

Year 2013

VCE Mathematical

Methods CAS

Trial Examination 2

Suggested Solutions



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SECTION 1

ANSWERS

1	A	B	C	D	E
2	A	B	C	D	E
3	A	B	C	D	E
4	A	B	C	D	E
5	A	B	C	D	E
6	A	B	C	D	E
7	A	B	C	D	E
8	A	B	C	D	E
9	A	B	C	D	E
10	A	B	C	D	E
11	A	B	C	D	E
12	A	B	C	D	E
13	A	B	C	D	E
14	A	B	C	D	E
15	A	B	C	D	E
16	A	B	C	D	E
17	A	B	C	D	E
18	A	B	C	D	E
19	A	B	C	D	E
20	A	B	C	D	E
21	A	B	C	D	E
22	A	B	C	D	E

SECTION 1

Question 1

Answer B

$$f(x) = \sqrt{b^2 - x^2}$$

$$f(0) = b \quad f\left(\frac{b}{2}\right) = \sqrt{b^2 - \frac{b^2}{4}} = \frac{\sqrt{3}b}{2}$$

$$\text{average rate of change } \bar{f} = \frac{f\left(\frac{b}{2}\right) - f(0)}{\frac{b}{2} - 0} = \frac{\frac{\sqrt{3}b}{2} - b}{\frac{b}{2}} = \frac{b\left(\frac{\sqrt{3}-2}{2}\right)}{\frac{b}{2}} = \sqrt{3} - 2$$

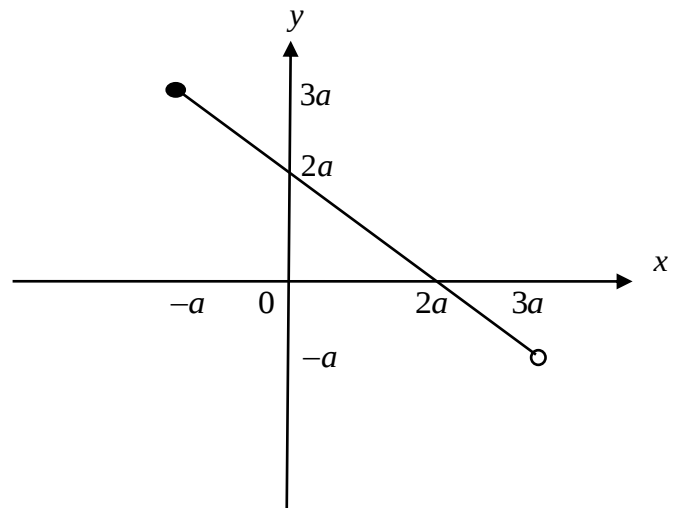
Question 2

Answer B

$$f: [-a, 3a] \rightarrow \mathbb{R}, \quad f(x) = 2a - x$$

$$f(3a) = -a$$

$$f(-a) = 3a$$

The range is $(-a, 3a]$ 

Question 3

Answer B

$$\text{The period } T = \frac{\pi}{n} = \frac{\pi}{\frac{b\pi}{3}} = \frac{3}{b}$$

Question 4

Answer A

$$f(x) = g(x) \log_e(h(x)) \text{ using the product rule}$$

$$f'(x) = g'(x) \log_e(h(x)) + \frac{g(x)h'(x)}{h(x)}$$

$$f'(2) = g'(2) \log_e(h(2)) + \frac{g(2)h'(2)}{h(2)}$$

$$\text{Now } g(2) = 3, \quad g'(2) = 4, \quad h(2) = e^2 \text{ and } h'(2) = 2$$

$$f'(2) = 4 \times \log_e(e^2) + \frac{3 \times 2}{e^2} = 8 + \frac{6}{e^2}$$

Question 5**Answer A**

$$\Delta = \begin{vmatrix} 1 & -k & 1 \\ 2 & -1 & k \\ 1 & 1 & 1 \end{vmatrix} = -k^2 + k + 2 = -(k-2)(k+1)$$

When $\Delta = 0$ or $k = 2$, $k = -1$ there is no unique solution.

When $k = -1$ there is no solution.

When $k = 2$ there is infinitely many solutions.

$\det \begin{pmatrix} 1 & -k & 1 \\ 2 & -1 & k \\ 1 & 1 & 1 \end{pmatrix}$	$-(k-2) \cdot (k+1)$
$\text{solve} \left(\det \begin{pmatrix} 1 & -k & 1 \\ 2 & -1 & k \\ 1 & 1 & 1 \end{pmatrix} = 0, k \right)$	$k = -1 \text{ or } k = 2$
$\text{eq1} := x - k \cdot y + z = 14$	$x - k \cdot y + z = 14$
$\text{eq2} := 2 \cdot x - y + k \cdot z = 10$	$2 \cdot x - y + k \cdot z = 10$
$\text{eq3} := x + y + z = -2 \cdot k$	$x + y + z = -2 \cdot k$
$\text{linSolve} \left(\begin{pmatrix} \text{eq1} \\ \text{eq2} \\ \text{eq3} \end{pmatrix}, \{x, y, z\} \right)$	$\left\{ -2 \cdot (k+1), \frac{-2 \cdot (k+7)}{k+1}, \frac{4 \cdot (k+4)}{k+1} \right\}$
$\text{linSolve} \left(\begin{pmatrix} \text{eq1} \\ \text{eq2} \\ \text{eq3} \end{pmatrix}, \{x, y, z\} \right) k = -1$	"No solution found"
$\text{linSolve} \left(\begin{pmatrix} \text{eq1} \\ \text{eq2} \\ \text{eq3} \end{pmatrix}, \{x, y, z\} \right) k = 2$	$\{ -(\mathbf{c1}-2), -6, \mathbf{c1} \}$

Question 6**Answer C**

$$f(x) = \tan(x) \quad f'(x) = \frac{1}{\cos^2(x)}$$

$$x = 60^\circ = \frac{\pi}{3} \quad h = 1^\circ = \frac{\pi}{180}$$

$$f\left(\frac{\pi}{3}\right) = \tan\left(\frac{\pi}{3}\right) = \sqrt{3} \quad f'\left(\frac{\pi}{3}\right) = \frac{1}{\cos^2\left(\frac{\pi}{3}\right)} = \frac{1}{\left(\frac{1}{2}\right)^2} = 4$$

$$f(x+h) \approx f(x) + hf'(x)$$

$$\tan(61^\circ) = \sqrt{3} + \frac{\pi}{180} \times 4 = 1.8019$$

Question 7**Answer A**

$$f(x) = -x^3 + 4x^2 + 3x - 2$$

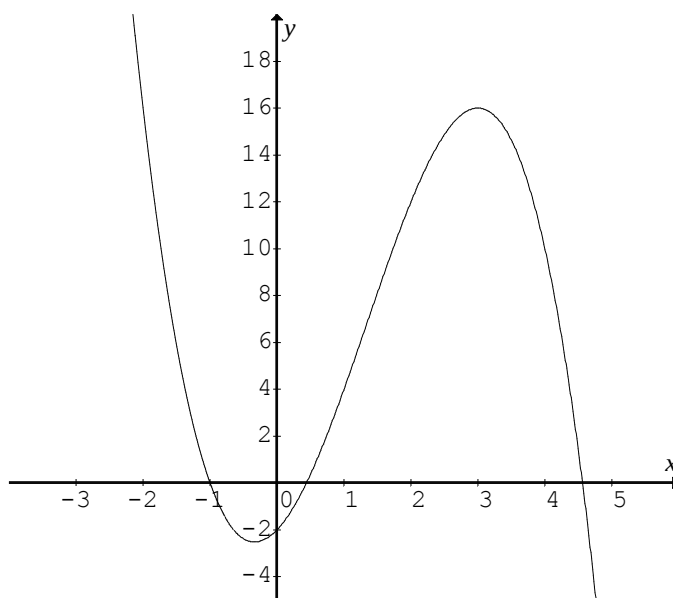
$$f'(x) = -3x^2 + 8x + 3$$

$$= -(3x^2 - 8x - 3)$$

$$= -(3x+1)(x-3)$$

Turning points at $x = -\frac{1}{3}$ and $x = 3$

To restrict the domain to make f a one-one function we require $a \leq -\frac{1}{3}$

