

Year 2013

VCE Mathematical

Methods CAS

Trial Examination 1

Suggested Solutions



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Question 1

- a. If $f(x) = \log_e(\cos(3x))$ using the chain rule

$$y = \log_e(u) \quad \text{where } u = \cos(3x)$$

$$\frac{dy}{du} = \frac{1}{u} \quad \frac{du}{dx} = -3\sin(3x)$$

$$f'(x) = \frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} = \frac{-3\sin(3x)}{\cos(3x)} = -3\tan(3x) \quad \text{M1}$$

$$f'\left(\frac{\pi}{18}\right) = -3\tan\left(\frac{\pi}{6}\right) = -3 \times \frac{\sqrt{3}}{3}$$

$$f'\left(\frac{\pi}{18}\right) = -\sqrt{3} \quad \text{A1}$$

- b. If $y = \frac{\sin(4x)}{2x^2}$ using the quotient rule

$$u = \sin(4x) \quad v = 2x^2$$

$$\frac{du}{dx} = 4\cos(4x) \quad \frac{dv}{dx} = 4x \quad \text{M1}$$

$$\frac{dy}{dx} = \frac{8x^2 \cos(4x) - 4x \sin(4x)}{(2x^2)^2}$$

$$\frac{dy}{dx} = \frac{4x(2x \cos(4x) - \sin(4x))}{4x^4} = \frac{2x \cos(4x) - \sin(4x)}{x^3} = \frac{g(x)}{x^3}$$

$$g(x) = 2x \cos(4x) - \sin(4x) \quad \text{A1}$$

Question 2

$$kx - 4y = 2$$

$$3x - (k+4)y = k+1$$

$$\Delta = \begin{vmatrix} k & -4 \\ 3 & -(k+4) \end{vmatrix} = -k(k+4) + 12 = -k^2 - 4k + 12 \quad \text{M1}$$

$$\Delta = -(k^2 + 4k - 12) = -(k+6)(k-2)$$

- i. There is a unique solution when $k \in \mathbb{R} \setminus \{2, -6\}$ A1

When $k = 2$ the equations become $\begin{matrix} 2x - 4y = 2 \\ 3x - 6y = 3 \end{matrix}$ these lines are both the same

line as $x - 2y = 1$, therefore we have an infinite number of solutions when $k = 2$

- ii. When $k = -6$ the equations become $\begin{matrix} -6x - 4y = 2 \\ 3x + 2y = -5 \end{matrix}$ these lines are parallel

with different y-intercepts, therefore there is no solution when $k = -6$ A1

Question 3

$$f'(x) = \frac{dy}{dx} = \frac{2}{\sqrt{4x+9}}$$

$$y = f(x) = \int \frac{2}{\sqrt{4x+9}} dx = \int 2 \times (4x+9)^{-\frac{1}{2}} dx \quad \text{A1}$$

$$y = f(x) = \frac{2}{4} \times 2 \times (4x+9)^{\frac{1}{2}} + c = \sqrt{4x+9} + c \quad \text{M1}$$

$$f(0) = 0 \Rightarrow 0 = \sqrt{9} + c \Rightarrow c = -3$$

$$y = f(x) = \sqrt{4x+9} - 3 \quad \text{A1}$$

Question 4

a. $f: \mathbb{R} \rightarrow \mathbb{R}, f(x) = 1 - 2\cos\left(\frac{\pi x}{6}\right)$

crosses the x-axis, when $y = 0$, finding the general solution

$$1 - 2\cos\left(\frac{\pi x}{6}\right) = 0$$

$$2\cos\left(\frac{\pi x}{6}\right) = 1$$

$$\cos\left(\frac{\pi x}{6}\right) = \frac{1}{2} \quad \text{M1}$$

$$\frac{\pi x}{6} = 2n\pi \pm \cos^{-1}\left(\frac{1}{2}\right) = 2n\pi \pm \frac{\pi}{3}$$

$$x = 12n \pm 2, \text{ where } n \in \mathbb{Z} \quad \text{A1}$$

b. $g: [0, 12] \rightarrow \mathbb{R}, g(x) = 1 - 2\cos\left(\frac{\pi x}{6}\right)$

amplitude is 2, period $T = \frac{2\pi}{\frac{\pi}{6}} = 12$ and the range is $[-1, 3]$ A1

crosses the x-axis at $x = 2$ and $x = 10$ $n = 0$ and $n = 1$ from a.

