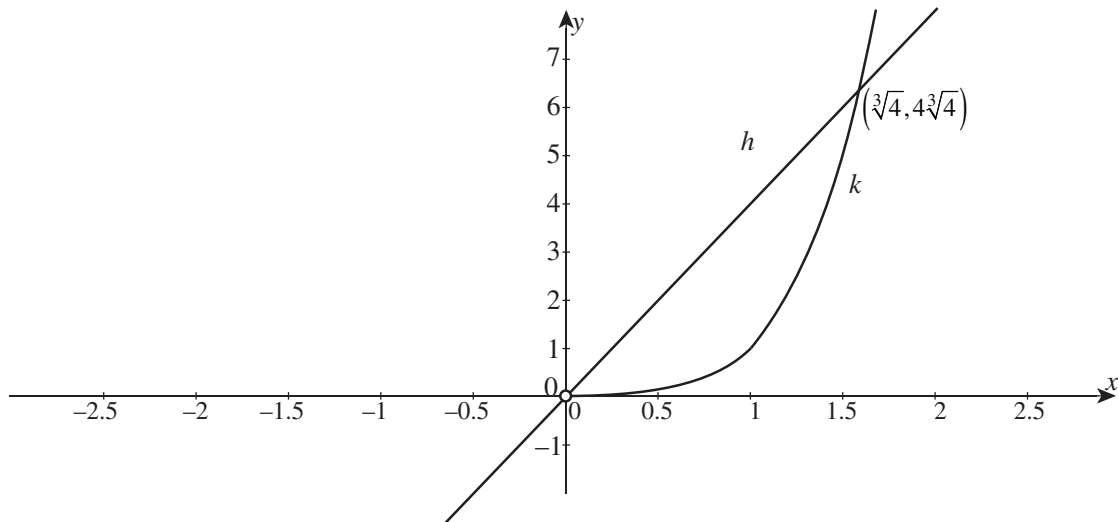
Question 6 (4 marks)

a. $h(x) = f(g(x)) = f(e^{2x}) = \log_e(e^{2x})^2 = \log_e(e^{4x}) = 4x$ A1

$$k(x) = g(f(x)) = g(\log_e(x^2)) = e^{2\log_e(x^2)}$$

$$k(x) = e^{\log_e(x^4)} = x^4$$
 A1

b.



$$x^4 = 4x \Rightarrow x(x^3 - 4) = 0$$

$x = \sqrt[3]{4}$ is the only solution in the domain

graphs of h and k on correct domain with $(0,0)$ excluded A1

$(\sqrt[3]{4}, 4\sqrt[3]{4})$ as intersection point A1

Question 7 (3 marks)

Given $2\log_3(x+4) - \log_3(-x) = 2$ we have $x+4 > 0$ and $-x > 0$, which together means $-4 < x < 0$.

$$\log_3(x+4)^2 - \log_3(-x) = 2$$

$$\log_3\left(\frac{(x+4)^2}{-x}\right) = 2 = \log_3(3^2)$$
 M1

$$(x+4)^2 = -9x$$

$$x^2 + 8x + 16 = -9x$$

$$x^2 + 17x + 16 = 0$$

$$(x+16)(x+1) = 0$$

$$x = -16, -1$$
 A1

Since $-4 < x < 0$, $x = -1$ is the only allowable solution. A1

Question 8 (5 marks)

- a. i. As X is a probability density function,

$$\int_0^2 kx(4-x^2)dx = 1$$

$$\left[2x^2 - \frac{x^4}{4} \right]_0^2 = \frac{1}{k}$$

$$4 = \frac{1}{k} \Rightarrow k = \frac{1}{4}$$

A1

- ii. The median, m , is such that $\int_0^m \frac{1}{4}x(4-x^2)dx = \frac{1}{2}$

M1

$$\left[2x^2 - \frac{x^4}{4} \right]_0^m = 2$$

$$2m^2 - \frac{m^4}{4} = 2$$

$$m^4 - 8m^2 + 8 = 0$$

A1

- b. $Y = aX + b$

Thus $E(Y) = aE(X) + b$ and $\sigma_Y = a\sigma_X$

M1

$$65 = 72a + b \text{ and } 5 = 8a$$

$$\text{So } a = \frac{5}{8} \text{ and } b = 20$$

A1

Question 9 (3 marks)

$$\text{We have } \frac{dN}{dt} = \frac{10\,000}{(t+3)^3} \text{ so that } N = \int_0^\infty \frac{10\,000}{(t+3)^3} dt$$

A1

$$N = \int_0^\infty 10\,000(t+3)^{-3} dt = \left[-\frac{5000}{(t+3)^2} \right]_0^\infty$$