**Question 8****Answer E**

$$y' = 2 - 2\log_e(2x' + 2)$$

$$\frac{y' - 2}{-2} = \log_e(2x' + 2) \quad y = \log_e(x)$$

$$\Rightarrow y = \frac{y' - 2}{-2} \text{ and } x = 2x' + 2 \Rightarrow y' = 2 - 2y \text{ and } x' = \frac{x}{2} - 1$$

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

Question 9**Answer C**

$$y = \cos(\pi bx) \quad \frac{dy}{dx} = -\pi b \sin(\pi bx) \text{ when } x = \frac{1}{6b} \quad m_T = -\pi b \sin\left(\frac{\pi}{6}\right) = -\frac{\pi b}{2}$$

$$\text{normal } m_N = \frac{2}{\pi b} = 4 \Rightarrow b = \frac{1}{2\pi}$$

Question 10**Answer E**

$$f(x) = ax^2 - 2bx = x(ax - 2b), \text{ the graph}$$

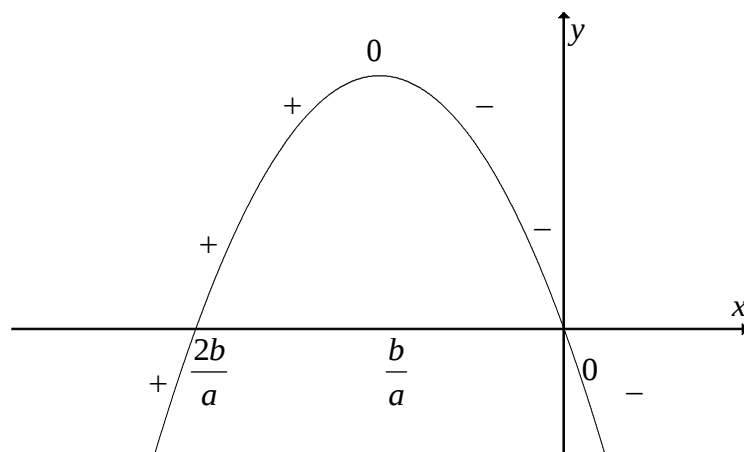
crosses the x -axis at $x = 0$ and $x = \frac{2b}{a}$.

$$f'(x) = 2ax - 2b.$$

A decreasing function has a negative gradient, so that $f'(x) < 0 \Rightarrow 2ax - 2b < 0$.

$$\Rightarrow ax < b \text{ if } a < 0 \text{ and } b > 0$$

$$\text{then } x > \frac{b}{a}.$$



Question 11**Answer C** T catches the train and D drives to work

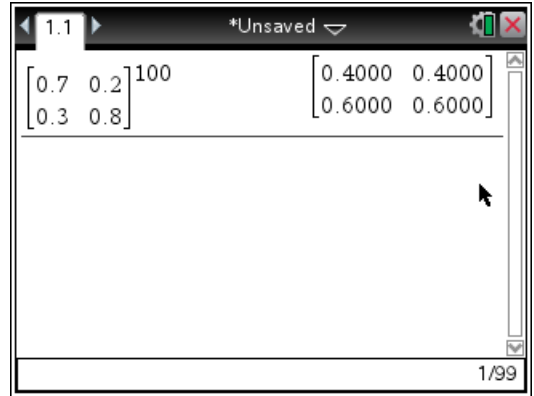
$$T \rightarrow T = 0.7 \Rightarrow T \rightarrow D = 0.3$$

$$D \rightarrow T = 0.2 \Rightarrow D \rightarrow D = 0.8$$

$$\begin{matrix} T & D \end{matrix}$$

$$\begin{matrix} T \\ D \end{matrix} \begin{bmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{bmatrix}^{100} = \begin{bmatrix} 0.4 & 0.4 \\ 0.6 & 0.6 \end{bmatrix}$$

long-term drives is $0.6 = \frac{3}{5}$

**Question 12****Answer C**

$$A = \frac{3\sqrt{3}}{2}x^2 \Rightarrow \frac{dA}{dx} = 3\sqrt{3}x \quad \text{given} \quad \frac{dA}{dt} = 36$$

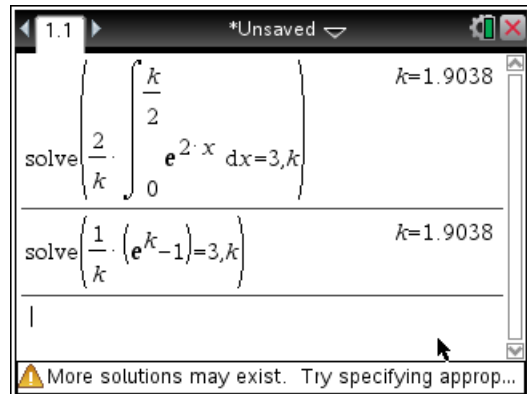
$$\frac{dx}{dt} = \frac{dx}{dA} \frac{dA}{dt} = \frac{36}{3\sqrt{3}x} \quad \left. \frac{dx}{dt} \right|_{x=2} = \frac{36}{3\sqrt{3} \times 2} = \frac{6}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = 2\sqrt{3} \text{ cm/min}$$

Question 13**Answer D**

$$\frac{1}{\frac{k}{2} - 0} \int_0^{\frac{k}{2}} e^{2x} dx = 3$$

$$\frac{2}{k} \left[\frac{1}{2} e^{2x} \right]_0^{\frac{k}{2}} = \frac{1}{k} (e^k - 1) = 3$$

solving for k gives, $k = 1.904$

**Question 14****Answer E**

$$f(x) = |x| \text{ and } g(x) = x^2 \Rightarrow g^{-1}(x) = \sqrt{x}$$

$$g^{-1}(x)g(x) = \sqrt{x} \times x^2 = x^{\frac{5}{2}} \neq f(x) \quad \text{Peter is incorrect}$$

$$g^{-1}(g(x)) = g^{-1}(x^2) = \sqrt{x^2} = |x| = f(x) \quad \text{Quentin is correct}$$

$$g(g^{-1}(x)) = g(\sqrt{x}) = (\sqrt{x})^2 = x \neq f(x) \quad \text{Sara is incorrect}$$

$$f(g(x)) = f(x^2) = |x^2| = x^2 = g(x) \quad \text{Tanya is correct}$$

Question 15**Answer D**

Total area under the curve is one $k \int_0^{\frac{\pi}{3}} \sin(3x) dx = 1$

$$k \left[-\frac{1}{3} \cos(3x) \right]_0^{\frac{\pi}{3}} = k \left[\left(-\frac{1}{3} \cos(\pi) + \frac{1}{3} \cos(0) \right) \right] = \frac{2k}{3} = 1 \Rightarrow k = \frac{3}{2}$$

The mean and median are both symmetrical at $x = \frac{\pi}{6}$

D. Is false $E(X^2) = \frac{\pi^2 - 4}{18}$

solve $k \cdot \int_0^{\frac{\pi}{3}} \sin(3 \cdot x) dx = 1, k$	$k = \frac{3}{2}$
$\frac{3}{2} \cdot \int_0^{\frac{\pi}{3}} (x \cdot \sin(3 \cdot x)) dx$	$\frac{\pi}{6}$
$\frac{3}{2} \cdot \int_0^{\frac{\pi}{3}} (x^2 \cdot \sin(3 \cdot x)) dx$	$\frac{\pi^2 - 4}{18}$
$\frac{3}{2} \cdot \int_0^{\frac{\pi}{4}} \sin(3 \cdot x) dx$	$\frac{\sqrt{2} + 2}{4}$

