INSIGHT

YEAR 12 *Trial Exam Paper*

2013

MATHEMATICAL METHODS (CAS)

Written examination 2

Worked solutions

This book presents:

- correct solutions with full working
- mark allocations
- tips

SECTION 1

Question 1

Answer is C

Worked solution

Period is $\frac{2\pi}{n}=\frac{2\pi}{2}=\pi$ and the amplitude is the coefficient of the sine term, which is 1.

Tip

- *Amplitude and period are never negative.*

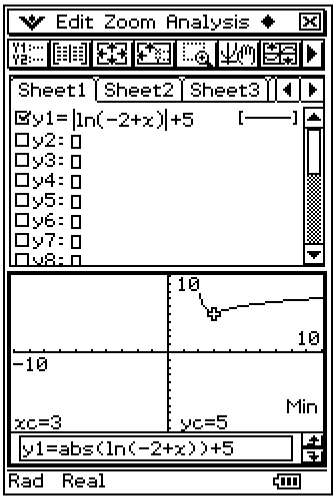
Question 2

Answer is D

Worked solution

Choose some sample values for *a* and *b*.

Using CAS, the graph of $f(x)=\left|\log_e(-2+x)\right|+5$ looks like



This shows the lowest point that the graph reaches is 5, so the range is $[5,\infty)$ or in the general sense $[b,\infty)$.

Question 3

Answer is D

Worked solution

The amplitude is $(-2-b)$, the period is 2 (so $2=\frac{2\pi}{n}$, so $n=\pi$), the graph has been reflected across the x-axis, and shifted down 2 units. These values give an equation of $y=-(-2-b)\sin(\pi x)-2$, which simplifies to give $y=(b+2)\sin(\pi x)-2$.

Question 4

Answer is E

Worked solution

The graph of $y=\tan(ax)$ has a period of $\frac{\pi}{a}$ and asymptotes at $x=\pm\frac{\pi}{2a}$.

So, in this case $\frac{\pi}{2a}=\frac{1}{12}$

$$12\pi=2a$$
$$a=6\pi$$

Question 5

Answer is B

Worked solution

$$z=3$$
$$x-z=-5$$
$$x+y=0$$

can be written as

$$0x+0y+1z=3$$
$$1x+0y-1z=-5$$
$$1x+1y+0z=0$$

Moving line 1 to the bottom gives

$$1x+0y-1z=-5$$
$$1x+1y+0z=0$$
$$0x+0y+1z=3$$

In matrix form, this is $\begin{bmatrix}1 & 0 & -1\\1 & 1 & 0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}-5\\0\\3\end{bmatrix}$.

Question 6

Answer is A

Worked solution

$y = e^{2x+4} - 3$ can be written as $y + 3 = e^{2x+4}$ and therefore $y = y' + 3$ and $x = 2x' + 4$.

So $y' = y - 3$ and $x' = \frac{1}{2}(x-4) = \frac{1}{2}x - 2$. The matrix equation that corresponds to these equations is **A**.

Question 7

Answer is E

Worked solution

Overall, the graph represents a negative polynomial of odd degree.

It has *x*-intercepts at $x = a,\ x = 0,\ x = b$. These give factors of $(x-a)$, x, and $(x-b)$. There is a stationary point of inflection at $x = a$, so the factor $(x-a)$ becomes $(x-a)^3$.

This means the equation of the graph could be $y = -x(x-a)^3(x-b)$ and a version of this is **E**.

Question 8

Answer is D

Worked solution

Reflect the graph in the line $y = x$ to get the graph of the inverse and then reflect in the *y*-axis to get the graph of $y = f^{-1}(-x)$.

Question 9

Answer is D

Worked solution

The period has been halved, therefore there is a dilation of factor $\frac{1}{2}$ from the *y*-axis. The graph has been reflected in the *x*-axis.

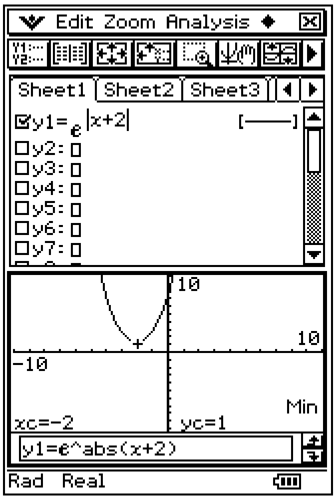
Question 10

Answer is E

Worked solution

Let b be any value, say $b = 2$.

Using CAS, the graph of $y = e^{|x+2|}$ looks like



The graph is one to one for the domain $[-b, \infty)$.