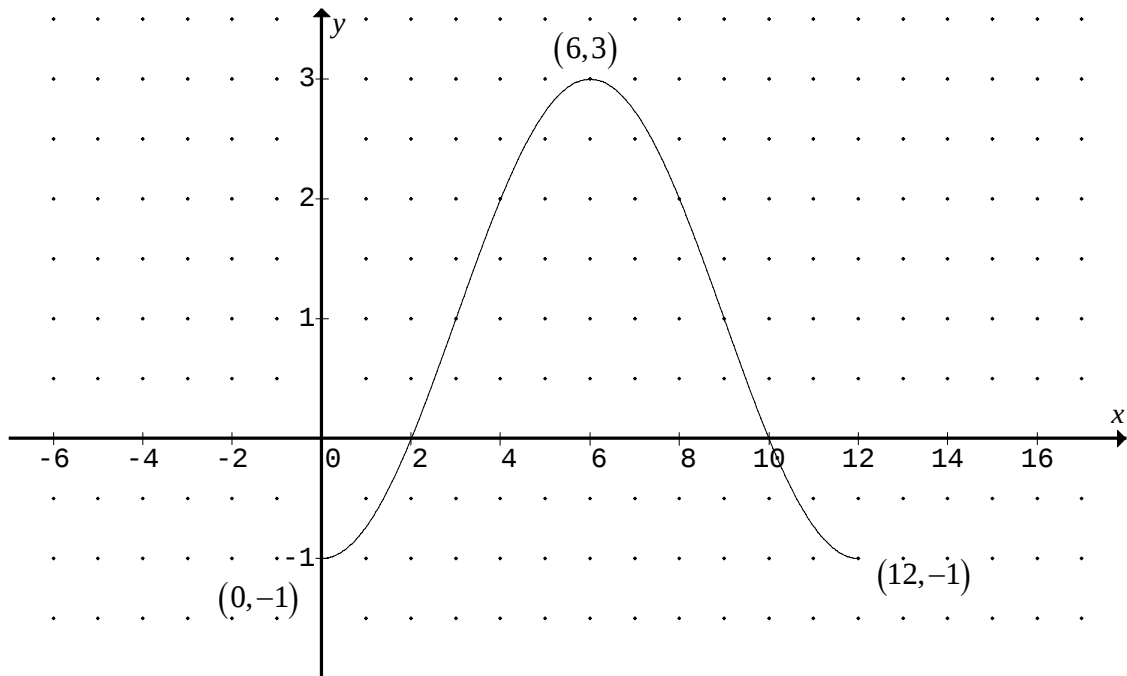
end-points $f(0) = -1$ $f(12) = -1$ $(0, -1)$ $(12, -1)$

maximum, when $y = 3$ when $\cos\left(\frac{\pi x}{6}\right) = -1$

$$\frac{\pi x}{6} = \pi \text{ so } x = 6 \quad (6, 3) \quad \text{A1}$$

correct graph, on restricted domain, end-points, shape.

G1



Question 5

$$\log_x 5 + \log_5 x = \frac{5}{2} \quad \text{to solve let } u = \log_x 5 \text{ then } \log_5 x = \frac{1}{\log_x 5} = \frac{1}{u}$$

$$u + \frac{1}{u} = \frac{5}{2} \quad \text{multiply both sides by } 2u$$

$$2u^2 + 2 = 5u$$

M1

$$2u^2 - 5u + 2 = 0$$

$$(2u - 1)(u - 2) = 0$$

$$u = \log_x 5 = \frac{1}{2}, 2$$

M1

$$\log_x 5 = \frac{1}{2} \quad \log_x 5 = 2$$

$$\text{in index form } x^{\frac{1}{2}} = \sqrt{x} = 5 \quad x^2 = 5 \text{ since } x > 0$$

$$x = 25, \sqrt{5}$$

A1

Question 6

$$f : y = 3e^{-2x} - 4 \quad \text{interchange } x \text{ and } y$$

$$f^{-1} : x = 3e^{-2y} - 4 \quad \text{re-arrange to make } y \text{ the subject}$$