Question 16**Answer E**

$$\Pr(X = k) = \frac{e^{-\lambda} \lambda^k}{k!}$$

$$\Pr(X = 0) = e^{-\lambda}, \Pr(X = 1) = \lambda e^{-\lambda}, \Pr(X = 2) = \frac{\lambda^2 e^{-\lambda}}{2!}$$

$$\Pr(X > 2) = \Pr(X \geq 3)$$

$$\Pr(X \geq 3) = 1 - [\Pr(X = 0) + \Pr(X = 1) + \Pr(X = 2)]$$

$$\Pr(X \geq 3) = 1 - \left[e^{-\lambda} + \lambda e^{-\lambda} + \frac{\lambda^2 e^{-\lambda}}{2!} \right]$$

$$\Pr(X \geq 3) = 1 - \frac{e^{-\lambda}}{2} (\lambda^2 + 2\lambda + 2)$$

Question 17**Answer D**

$$v(t) = \frac{27}{(3t+4)^2} \text{ m/s initial velocity } v(0) = \frac{27}{16} \text{ m/s A. is true.}$$

$$\int_0^2 \frac{27}{(3t+4)^2} dt = \frac{27}{20} \text{ distance travelled in the first two seconds, B. is true.}$$

$$a = \frac{dv}{dt} = \frac{d}{dt} \left[\frac{27}{(3t+4)^2} \right] = \frac{-162}{(3t+4)^3} \text{ m/s}^2 \text{ acceleration C. is true.}$$

$$x(t) = \int \frac{27}{(3t+4)^2} dt = \frac{-27}{3(3t+4)} + c = \frac{-9}{3t+4} + c$$

$$x(0) = 0 \Rightarrow 0 = -\frac{9}{4} + c \quad c = \frac{9}{4}$$

$$x(t) = \frac{9}{4} - \frac{9}{3t+4} = \frac{9(3t+4) - 9 \times 4}{4(3t+4)} = \frac{27t}{4(3t+4)} \text{ position E. is true, D. is false}$$

Define $v(t) = \frac{27}{(3 \cdot t + 4)^2}$	Done
$v(0)$	$\frac{27}{16}$
$\int_0^2 v(t) dt$	$\frac{27}{20}$
$\frac{d}{dt}(v(t))$	$\frac{-162}{(3 \cdot t + 4)^3}$
$\int_0^t v(t) dt$	$\frac{9}{4} - \frac{9}{3 \cdot t + 4}$
comDenom $\left(\frac{9}{4} - \frac{9}{3 \cdot t + 4} \right)$	$\frac{27 \cdot t}{12 \cdot t + 16}$

Question 18**Answer B**

A cubic with no turning points can be expressed in the form $f(x) = a(x+h)^3 + k$

$$f(x) = a(x+h)^3 + k = a(x^3 + 3x^2h + 3xh^2 + h^3) + k = ax^3 + 3ahx^2 + 3ah^2x + ah^3 + k$$

$$f(x) = ax^3 + bx^2 + cx + d \Rightarrow b = 3ah, \quad c = 3ah^2 \text{ and } d = ah^3 + k$$

The point of inflexion is at $(-h, k)$ and $f'(x) = 3ax^2 + 2bx + c = 3a(x+h)^2$

$f'(x) = 0$ when $x = -h$ and this equation has only one root, or for no turning points, no real factors, when $\Delta \leq 0$, thus

$$\Delta = (2b)^2 - 4 \times 3a \times c = 4b^2 - 12ac = 4(b^2 - 3ac) \leq 0 \Rightarrow b^2 \leq 3ac$$

Question 19**Answer A**

The area bounded by the graph of $y = f(x)$ between the x -values of $x = a$ and $x = b$, is the same if the area is translated b units to the right, that is the area bounded by the graph of $y = f(x - b)$ between the x -values of $x = a + b$ and $x = 2b$. This is also the same area if the area is translated b units to the left, that is the area bounded by the graph of $y = f(x + b)$ between the x -values of $x = a - b$ and $x = 0$, since $f(x) > 0$ for $x \in [a - b, 2b]$ where $a, b \in \mathbb{R}$ and $b > a > 0$.

Question 20**Answer D**

$$X \stackrel{d}{=} N(\mu = 15, \sigma^2 = 16) \quad \sigma = 4 \quad Z \stackrel{d}{=} N(\mu = 0, \sigma^2 = 1) \quad Z = \frac{x - \mu}{\sigma}$$

X	7	11	15	19	23
Z	-2	-1	0	1	2

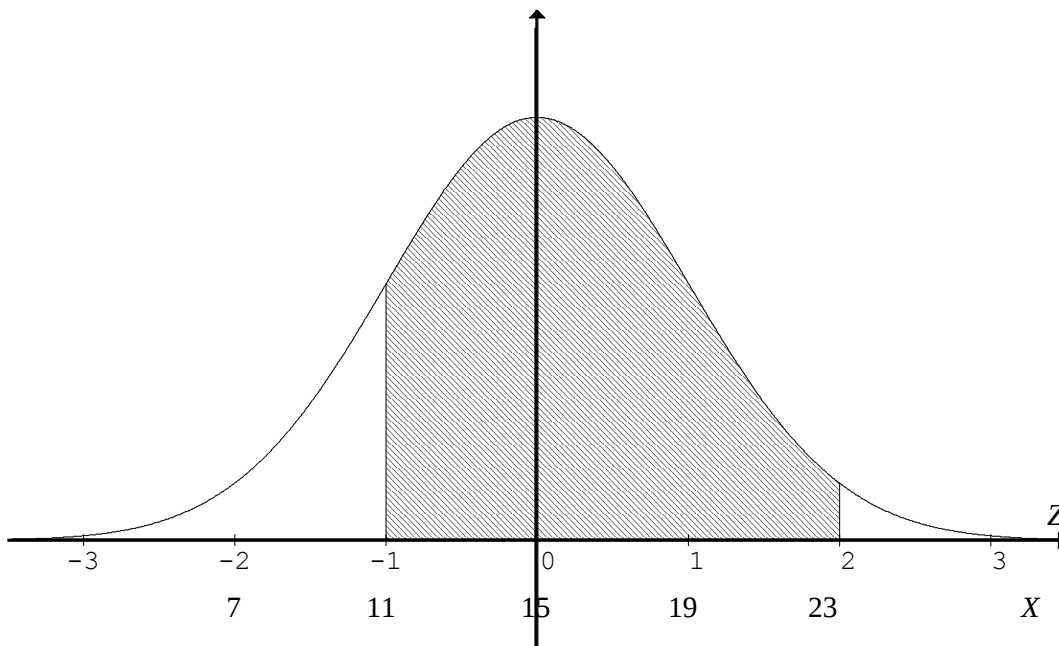
$$\Pr(-1 < Z < 2) = 1 - [\Pr(Z > 2) + \Pr(Z < -1)] \quad \text{A. is true}$$

$$= \Pr(-1 < Z < 0) + \Pr(0 < Z < 2) = \Pr(0 < Z < 1) + \Pr(0 < Z < 2) \quad \text{B. is true}$$

$$= \Pr(11 < X < 23) = 1 - [\Pr(X > 23) + \Pr(X < 11)] \quad \text{C. is true}$$

$$= 1 - [\Pr(X < 7) + \Pr(X > 19)] \quad \text{by symmetry, E. is true}$$

D. is false, all of A. B. C. E. are true



Question 21**Answer E**

Since it is a discrete random variable, the probabilities add to one, so that

$$a + \frac{a}{2} + \frac{a}{3} = \frac{11a}{6} = 1 \Rightarrow a = \frac{6}{11}$$

$$E(X) = \sum x \Pr(X = x) = 1 \times a + 2 \times \frac{a}{2} + 3 \times \frac{a}{3} = 3a = \frac{18}{11}$$

$$E(X^2) = \sum x^2 \Pr(X = x) = 1 \times a + 4 \times \frac{a}{2} + 9 \times \frac{a}{3} = 6a = \frac{36}{11}$$

$$\text{var}(X) = E(X^2) - (E(X))^2 = \frac{36}{11} - \left(\frac{18}{11}\right)^2 = \frac{72}{121}$$