Question 11

Answer is D

Worked solution

$g(f(x)) = \dfrac{1}{\sqrt{2e^x}}$ and with $x = 0$, $g(f(x)) = \dfrac{1}{\sqrt{2e^0}} = \dfrac{1}{\sqrt{2}}$

Question 12

Answer is D

Worked solution

Using the chain rule

$$\frac{dy}{dx} = \frac{1}{\sqrt{f(x)}} \times \frac{1}{2\sqrt{f(x)}} \times f'(x)$$

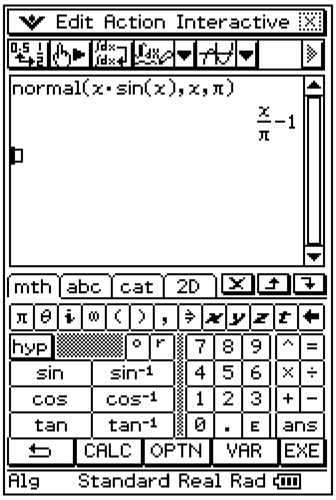
$$= \frac{f'(x)}{2f(x)}$$

Question 13

Answer is D

Worked solution

Using CAS



So the equation of the normal is $y = \dfrac{x}{\pi} - 1$. Rearranging, this gives $(y+1)\pi = x$.

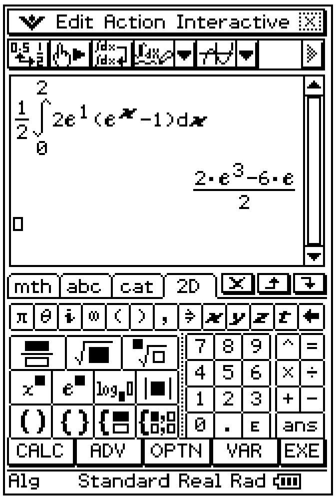
Question 14

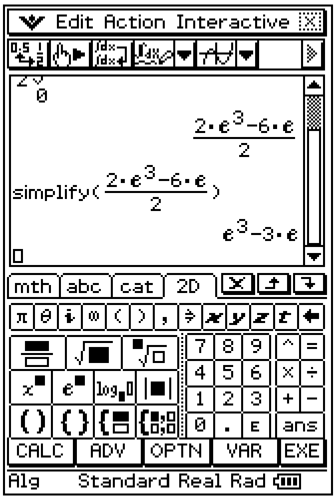
Answer is D

Worked solution

The average value of a function is calculated as $\dfrac{1}{b-a}\int_a^b f(x)\,dx = \dfrac{1}{2-0}\int_0^2 2e(e^x - 1)\ dx$.

Using CAS, this gives





Question 15

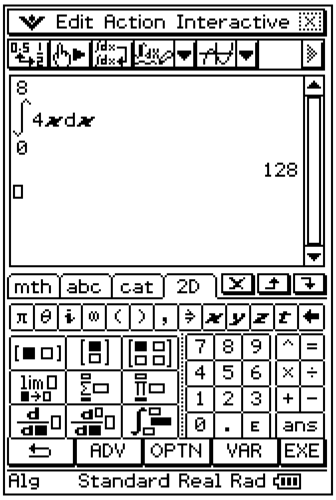
Answer is C

Worked solution

This question represents the fundamental theorem of integral calculus with

$$\lim_{\delta x \to 0} \sum_{i=1}^{n} (4x_i\,\delta x) = \int_0^8 4x\ dx$$

Using CAS, this is



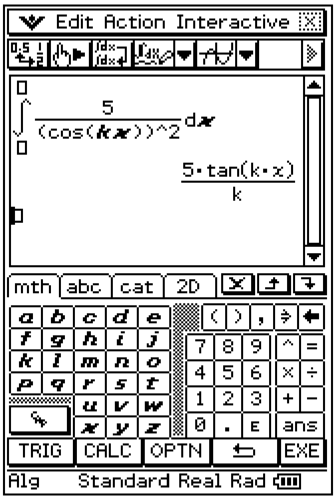
Question 16

Answer is D

Worked solution

For $f'(x) = \dfrac{5}{\cos^2(kx)}$, $f(x) = \displaystyle\int \frac{5}{\cos^2(kx)}\,dx$

Using CAS, this gives



Question 17

Answer is C

Worked solution

As $q'(x) = p(x)$**,** this means that $q(x) = \int p(x)\ dx$.

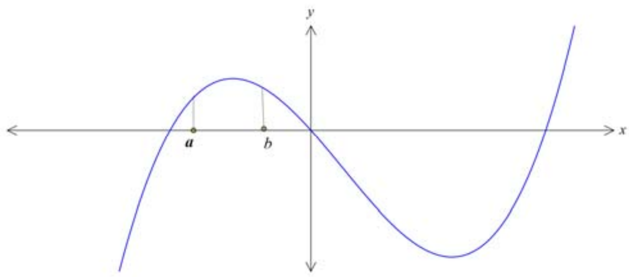
So $\displaystyle\int_a^b p(2x)\,dx = \left[\frac{1}{2}q(2x)\right]_a^b = \frac{1}{2}\big(q(2b) - q(2a)\big)$**.**

Question 18

Answer is D

Worked solution

The graph of $y = -2f(x)$ will give the gradient function of g.
A quick sketch of $y = -2f(x)$ gives



Over interval $[a, b]$ the y-values are positive, therefore the gradient values are positive.

Question 19

Answer is B

Worked solution

Let the number of defectives in the box be X.

$X \sim Bi(n, p)$

We are given $E(X) = 12 = np$ and $\mathrm{Var}(X) = 10 = npq$.