M1

$$N = 0 + \frac{5000}{9} = 556$$

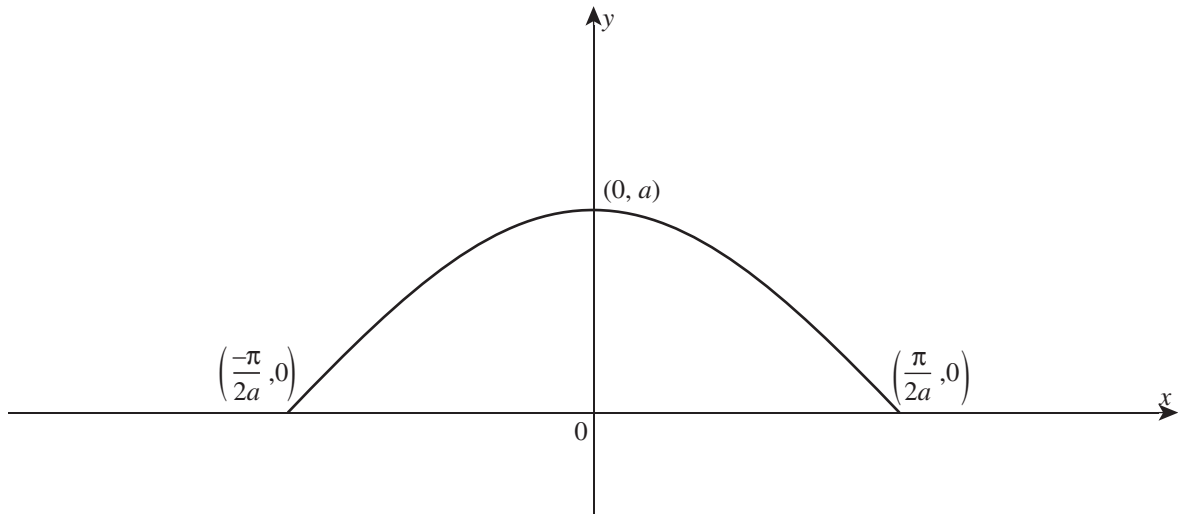
A1

Question 10 (7 marks)

a. $f(x) = a \cos(ax)$

The x -intercepts are given by $a \cos(ax) = 0$

So the closest x -intercepts to the y -axis are $x = \pm \frac{\pi}{2a}$.



Cosine graph between $\left[-\frac{\pi}{2a}, \frac{\pi}{2a}\right]$ A1

y -intercept at $(0, a)$ A1

b. $f'(x) = -a^2 \sin(ax)$

$$f'\left(-\frac{\pi}{2a}\right) = -a^2 \sin\left(-\frac{\pi}{2}\right) = -a^2(-1) = a^2$$

M1

Equation of tangent at $\left(-\frac{\pi}{2a}, 0\right)$:

$$y - 0 = a^2 \left(x + \frac{\pi}{2a}\right)$$

$$y = \frac{a\pi}{2} + a^2x$$

A1

$$f'\left(\frac{\pi}{2a}\right) = -a^2 \sin\left(\frac{\pi}{2}\right) = -a^2$$

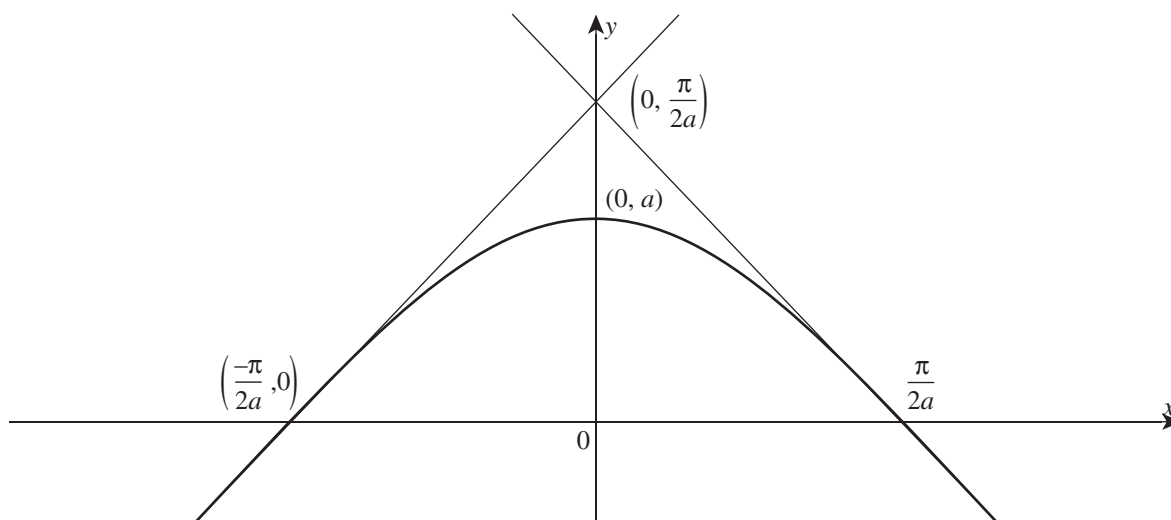
Equation of tangent at $\left(\frac{\pi}{2a}, 0\right)$:

$$y - 0 = -a^2 \left(x - \frac{\pi}{2a}\right)$$

$$y = \frac{a\pi}{2} - a^2x$$

A1

c.



Area required = area of triangle formed by tangents and x -axis – area under curve

$$A = \frac{1}{2} \times \frac{\pi}{a} \times \frac{a\pi}{2} - 2 \int_0^{\frac{\pi}{2a}} a \cos(ax) dx \quad \text{M1}$$

$$A = \frac{\pi^2}{4} - 2 \left[\sin(ax) \right]_0^{\frac{\pi}{2a}}$$

$$A = \frac{\pi^2}{4} - 2 \sin\left(\frac{\pi}{2}\right) = \frac{\pi^2}{4} - 2, \text{ which is independent of } a. \quad \text{A1}$$