All of **A. B. C.** and **D.** are true, **E.** is false

$$E\left(\frac{1}{X}\right) = \sum \frac{1}{x} \Pr(X = x) = 1 \times a + \frac{1}{2} \times \frac{a}{2} + \frac{1}{3} \times \frac{a}{3} = \frac{49}{36}a = \frac{49}{36} \times \frac{6}{11} = \frac{49}{66}$$

Question 22**Answer B**

$$f(x) = \log_e(3x + 4)$$

x	0	1	2	3	4
$f(x)$	$\log_e(4)$	$\log_e(7)$	$\log_e(10)$	$\log_e(13)$	$\log_e(16)$

consider four right rectangles, each of width one unit.

$$A_L = 1 \times \log_e(7) + 1 \times \log_e(10) + 1 \times \log_e(13) + 1 \times \log_e(16)$$

$$A_L = \log_e(7 \times 10 \times 13 \times 16) = \log_e(14560)$$

END OF SECTION 1 SUGGESTED ANSWERS

SECTION 2

Question 1

a.i. Let R pays by credit card, and C pays using cash
 $C \rightarrow C = 0.35 \Rightarrow C \rightarrow R = 0.65$ and $R \rightarrow R = 0.45 \Rightarrow R \rightarrow C = 0.55$
 $\Pr(\text{uses credit card three times and the first is credit card})$
 $= \Pr(RRRC) + \Pr(RRCR) + \Pr(RCRR)$ M1
 $= 0.45^2 \times 0.55 + 0.45 \times 0.55 \times 0.65 + 0.55 \times 0.65 \times 0.45$
 $= 0.433$ A1

ii. $\Pr(\text{first and fourth are credit cards})$
 $= \Pr(RCCR) + \Pr(RCRR) + \Pr(RRCR) + \Pr(RRRR)$
 $= 0.55 \times 0.35 \times 0.65 + 0.55 \times 0.65 \times 0.45 + 0.45 \times 0.55 \times 0.65 + 0.45^3$
 $= 0.538$ A1

$C \quad R$

alternatively $\begin{matrix} C \\ R \end{matrix} \begin{bmatrix} 0.35 & 0.55 \\ 0.65 & 0.45 \end{bmatrix}^3 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.462 \\ 0.538 \end{bmatrix}$ fourth is credit card 0.538

b.i. $X \stackrel{d}{=} Bi(n=16, p=?)$
 $\Pr(X=8) + \Pr(X=9) = 0.25$
 $\binom{16}{8} p^8 (1-p)^8 + \binom{16}{9} p^9 (1-p)^7 = 0.25$ by CAS M1
 $1430 p^8 (p-9)(p-1)^7 = 0.25$
solving with $0 < p < 1$ gives $p = 0.4132$ or 0.6473
but since $E(X) > 8 \Rightarrow p > 0.5$ so $p = 0.6473$ A1

ii. $X \stackrel{d}{=} Bi(n=16, p=0.65)$
 $\Pr(X > 8) = \Pr(X \geq 9) = 0.8406$ A1

$nCr(16,8) \cdot p^8 \cdot (1-p)^8 + nCr(16,9) \cdot p^9 \cdot (1-p)^7$	$1430 \cdot p^8 \cdot (p-9) \cdot (p-1)^7$
$\text{solve}(1430 \cdot p^8 \cdot (p-9) \cdot (p-1)^7 = 0.25, p) 0 < p < 1$	$p = 0.4132 \text{ or } p = 0.6473$
$\text{binomcdf}(16, 0.65, 9, 16)$	0.8406

c. X is the time in hours spent shopping, $X \stackrel{d}{=} N(\mu = ?, \sigma^2 = ?)$

$$(1) \quad \Pr(X > 3700) = 0.14$$

$$(2) \quad \Pr(X < 2990) = 0.26$$

$$(1) \Rightarrow \frac{3700 - \mu}{\sigma} = 1.0803 \quad \text{M1}$$

$$(2) \Rightarrow \frac{2990 - \mu}{\sigma} = -0.6433 \quad \text{A1}$$

$$(1) \quad 3700 - \mu = 1.0803\sigma$$

$$(2) \quad 2990 - \mu = -0.6433\sigma$$

now subtract equations (1) – (2) solving gives

$$\sigma = \$412 \text{ substituting gives } \mu = \$3,255 \quad \text{A1}$$

$eq1: \frac{3700-m}{s} = \text{invNorm}(0.86, 0, 1)$	$\frac{3700-m}{s} = 1.0803$
$eq2: \frac{2990-m}{s} = \text{invNorm}(0.26, 0, 1)$	$\frac{2990-m}{s} = -0.6433$
$\text{solve}\left(\begin{Bmatrix} eq1 \\ eq2 \end{Bmatrix}, \{m, s\}\right)$	$s=411.9130 \text{ and } m=3255.002$

d.i. Since the function is continuous at $t = 3$ $ae^1 = b \sin\left(\frac{3\pi}{6}\right) = b \Rightarrow b = ae \quad \text{A1}$

$$\int_0^3 ae^{\frac{t}{3}} dt + \int_3^6 b \sin\left(\frac{\pi t}{6}\right) dt = 1$$

Since the total area under the curve is equal to one, substituting $b = ae$

$$a \left[\int_0^3 e^{\frac{t}{3}} dt + \int_3^6 e \sin\left(\frac{\pi t}{6}\right) dt \right] = 1$$

$$a \left(\left[3e^{\frac{t}{3}} \right]_0^3 - \left[\frac{6e}{\pi} \cos\left(\frac{\pi t}{6}\right) \right]_3^6 \right) = 1 \quad \text{A1}$$

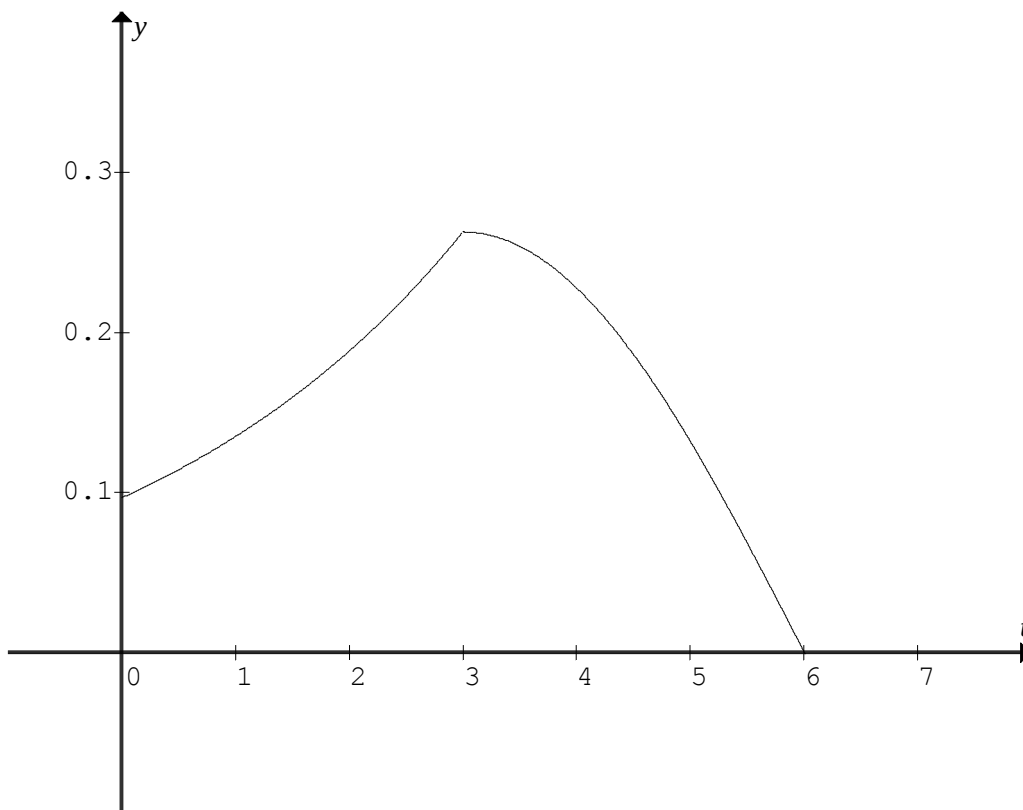
$$a \left(3(e-1) - \frac{6e}{\pi} \left(\cos(\pi) - \cos\left(\frac{\pi}{2}\right) \right) \right) = a \left(3(e-1) + \frac{6e}{\pi} \right) = 1$$