So, $12q = 10$, $q = \frac{5}{6}$ and $p = \frac{1}{6}$.

The chance of not being defective is given by q and $q = \frac{5}{6}$.

Question 20

Answer is D

Worked solution

Only in a symmetrical distribution is the probability of being less than the mean equal to 0.5.

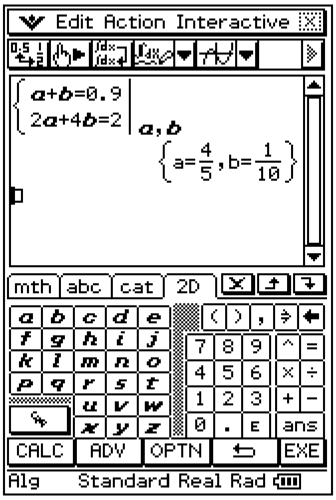
Question 21

Answer is B

Worked solution

$\sum p(x) = a + b + 0.1 = 1$ and $E(X) = 2a + 4b + 0.6 = 2.6$

Solving the simultaneous equations gives



Question 22

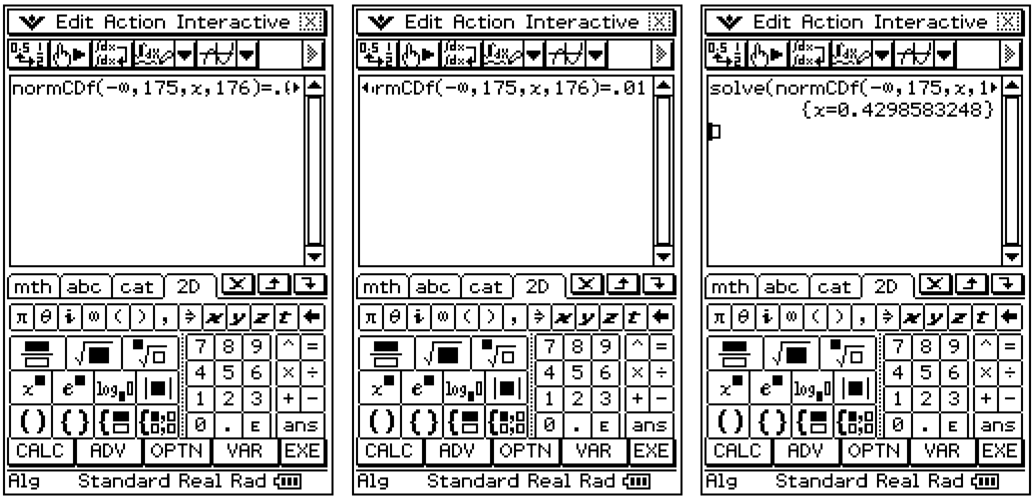
Answer is C

Worked solution

$X \sim N(\mu = 176,\ \sigma \text{ unknown})$

$\Pr(X < 175) = 0.01$

This can be solved using CAS.



SECTION 2

Instructions for Section 2

Answer **all** questions in the spaces provided.
A decimal answer will not be accepted if an **exact** answer is required to a question.
For questions where more than one mark is available, appropriate working must be shown.
Unless otherwise stated, diagrams are not drawn to scale.

Question 1a.

Worked solution

The information given, together with the transition matrix, suggests

$$\begin{bmatrix} \frac{3}{5} & \frac{1}{3} \\ \frac{2}{5} & \frac{2}{3} \end{bmatrix} = \begin{bmatrix} \Pr(L_{i+1} \mid L_i) & \Pr(L_{i+1} \mid R_i) \\ \Pr(R_{i+1} \mid L_i) & \Pr(R_{i+1} \mid R_i) \end{bmatrix},$$

where L_i is the event of buying lilies one week and R_i is the event of buying roses one week.

So the chance that Rebecca buys lilies one week, given that she bought roses the previous week is $\frac{1}{3}$.

Mark allocation

- 1 mark for the correct answer.

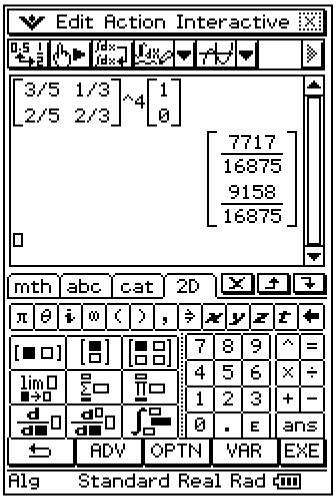
Question 1b.

Worked solution

i. The fifth Saturday means that there will be four transitions. The probabilities will be

$$\begin{bmatrix} \frac{3}{5} & \frac{1}{3} \\ \frac{2}{5} & \frac{2}{3} \end{bmatrix}^4 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Using CAS, this gives



So, in 4 weeks' time, the probability that Rebecca buys lilies is $\frac{7717}{16\,875}$.

Mark allocation: 1 + 1 = 2 marks

- 1 mark for setting the transition matrix to the power of 4.
- 1 mark for the correct answer.

Tip

- *This needs to be the exact answer. A decimal approximation, regardless of the number of decimal places, is not acceptable.*

Question 1b.