$$3e^{-2y} = x + 4 \quad \text{M1}$$

$$e^{-2y} = \frac{x+4}{3} \quad \Rightarrow \quad -2y = \log_e \left(\frac{x+4}{3} \right)$$

$$y = -\frac{1}{2} \log_e \left(\frac{x+4}{3} \right) \quad \text{or} \quad y = \frac{1}{2} \log_e \left(\frac{3}{x+4} \right) \quad \text{A1}$$

$$\text{but domain } f = \text{range } f^{-1} = R \quad \text{and} \quad \text{range } f = \text{domain } f^{-1} = (-4, \infty)$$

we must state the maximal domain of the inverse function

$$f^{-1} : (-4, \infty) \rightarrow R, \quad f^{-1}(x) = -\frac{1}{2} \log_e \left(\frac{x+4}{3} \right) \quad \text{A1}$$

Question 7

$$\text{a.} \quad y = 3 - \frac{12}{(x+2)^2} \quad \text{when } x=0 \quad y = 3 - \frac{12}{2^2} = 0$$

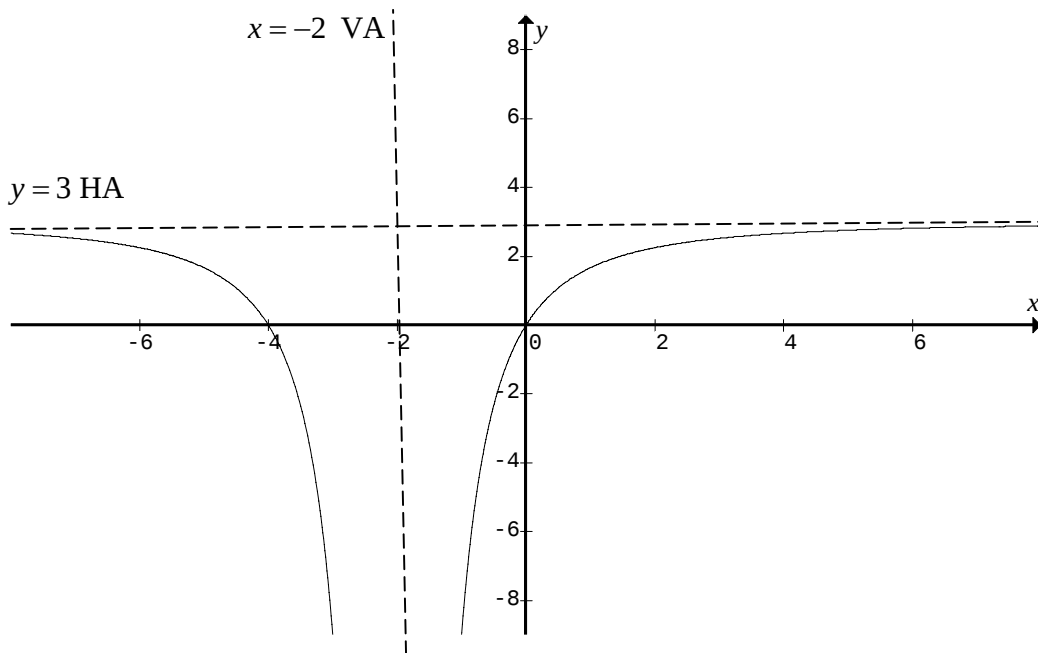
$$\text{crosses the } x\text{-axis when } y=0 \Rightarrow 3 - \frac{12}{(x+2)^2} = 0 \Rightarrow (x+2)^2 = 4$$

$$x+2 = \pm 2 \Rightarrow x=0 \quad \text{and} \quad x=-4 \quad (0,0) \quad (-4,0) \quad \text{A1}$$

$x = -2$ is a vertical asymptote and $y = 3$ is a horizontal asymptote

$$\text{domain } R \setminus \{-2\} \quad \text{range } (-\infty, 3) \quad \text{A1}$$

correct graph, shape asymptotes, correct axial intercepts G1



b. $y = 3 - \frac{12}{(x+2)^2}$ from the graph of $y = \frac{1}{x^2}$

$\frac{1}{2}$ mark for each correct transformation, the translations must come last.