$$a \left(\frac{3\pi(e-1) + 6e}{\pi} \right) = 1 \Rightarrow a = \frac{\pi}{3\pi e + 6e - 3\pi}$$

$$a = \frac{\pi}{3(e(\pi+2) - \pi)}$$

- ii. Graph passes through $(0, 0.097)$ $(3, 0.26)$ $(6, 0)$ and is continuous and zero elsewhere

G2



$$\begin{aligned}
 \text{iii. } \Pr(X > 4) &= b \int_4^6 \sin\left(\frac{\pi t}{6}\right) dt \\
 &= \left[-\frac{6b}{\pi} \cos\left(\frac{\pi t}{6}\right) \right]_4^6 = \left(-\frac{6b}{\pi} \left(\cos(\pi) - \cos\left(\frac{2\pi}{3}\right) \right) \right) \text{ with } b = \frac{\pi e}{3(e(\pi+2) - \pi)} \\
 &= \frac{e}{e(\pi+2) - \pi}
 \end{aligned}$$

A1

$$\text{iv. } \Pr(T < 2 | T < 3) = \frac{\Pr(T < 2)}{\Pr(T < 3)}$$

M1

$$\begin{aligned}
 &= \frac{\int_0^2 a e^{\frac{t}{3}} dt}{\int_0^3 a e^{\frac{t}{3}} dt} = \frac{\left[3e^{\frac{t}{3}} \right]_0^2}{\left[3e^{\frac{t}{3}} \right]_0^3} \\
 &= \frac{e^{\frac{2}{3}} - 1}{e - 1}
 \end{aligned}$$

A1

v. $E(T) = a \int_0^3 t e^{\frac{t}{3}} dt + \int_3^6 et \sin\left(\frac{\pi t}{6}\right) dt$ with $a = \frac{\pi}{3(e(\pi+2) - \pi)}$ M1

mean time $E(T) = 2.92$ hours A1

Define $f(t) = \begin{cases} a \cdot e^{\frac{t}{3}}, & 0 \leq t \leq 3 \\ b \cdot \sin\left(\frac{\pi \cdot t}{6}\right), & 3 \leq t \leq 6 \end{cases}$	Done
$f(3)$	$a \cdot e$
$\text{solve}\left(\int_0^6 f(t) dt = 1, a\right) b = a \cdot e$	$a = \frac{\pi}{3 \cdot (e \cdot (\pi+2) - \pi)}$
$a := \frac{\pi}{3 \cdot (e \cdot (\pi+2) - \pi)}$	$\frac{\pi}{3 \cdot (e \cdot (\pi+2) - \pi)}$
$b := a \cdot e$	$\frac{e \cdot \pi}{3 \cdot (e \cdot (\pi+2) - \pi)}$
Define $f(t) = \begin{cases} a \cdot e^{\frac{t}{3}}, & 0 \leq t \leq 3 \\ b \cdot \sin\left(\frac{\pi \cdot t}{6}\right), & 3 \leq t \leq 6 \end{cases}$	Done

$\int_4^6 f(t) dt$	$\frac{e}{e \cdot (\pi+2) - \pi}$
$\frac{\int_0^2 f(t) dt}{\int_0^3 f(t) dt}$	$\frac{2}{e^3 - 1}$
$\int_0^6 (t \cdot f(t)) dt$	2.9222

Question 2

i. $f(x) = x^3 + bx^2 + cx + 6$
 $f'(x) = 3x^2 + 2bx + c$ A1

ii. $f(2) = (2)^3 + b(2)^2 + 2c + 6 = 8 + 4b + 2c + 6 = 4b + 2c + 14$
 $f'(2) = 3(2)^2 + 4b + c = 4b + c + 12$ M1

Equation of the tangent at P

$$y - (4b + 2c + 14) = (4b + c + 12)(x - 2)$$

$$tp(x) = y = (4b + c + 12)x - 2(4b + c + 12) + (4b + 2c + 14)$$

$$tp(x) = y = (4b + c + 12)x - 4b - 10$$
 A1

iii. Solving $tp(x) = f(x)$ when $x = -1$ $tp(-1) = 6 = f(-1)$
 $tp(-1) = -(4b + c + 12) - 4b - 10 = -8b - c - 22 = 6$
 $\Rightarrow (1) \quad -8b - c = 28$ M1

$$f(-1) = 6 \Rightarrow -1 + b - c + 6 = 6$$

$$\Rightarrow (2) \quad b - c = 1$$
 A1

solving these simultaneous equations,

$$(2) - (1) \Rightarrow 9b = -27 \Rightarrow b = -3 \text{ and } c = -4 \quad P(2, -6)$$

iv. so substitute $b = -3$ and $c = -4$
 $f(x) = x^3 - 3x^2 - 4x + 6$ and $tp(x) = -4x + 2$ A1

$$A_1 = \int_{-1}^2 (f(x) - tp(x)) dx$$

$$A_1 = \int_{-1}^2 (x^3 - 3x^2 + 4) dx$$
 A1

v. Equation of the tangent at Q

$$f(-1) = 6 \text{ and } f'(x) = 3x^2 - 6x - 4 \Rightarrow f'(-1) = 3 + 6 - 4 = 5$$
 M1

$$y - 6 = 5(x + 1)$$

$$tq(x) = y = 5x + 11$$
 A1

vi. $tq(x) = f(x)$

$$5x + 11 = x^3 - 3x^2 - 4x + 6 \text{ solving gives } \Rightarrow x = 5 \text{ or } x = -1$$