Worked solution

ii. This means that Rebecca buys lilies the first week, then roses for the next 3 weeks, and, finally, lilies the last week.

i.e. $LRRRL = 1\times\frac{2}{5}\times\frac{2}{3}\times\frac{2}{3}\times\frac{1}{3} = \frac{8}{135}$

Mark allocation

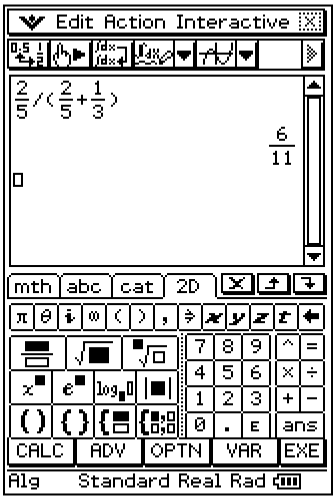
- 1 mark for the correct answer. (**Note:** Answer must be exact.)

Question 1c.

Worked solution

This can be calculated using the probabilities shown on the diagonals

$\begin{bmatrix} \frac{3}{5} & \frac{1}{3} \\ \frac{2}{5} & \frac{2}{3} \end{bmatrix}$. So the long term probability of buying roses is $\dfrac{\frac{2}{5}}{\frac{2}{5}+\frac{1}{3}} = \dfrac{6}{11}$



Mark allocation: 1 + 1 = 2 marks

- 1 mark for method.
- 1 mark for the correct answer.

Tip

- *An alternative method is to raise the power of the transition matrix to a sufficiently large power and then evaluate.*

Question 1d.

Worked solution

As the function is symmetrical around $x = 6,$ the mean, median (and mode) are concurrent and occur at $x = 6.$

Mark allocation: 1 + 1 = 2 marks

- 1 mark for the mean.
- 1 mark for the median.

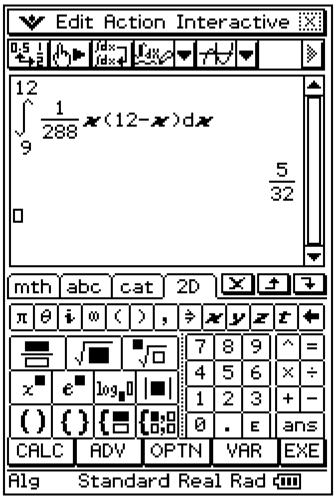
Question 1e.

Worked solution

This is a conditional probability question; i.e. $\Pr(T_R > 9 \mid T_R > 6)$.

$$\Pr(T_R > 9 \mid T_R > 6) = \frac{\Pr(T_R > 9)}{\Pr(T_R > 6)}$$

The probability of lasting longer than 9 days = $\int_9^{12} \frac{1}{288} t(12-t)\, dt = \frac{5}{32}$.



$$\Pr(T_R > 9 \mid T_R > 6) = \frac{\frac{5}{32}}{0.5} = \frac{5}{16}$$

Mark allocation: 1 + 1 = 2 marks

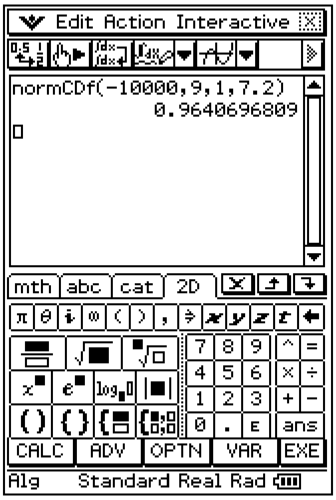
- 1 mark for finding $\Pr(T_R \geq 9) = \frac{5}{32}$.
- 1 mark for the correct answer.

Question 1f.

Worked solution

For roses, $\Pr(T_R \le 9) = \dfrac{27}{32} = 0.8438,$ correct to 4 decimal places.

For lilies, $\Pr(T_L \le 9) = 0.9641$, correct to 4 decimal places.



Mark allocation: 1 + 1 = 2 marks

- 1 mark for each correct answer.

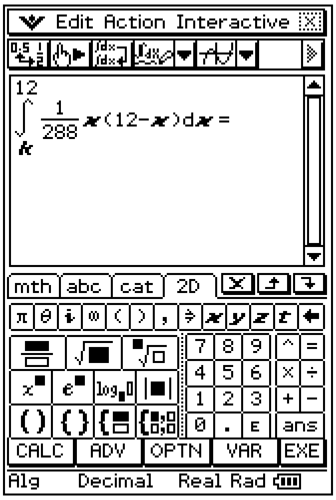
Tip

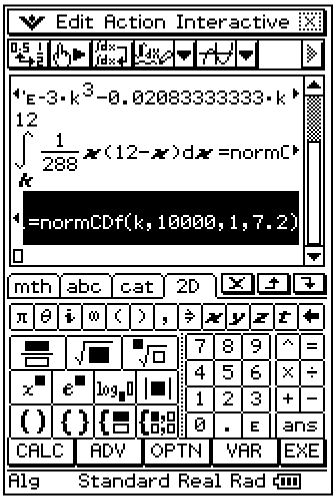
- *Answers must be in decimal form for this question;* $\dfrac{27}{32}$ *is not acceptable.*

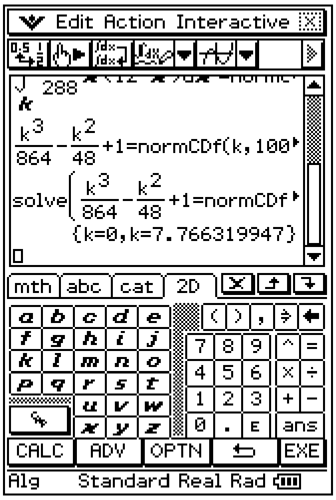
Question 1g.