- reflect in the x -axis $y = -\frac{1}{x^2}$
- dilate by a factor of 12 parallel to the y -axis (or away from the x -axis) $y = -\frac{12}{x^2}$
- translate 2 units to the left parallel to the x -axis (or away from the y -axis) $y = -\frac{12}{(x+2)^2}$
- translate 3 units up parallel to the y -axis (or away from the x -axis) $y = 3 - \frac{12}{(x+2)^2}$

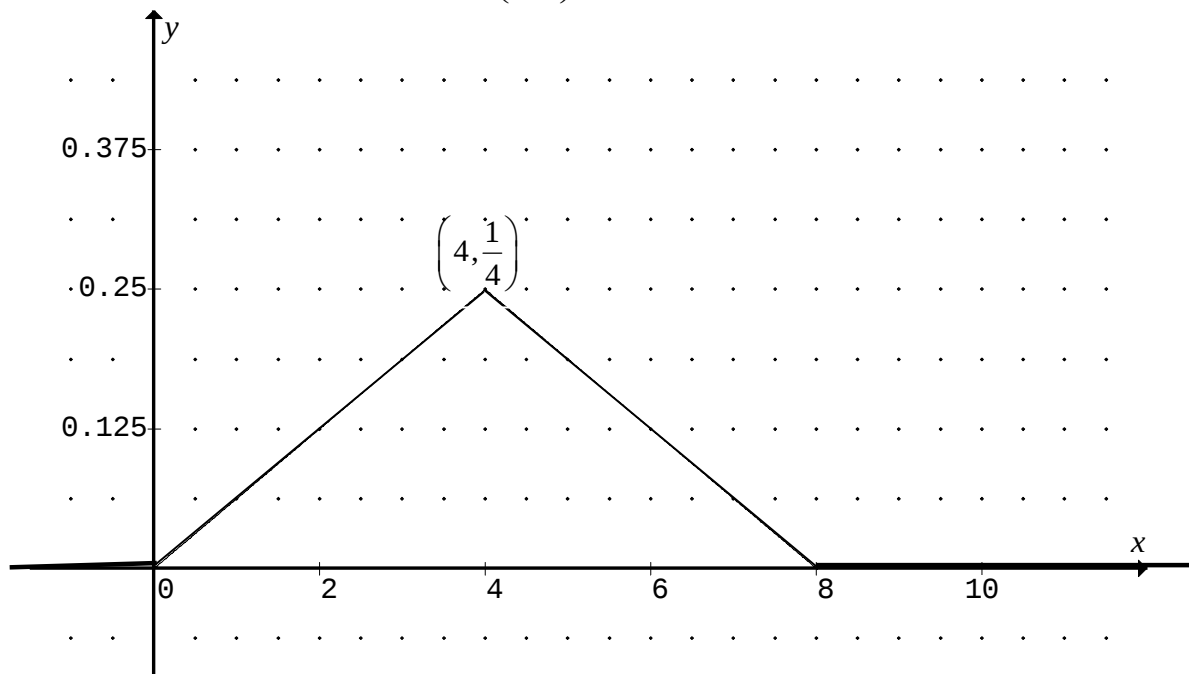
Question 8

a. $f(x) = \begin{cases} \frac{1}{16}(4 - |x-4|) & \text{for } 0 \leq x \leq 8 \\ 0 & \text{elsewhere} \end{cases}$

$f(0) = f(8) = 0$ endpoints $(0,0)$ $(8,0)$

correct graph, shape, point at $\left(4, \frac{1}{4}\right)$, zero elsewhere

G2



$$\begin{aligned}
 \text{b.} \quad \Pr(X < 2 \mid X < 6) &= \frac{\Pr(X < 2)}{\Pr(X < 6)} \\
 \Pr(X < 2 \mid X < 6) &= \frac{\frac{1}{2} \times 2 \times \frac{1}{8}}{1 - \frac{1}{2} \times 2 \times \frac{1}{8}} = \frac{\frac{1}{8}}{\frac{7}{8}} \quad \text{by area of triangles} \\
 \Pr(X < 2 \mid X < 6) &= \frac{1}{7} \quad \text{A1}
 \end{aligned}$$