$$f(5) = (5)^3 - 3(5)^2 - 20 + 6 = 36 \text{ or } tq(5) = 25 + 11 = 36$$

$$R(5, 36)$$
 A1

vii. $A_2 = \int_{-1}^5 (tq(x) - f(x)) dx$

$$A_2 = \int_{-1}^5 (-x^3 + 3x^2 + 9x + 5) dx$$

A1

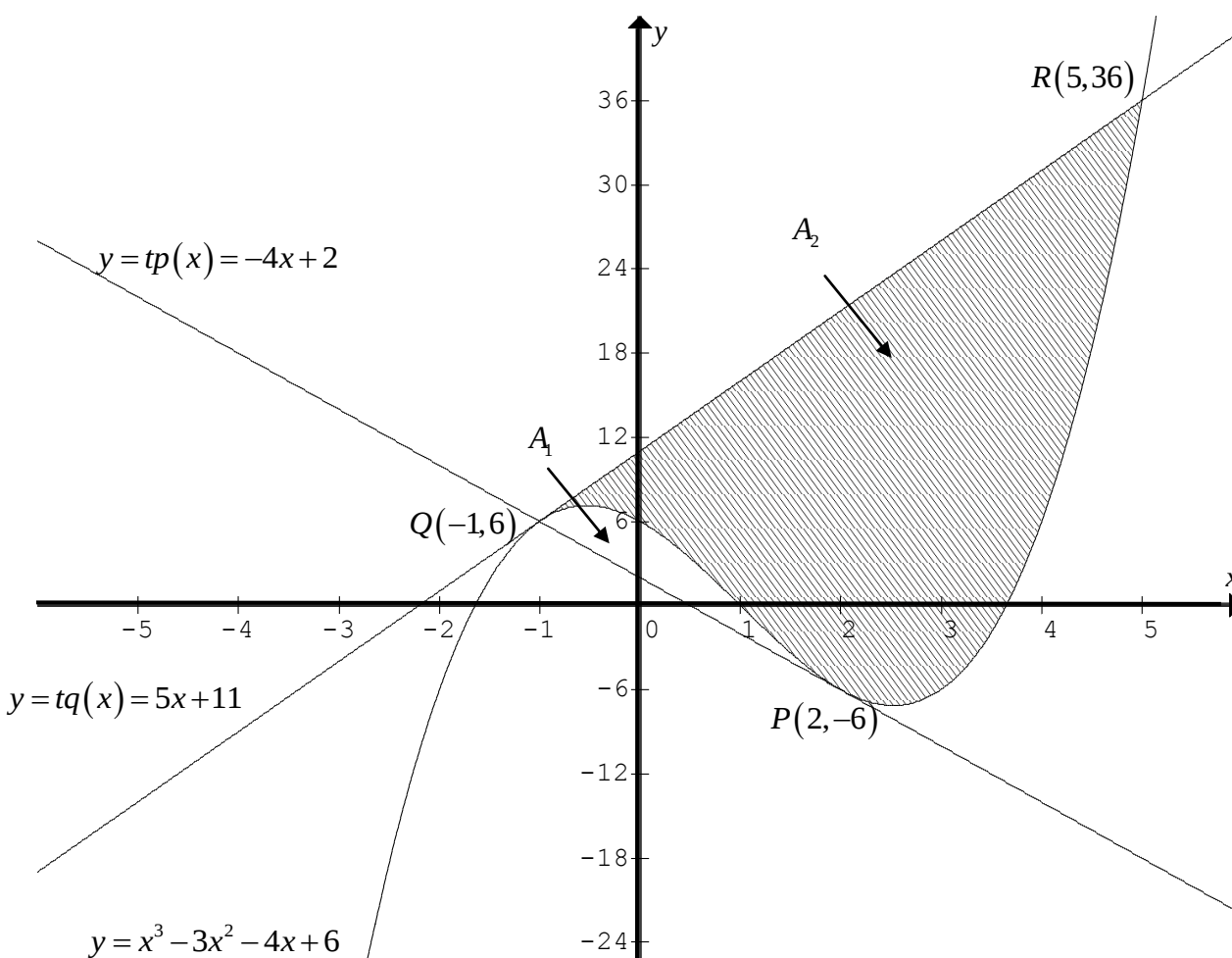
viii. $A_1 = \frac{27}{4}$ $A_2 = 108$

$$\frac{A_2}{A_1} = 16$$

A1

ix. correct graphs, tangents at points P , and Q

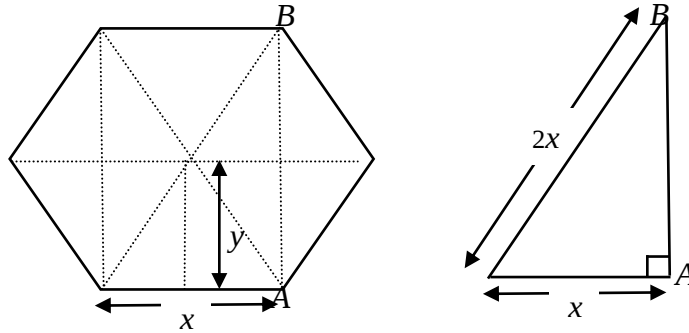
G2



Define $f(x)=x^3+b \cdot x^2+c \cdot x+6$	Done
Define $df(x)=\frac{d}{dx}(f(x))$	Done
$f(2)$	$4 \cdot b+2 \cdot c+14$
$df(2)$	$4 \cdot b+c+12$
$\text{solve}(y-f(2)=df(2) \cdot (x-2),y)$	$y=(4 \cdot b+c+12) \cdot x-4 \cdot b-10$
Define $tp(x)=(4 \cdot b+c+12) \cdot x-4 \cdot b-10$	Done
$\text{solve}(tp(x)=f(x),x)$	$x=-(b+4)$ or $x=2$
$\text{solve}(-(b+4)=-1 \text{ and } f(-1)=6, \{b,c\})$	$b=-3$ and $c=-4$
Define $f(x)=x^3+b \cdot x^2+c \cdot x+6 b=-3 \text{ and } c=-4$	Done
Define $df(x)=\frac{d}{dx}(f(x))$	Done
Define $tp(x)=(4 \cdot b+c+12) \cdot x-4 \cdot b-10 b=-3 \text{ and } c=-4$	Done
$f(x)-tp(x)$	$x^3-3 \cdot x^2+4$
$\int_{-1}^2 (f(x)-tp(x))dx$	$\frac{27}{4}$
$\text{solve}(y-f(-1)=df(-1) \cdot (x+1),y)$	$y=5 \cdot x+11$
Define $tq(x)=5 \cdot x+11$	Done
$\text{solve}(tq(x)=f(x),x)$	$x=-1$ or $x=5$
$tq(x)-f(x)$	$-x^3+3 \cdot x^2+9 \cdot x+5$
$tq(5)$	36
$f(5)$	36
$tq(x)-f(x)$	$-x^3+3 \cdot x^2+9 \cdot x+5$
$\int_{-1}^5 (tq(x)-f(x))dx$	108
$\frac{108}{\frac{27}{4}}$	16

Question 3

- a. Consider the hexagonal base of the vase



M1

$$d(AB) = \sqrt{3}x \quad y = \frac{1}{2}d(AB) = \frac{\sqrt{3}x}{2}$$

$$\text{One triangle has area } \frac{1}{2}xy = \frac{1}{2}x \frac{\sqrt{3}x}{2} = \frac{\sqrt{3}}{4}x^2$$

$$\text{or area of one equilateral triangle } \frac{1}{2}x^2 \sin(60^\circ) = \frac{1}{2}x^2 \times \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{4}x^2$$

$$\text{Area of the base is six triangles } 6 \times \frac{\sqrt{3}}{4}x^2 = \frac{3\sqrt{3}}{2}x^2$$

A1

Total surface area is the area of the base and six rectangles of length x and height h .

$$S = \frac{3\sqrt{3}}{2}x^2 + 6xh$$

- b. Volume is area of the base multiplied by the height

$$V = \frac{3\sqrt{3}}{2}x^2h$$

A1

c.i. $S = \frac{3\sqrt{3}}{2}x^2 + 6xh \Rightarrow 6xh = S - \frac{3\sqrt{3}}{2}x^2 = \frac{2S - 3\sqrt{3}x^2}{2}$

$$h = \frac{2S - 3\sqrt{3}x^2}{12x}$$

M1

$$V(x) = \frac{3\sqrt{3}}{2}x^2h = \frac{3\sqrt{3}}{2}x^2 \left(\frac{2S - 3\sqrt{3}x^2}{12x} \right)$$

$$V(x) = \frac{\sqrt{3}}{8}(2Sx - 3\sqrt{3}x^3)$$

M1

$$V(x) = \frac{\sqrt{3}}{8}x(2S - 3\sqrt{3}x^2) > 0 \Rightarrow 2S - 3\sqrt{3}x^2 > 0 \quad 2S > 3\sqrt{3}x^2 \text{ and } x > 0$$

$$\text{so that } 0 < x < \sqrt{\frac{2S}{3\sqrt{3}}} \text{ or } 0 < x < \frac{\sqrt{2\sqrt{3}S}}{3}$$