Worked solution

For the probabilities to be equal, $\int_k^{12} \frac{1}{288} t(12-t)\ dt = \text{normCDf}(k, 100\,000,\ \sigma = 1,\ \mu = 7.2)$.

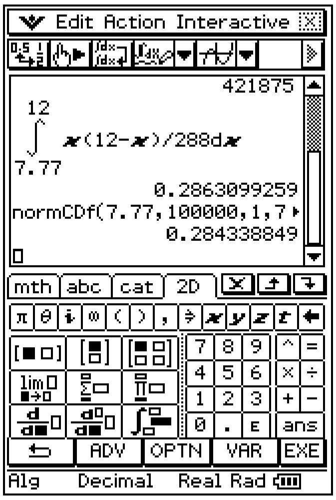




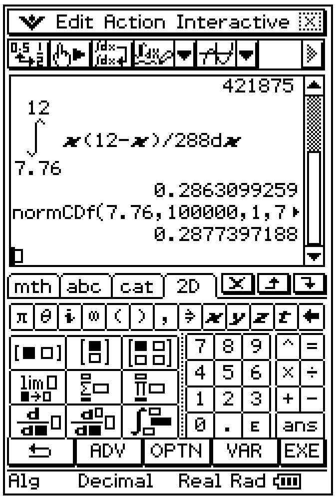


Check whether to let k = 7.77 or 7.76.

If $k = 7.77$ then correct to 2 decimal places the integral probability is 0.29, whereas the normal distribution probability is 0.28 (so the two are not equal at the 2 decimal places level).



However if $k = 7.76$, the integral probability gives 0.29 and the normal distribution probability gives 0.29 (correct to 2 decimal places) and so they are equal.



So, k = 7.76.

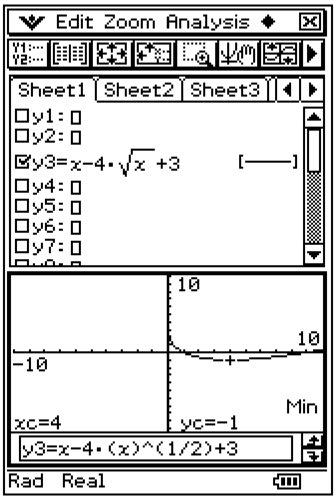
Mark allocation: 1 + 1 = 2 marks

- 1 mark for method.
- 1 mark for the correct answer.

Question 2a.

Worked solution

f is strictly increasing for $x: f'(x) \geq 0$. The minimum turning point occurs at $(4, -1)$.



So f is strictly increasing for $[4, \infty)$.

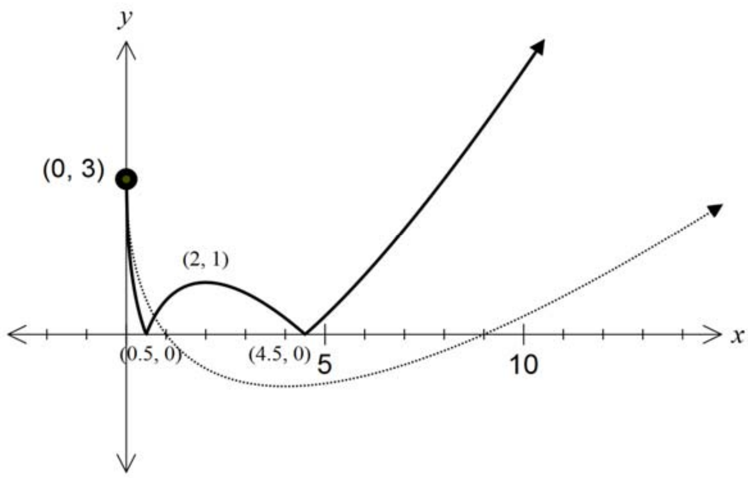
Mark allocation: 1 + 1 = 2 marks

- 1 mark for finding the turning point.
- 1 mark for the correct answer.

Note: Must have square bracket at 4.

Question 2b.

Worked solution



Mark allocation: 1 + 1 = 2 marks

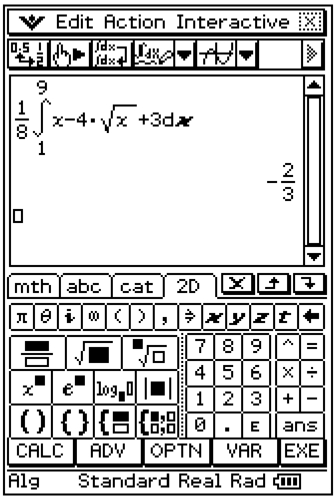
- 1 mark for the correct shape.
- 1 mark for co-ordinates labelled correctly.