Question 9

$$\begin{aligned}
 \Pr(A) &= \frac{1}{3} \quad \text{and} \quad \Pr(B) = \frac{2}{5} \\
 \text{Since } A \text{ and } B \text{ are independent, then } A' \text{ and } B' \text{ are also independent} \\
 \Pr(A' \cap B') &= \Pr(A')\Pr(B') = \frac{2}{3} \times \frac{3}{5} = \frac{2}{5} \quad \text{M1} \\
 \Pr(A' \cup B') &= \Pr(A') + \Pr(B') - \Pr(A' \cap B') \\
 &= \frac{2}{3} + \frac{3}{5} - \frac{2}{5} = \frac{10 + 9 - 6}{15} \\
 &= \frac{13}{15} \quad \text{A1}
 \end{aligned}$$

Question 10

$$\begin{aligned}
 \text{a.} \quad f(x) &= xe^{-2x} \quad \text{product rule} \\
 u &= x \quad v = e^{-2x} \\
 \frac{du}{dx} &= 1 \quad \frac{dv}{dx} = -2e^{-2x} \\
 f'(x) &= e^{-2x} - 2xe^{-2x} \quad \text{does not need to be simplified} \quad \text{A1}
 \end{aligned}$$

$$\begin{aligned}
 \text{b.} \quad &\text{for a stationary point } f'(x) = 0 \\
 &f'(x) = e^{-2x}(1 - 2x) = 0 \quad \text{A1} \\
 &\text{since } e^{-2x} \neq 0 \Rightarrow 1 - 2x = 0 \Rightarrow x = \frac{1}{2} \quad \text{and} \quad f\left(\frac{1}{2}\right) = \frac{1}{2}e^{-1} \\
 &\text{stationary point is a maximum at } \left(\frac{1}{2}, \frac{1}{2e}\right) \quad \text{A1}
 \end{aligned}$$

c. from a. $\frac{d}{dx}(xe^{-2x}) = e^{-2x} - 2xe^{-2x}$

$$\int (e^{-2x} - 2xe^{-2x}) dx = xe^{-2x}$$

$$\int e^{-2x} dx - \int 2xe^{-2x} dx = xe^{-2x}$$

$$-\frac{1}{2} \int 2xe^{-2x} dx = xe^{-2x} - \int e^{-2x} dx = xe^{-2x} + \frac{1}{2} e^{-2x} \quad \text{M1}$$

$$\int xe^{-2x} dx = -\frac{1}{2} xe^{-2x} - \frac{1}{4} e^{-2x} = -\frac{1}{4} e^{-2x} (2x+1)$$

The required area is $A = \int_0^2 xe^{-2x} dx$ A1

$$A = -\left[\frac{1}{4} (2x+1) e^{-2x} \right]_0^2$$

$$A = \left(-\frac{5}{4} e^{-4} + \frac{1}{4} \right)$$

$$A = \frac{1}{4} (1 - 5e^{-4}) \quad \text{A1}$$

Question 11

Since the mode is 5, $\Pr(X=5) > \Pr(X=4) \Rightarrow \frac{\Pr(X=5)}{\Pr(X=4)} > 1$

substitute into $\frac{\Pr(X=k+1)}{\Pr(X=k)} = \frac{(n-k)p}{(k+1)(1-p)}$ with $k=4$ and $n=8$

$$\frac{4p}{5(1-p)} > 1 \Rightarrow 4p > 5 - 5p \Rightarrow 9p > 5 \Rightarrow p > \frac{5}{9} \quad \text{M1}$$

Also since the mode is 5, $\Pr(X=6) < \Pr(X=5) \Rightarrow \frac{\Pr(X=6)}{\Pr(X=5)} < 1$

substitute into $\frac{\Pr(X=k+1)}{\Pr(X=k)} = \frac{(n-k)p}{(k+1)(1-p)}$ with $k=5$ and $n=8$

$$\frac{3p}{6(1-p)} < 1 \Rightarrow p < 2(1-p) \Rightarrow p < 2 - 2p \quad \text{M1}$$

$$3p < 2 \Rightarrow p < \frac{2}{3}$$

so $\frac{5}{9} < p < \frac{2}{3}$ $p_1 = \frac{5}{9}$ and $p_2 = \frac{2}{3}$ A1

END OF SUGGESTED SOLUTIONS