A1

$$\text{ii.} \quad \frac{dV}{dx} = \frac{\sqrt{3}}{8} (2S - 9\sqrt{3}x^2) \quad \text{A1}$$

$$\text{for a maximum volume} \quad \frac{dV}{dx} = 0 \Rightarrow 2S = 9\sqrt{3}x^2$$

$$x = \sqrt{\frac{2S}{9\sqrt{3}}} = \frac{1}{3}\sqrt{\frac{2S}{\sqrt{3}}} \quad \text{A1}$$

$$\text{d.i.} \quad V = \frac{3\sqrt{3}}{2} x^2 h \Rightarrow h = \frac{2V}{3\sqrt{3}x^2} \quad \text{A1}$$

$$S = \frac{3\sqrt{3}}{2} x^2 + 6xh$$

$$S = \frac{3\sqrt{3}}{2} x^2 + 6x \left(\frac{2V}{3\sqrt{3}x^2} \right)$$

$$S(x) = \frac{3\sqrt{3}}{2} x^2 + \frac{4V}{\sqrt{3}x} = \frac{3\sqrt{3}}{2} x^2 + \frac{4V}{\sqrt{3}} x^{-1} = \sqrt{3} \left(\frac{3}{2} x^2 + \frac{4V}{3} x^{-1} \right) \quad \text{M1}$$

$$\text{ii.} \quad \frac{dS}{dx} = \sqrt{3} \left(3x - \frac{4V}{3} x^{-2} \right) = \sqrt{3} \left(3x - \frac{4V}{3x^2} \right) \quad \text{A1}$$

for a minimum surface area

$$\frac{dS}{dx} = 0 \Rightarrow 3x - \frac{4V}{3x^2} = 0 \Rightarrow x^3 = \frac{4V}{9}$$

$$x = \sqrt[3]{\frac{4V}{9}} \quad \text{A1}$$

Note that from CAS, there are many equivalent forms, for all these answers and equations.

e. solving the five non-linear equations using CAS, for four unknowns, S, V, h, x

$$(1) \quad S = \frac{3\sqrt{3}}{2} x^2 + 6xh \quad (2) \quad V = \frac{3\sqrt{3}}{2} x^2 h \quad \text{M1}$$

$$(3) \quad x = \sqrt{\frac{2S}{9\sqrt{3}}} \quad (4) \quad x = \sqrt[3]{\frac{4V}{9}} \quad \text{and} \quad (5) \quad V = 9S$$

$$\text{gives} \quad x = 18\sqrt{3} \text{ and } h = 27 \text{ cm} \quad \text{A1}$$

$s = \frac{3 \cdot \sqrt{3}}{2} \cdot x^2 + 6 \cdot x \cdot h$	$s = \frac{3 \cdot \sqrt{3} \cdot x^2}{2} + 6 \cdot h \cdot x$
$v = \frac{3 \cdot \sqrt{3}}{2} \cdot x^2 \cdot h$	$v = \frac{3 \cdot h \cdot \sqrt{3} \cdot x^2}{2}$
$\text{solve}\left(s = \frac{3 \cdot \sqrt{3}}{2} \cdot x^2 + 6 \cdot x \cdot h, h\right)$	$h = \frac{2 \cdot s - 3 \cdot \sqrt{3} \cdot x^2}{12 \cdot x}$
$v = \frac{3 \cdot \sqrt{3}}{2} \cdot x^2 \cdot h \mid h = \frac{2 \cdot s - 3 \cdot \sqrt{3} \cdot x^2}{12 \cdot x}$	$v = \frac{\sqrt{3} \cdot s \cdot x}{4} - \frac{9 \cdot x^3}{8}$
$\frac{d}{dx}\left(\frac{\sqrt{3} \cdot s \cdot x}{4} - \frac{9 \cdot x^3}{8}\right)$	$\frac{\sqrt{3} \cdot s}{4} - \frac{27 \cdot x^2}{8}$
$\text{solve}\left(\frac{\sqrt{3} \cdot s}{4} - \frac{27 \cdot x^2}{8} = 0, x\right)$	$x = \frac{\sqrt[3]{\frac{3}{4} \cdot \sqrt{2} \cdot s}}{9} \text{ and } s \geq 0 \text{ or } x = -\frac{\sqrt[3]{\frac{3}{4} \cdot \sqrt{2} \cdot s}}{9} \text{ and } s \geq 0$
$\text{solve}\left(v = \frac{3 \cdot \sqrt{3}}{2} \cdot x^2 \cdot h, h\right)$	$h = \frac{2 \cdot \sqrt{3} \cdot v}{9 \cdot x^2}$
$s = \frac{3 \cdot \sqrt{3}}{2} \cdot x^2 + 6 \cdot x \cdot h \mid h = \frac{2 \cdot \sqrt{3} \cdot v}{9 \cdot x^2}$	$s = \frac{4 \cdot \sqrt{3} \cdot v}{3 \cdot x} + \frac{3 \cdot \sqrt{3} \cdot x^2}{2}$
$\frac{d}{dx}\left(4 \cdot \sqrt{3} \cdot v - \frac{3 \cdot \sqrt{3} \cdot x^2}{2}\right)$	$3 \cdot \sqrt{3} \cdot x - \frac{4 \cdot \sqrt{3} \cdot v}{2}$

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$\frac{d}{dx}\left(\frac{4 \cdot \sqrt{3} \cdot v}{3 \cdot x} + \frac{3 \cdot \sqrt{3} \cdot x^2}{2}\right)$	$3 \cdot \sqrt{3} \cdot x - \frac{4 \cdot \sqrt{3} \cdot v}{2}$
$\text{solve}\left(3 \cdot \sqrt{3} \cdot x - \frac{4 \cdot \sqrt{3} \cdot v}{2} = 0, x\right)$	$x = \frac{\frac{2}{3} \cdot (3 \cdot v)^{\frac{1}{3}}}{3}$
$eq1: s = \frac{3 \cdot \sqrt{3}}{2} \cdot x^2 + 6 \cdot x \cdot h$	$s = \frac{3 \cdot \sqrt{3} \cdot x^2}{2} + 6 \cdot h \cdot x$
$eq2: v = \frac{3 \cdot \sqrt{3}}{2} \cdot x^2 \cdot h$	$v = \frac{3 \cdot h \cdot \sqrt{3} \cdot x^2}{2}$
$eq3: x = \frac{1}{3} \cdot \sqrt[3]{\frac{2 \cdot s}{\sqrt{3}}}$	$x = \frac{\sqrt[3]{\frac{3}{4} \cdot \sqrt{2} \cdot s}}{9}$
$eq4: x = \sqrt[3]{\frac{4 \cdot v}{9}}$	$x = \frac{\frac{2}{3} \cdot (3 \cdot v)^{\frac{1}{3}}}{3}$
$eq5: v = 9 \cdot s$	$v = 9 \cdot s$
$\text{solve}\left(\begin{cases} eq1 \\ eq2 \\ eq3, \{x, h, s, v\} \\ eq4 \\ eq5 \end{cases} \mid x > 0 \text{ and } h > 0 \text{ and } s > 0 \text{ and } v > 0\right)$	$x = 18 \cdot \sqrt{3} \text{ and } h = 27 \cdot \sqrt{3} \text{ and } s = 3 \cdot \sqrt{3} \text{ and } v = 27 \cdot \sqrt{3}$