Question 2c.

Worked solution

Let the width of the rectangle be w, so $\left|\int_1^9 x - 4\sqrt{x} + 3\ dx\right| = w \times (9-1)$.

$w = \frac{1}{8}\left|\int_1^9 (x - 4\sqrt{x} + 3)\ dx\right| = \frac{2}{3}$



So $w = \frac{2}{3}$, therefore D has the co-ordinates of $(1, -\frac{2}{3})$.

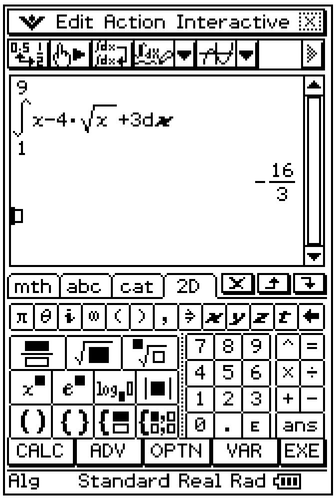
Mark allocation: 1 + 1 = 2 marks

- 1 mark for method.
- 1 mark for the correct answer.

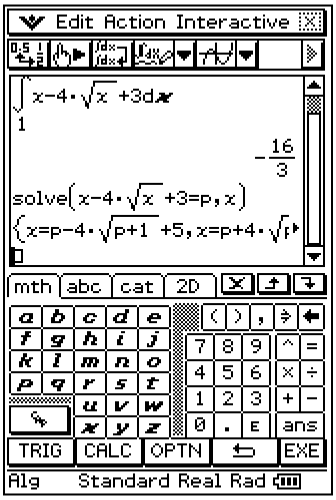
Question 2d.

Worked solution

The shaded area is $\dfrac{16}{3}$, therefore half the area is $\dfrac{8}{3}$.

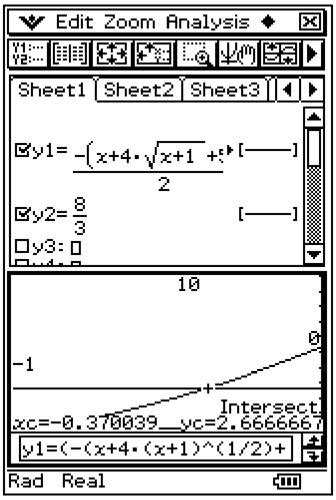


The graph of $y = f(x)$ intersects the line $y = p$ at
$x = p - 4\sqrt{p+1} + 5$ and $x = p + 4\sqrt{p+1} + 5$.



So the integral is $\int_{p-4\sqrt{p+1}+5}^{p+4\sqrt{p+1}+5} (p-(x-4\sqrt{x}+3))\,dx = \dfrac{8}{3}$ and we need to find p.

Using the graph screen of CAS, this is found to be $p = -0.370$.



Mark allocation: 1 + 1 + 1 = 3 marks

- 1 mark for stating the area as $\frac{16}{6} = \frac{8}{3}$.
- 1 mark for determining the points of intersection of the line $y = p$ with the graph $y = f(x)$.
- 1 mark for finding the value of *p*.

Question 2e.

Worked solution

i. $f(g(x)) = x^2 - 4\sqrt{x^2} + 3$

$= x^2 - 4|x| + 3$

Mark allocation: 1 + 1 = 2 marks

- 1 mark for method.
- 1 mark for the correct answer.

Note: Must show a working step and not just the answer.

Question 2e.

Worked solution

ii. $f(g(x)) = x^2 - 4\sqrt{x^2} + 3$

$= x^2 - 4|x| + 3$

$= \begin{cases} x^2 - 4x + 3 & x \geq 0 \\ x^2 + 4x + 3, & x < 0 \end{cases}$

So $\frac{d}{dx}(f(g(x)) = \begin{cases} 2x - 4, & x > 0 \\ 2x + 4, & x < 0 \end{cases}$

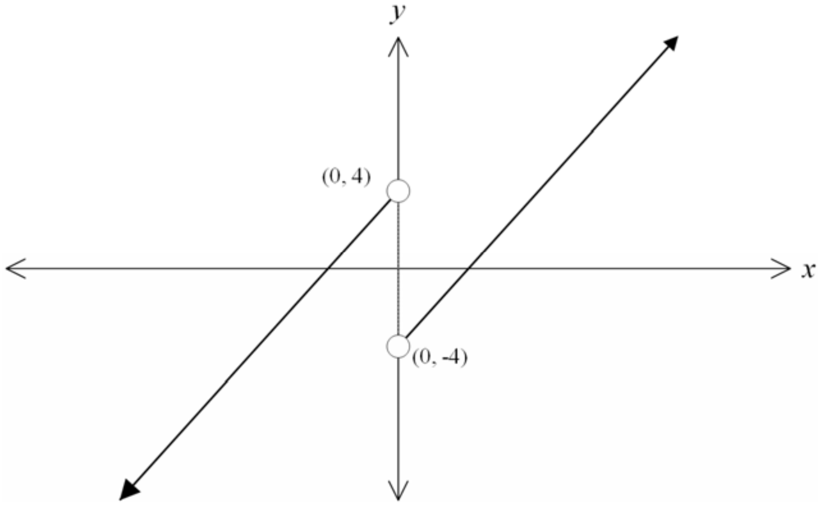
Mark allocation: 1 + 1 = 2 marks

- 1 mark for setting up the hybrid function of $f(g(x))$.
- 1 mark for the correct derivative with domains specified.

Question 2e.

Worked solution

iii.



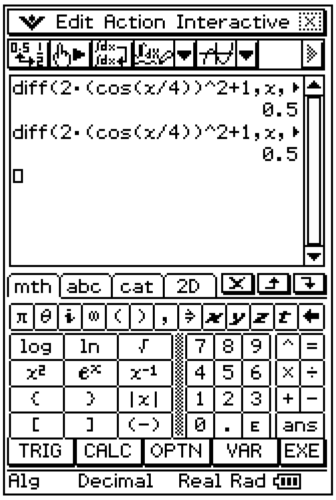
Mark allocation: 1 + 1 + 1 = 3 marks

- 1 mark for $y = 2x + 4,\ x < 0$ section.
- 1 mark for $y = 2x - 4,\ x > 0$ section.
- 1 mark for open circles and correctly labelled intercepts.

Question 3a.

Worked solution

Using CAS, $f'(\pi) = 0.5$ and $f'(3\pi) = 0.5$



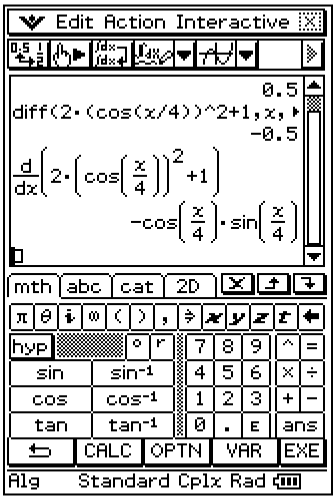
Mark allocation: 1 + 1 = 2 marks

- 1 mark for each correct answer.

Question 3b.

Worked solution

i. Using CAS gives $\frac{d}{dx}(2\cos^2\left(\frac{x}{4}\right)+1) = -\cos(\frac{x}{4})\sin(\frac{x}{4})$.



Mark allocation

- 1 mark for the correct answer.

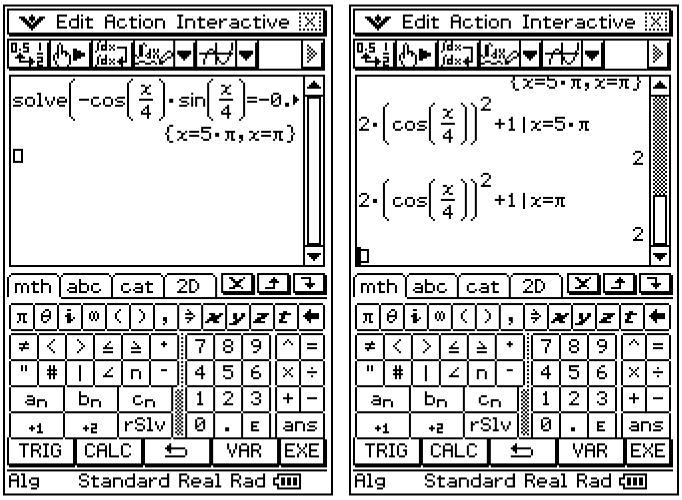
Question 3b.

Worked solution

ii. Gradient of normal is 2, therefore gradient of tangent is $-\frac{1}{2}$.

$$-\cos\left(\frac{x}{4}\right)\sin\left(\frac{x}{4}\right)=-\frac{1}{2}$$

Using CAS to solve over the domain $[-\pi,\ 6\pi]$ gives



So there are *two* points on the curve where the gradient of the normal is 2;
i.e. at $(\pi,\ 2)$ and $(5\pi,\ 2)$.