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Question 4

a.i. $g(x) = \int_0^x f(t) dt - 2x$

$$g(-2) = \int_0^{-2} f(t) dt + 4$$

$$= 4 - \int_{-2}^0 f(t) dt$$

Now $\int_{-2}^0 f(t) dt$ is the area of a triangle of side lengths 2 and 4

$$= 4 - \frac{1}{2} \times 2 \times 4$$

$$= 0$$

A1

ii. $g(4) = \int_0^4 f(t) dt - 8$

Now $\int_0^4 f(t) dt$ represents one-quarter of the area of a circle of radius 4

$$= \frac{1}{4} \pi \times 4^2 - 8$$

$$= 4\pi - 8$$

A1

iii. $g(-3) = \int_0^{-3} f(t) dt + 6$

$$= 6 - \int_{-3}^0 f(t) dt = 6 - \int_{-3}^0 (2t + 4) dt$$

$$= 6 - \left[t^2 + 4t \right]_{-3}^0 = 6 - [0 - (9 - 12)]$$

$$= 3$$

A1

b. $g'(x) = f(x) - 2$

where $f(x) = \begin{cases} 2x + 4 & \text{for } -3 \leq x \leq 0 \\ \sqrt{16 - x^2} & \text{for } 0 \leq x \leq 4 \end{cases}$

M1

$$g'(x) = \begin{cases} 2x + 2 & \text{for } -3 \leq x \leq 0 \\ \sqrt{16 - x^2} - 2 & \text{for } 0 \leq x \leq 4 \end{cases}$$

A1

- c. The function $g(x)$ has a maximum or minimum turning point when

$$g'(x) = 0 \text{ or } f(x) = 2$$

$$(1) \text{ for } -3 \leq x \leq 0 \Rightarrow 2x + 2 = 0 \Rightarrow x = -1 \quad \text{M1}$$

$$\begin{aligned} g(-1) &= \int_0^{-1} f(t) dt + 2 = 2 - \int_{-1}^0 (2t + 4) dx = 2 - [t^2 + 4t]_{-1}^0 \\ &= 2 - [(0) - (1 - 4)] \\ &= -1 \end{aligned}$$

$$(2) \text{ for } 0 \leq x \leq 4 \Rightarrow \sqrt{16 - x^2} - 2 = 0$$

$$16 - x^2 = 4 \Rightarrow x^2 = 12 \Rightarrow x = 2\sqrt{3} \text{ since } 0 \leq x \leq 4 \quad \text{M1}$$

$$\begin{aligned} g(2\sqrt{3}) &= \int_0^{2\sqrt{3}} f(t) dt - 4\sqrt{3} = \int_0^{2\sqrt{3}} \sqrt{16 - t^2} dt - 4\sqrt{3} \\ &= \frac{8\pi}{3} + 2\sqrt{3} - 4\sqrt{3} = \frac{8\pi}{3} - 2\sqrt{3} \approx 4.9135 \end{aligned}$$

$$\left(2\sqrt{3}, \frac{8\pi}{3} - 2\sqrt{3}\right) \text{ is the maximum turning point} \quad \text{A1}$$

$$(-1, -1) \text{ is the minimum turning point} \quad \text{A1}$$

- d. The point $(0, 0)$ is the inflexion point A1

- e. Now g has a domain $[-3, 4]$

$$g(4) = 4\pi - 8 \approx 4.5664 \text{ and } g(-3) = 3 \text{ are endpoints.}$$

$$\text{The range is } \left[-1, \frac{8\pi}{3} - 2\sqrt{3}\right], \text{ crosses } x\text{-axis at } x = -2 \text{ and } x = 0, (-2, 0) (0, 0)$$

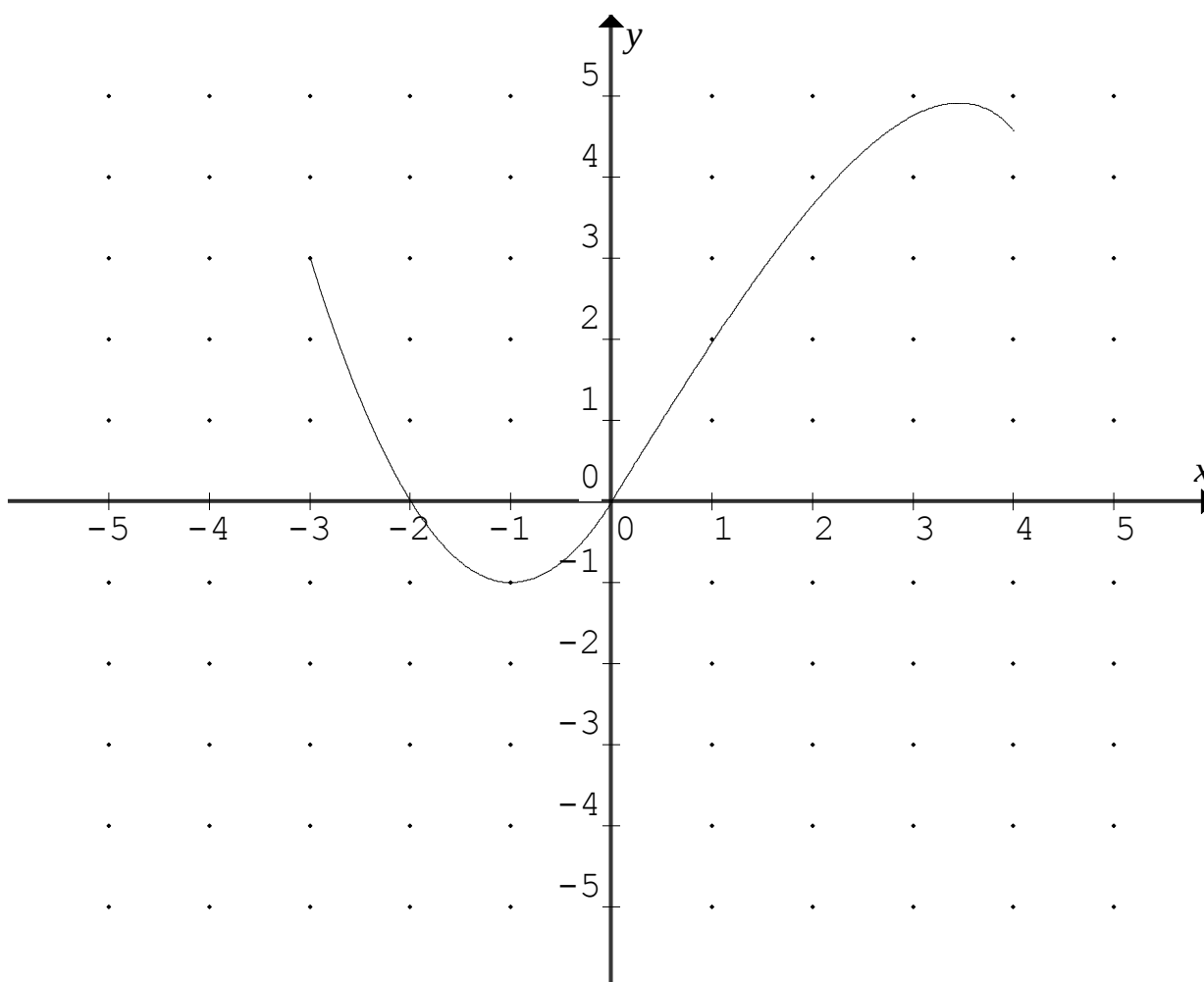
the point $(0, 0)$ is the inflexion point

correct graph over correct domain and range,

all stationary points and intercepts

G1

G1



Define $f(x) = \begin{cases} 2x+4, & -3 \leq x \leq 0 \\ \sqrt{16-x^2}, & 0 \leq x \leq 4 \end{cases}$	Done
Define $g(x) = \int_0^x f(t) dt - 2 \cdot x$	Done
$g(-2)$	0
$g(4)$	$4 \cdot \pi - 8$
$g(-3)$	3
$g(-1)$	-1
$g(2 \cdot \sqrt{3})$	$\frac{8 \cdot \pi}{3} - 2 \cdot \sqrt{3}$
$g(0)$	0

END OF SECTION 2 SUGGESTED ANSWERS