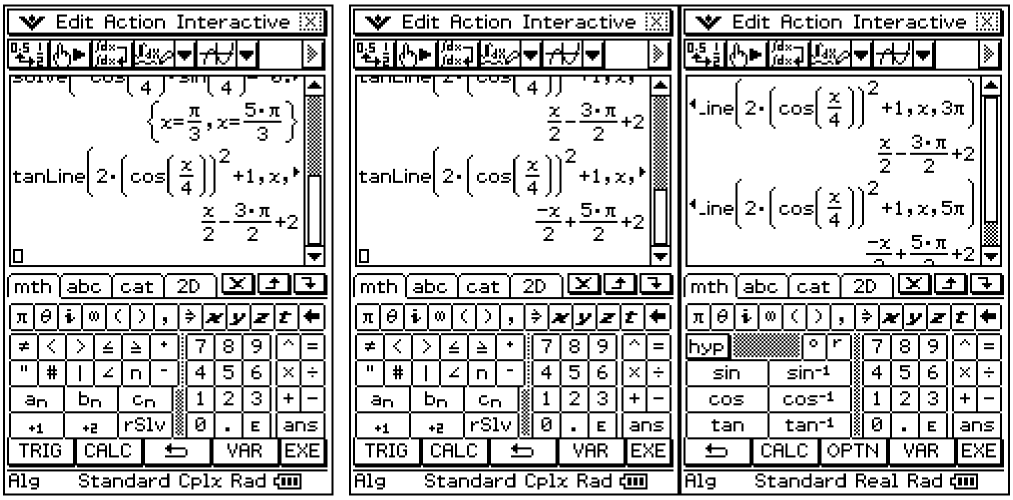
Mark allocation: 1 + 1 = 2 marks

- 1 mark for gradient of the tangent equals $-\frac{1}{2}$.
- 1 mark for giving answer as co-ordinates.

Question 3c.

Worked solution

Using CAS gives



$\therefore \quad y = \frac{x}{2} - \frac{3\pi}{2} + 2$ and $y = -\frac{x}{2} + \frac{5\pi}{2} + 2$

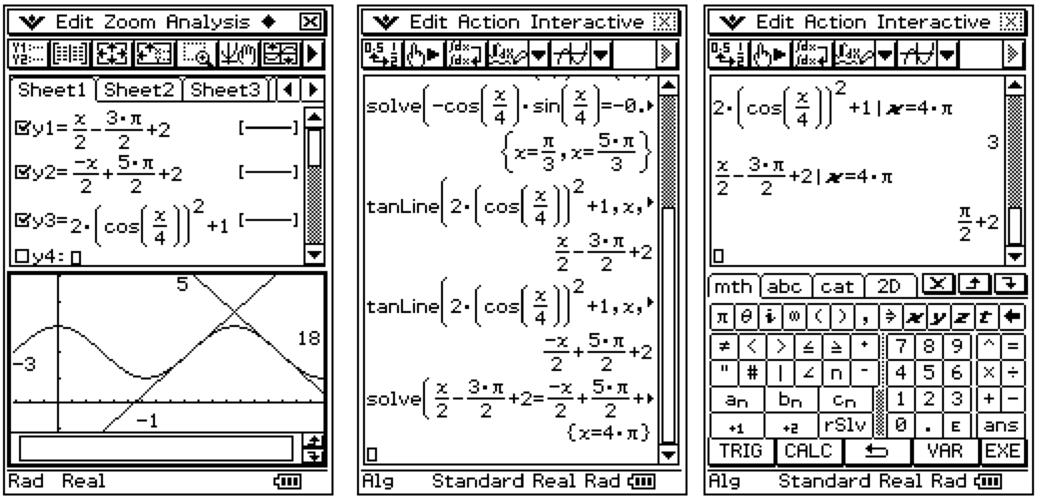
Mark allocation: 1 + 1 = 2 marks

- 1 mark for each correct answer.

Question 3d.

Worked solution

The tangent lines intersect at $x = 4\pi,\ \ y = \dfrac{\pi}{2} + 2$.



So if the graph of $y = f(x)$ is shifted down by $\dfrac{\pi}{2} + 2$ units, then the tangents will intersect at the x-axis, therefore $b = -\dfrac{\pi}{2} - 2$.

Mark allocation: 1 + 1 = 2 marks

- 1 mark for finding the point of intersection.
- 1 mark for the correct answer.

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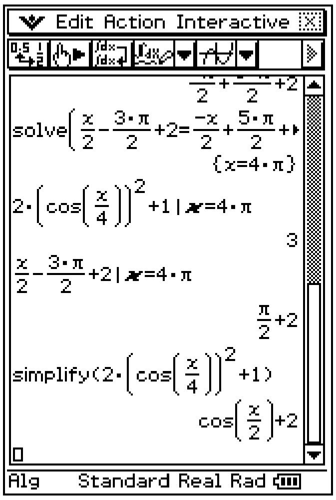
Question 3e.

Worked solution

Using CAS to simplify gives $2\cos^2\left(\frac{x}{4}\right)+1=\cos\left(\frac{x}{2}\right)+2$.

So $a=\frac{1}{2}$, $b=2$.



Period $=\frac{2\pi}{n}=\frac{2\pi}{\frac{1}{2}}=4\pi$

Mark allocation: 1 + 1 = 2 marks

- 1 mark for finding *a* and *b* values.
- 1 mark for showing period.

Question 3f.

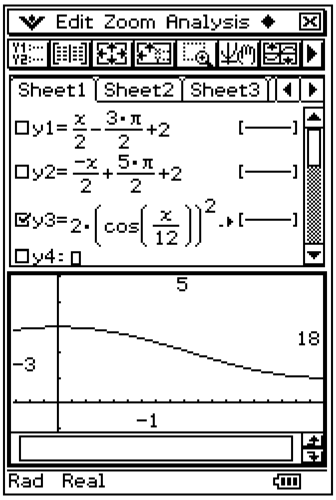
Worked solution

i. Need the lowest point in the cycle to occur at $x = 6\pi,$ so half the period needs to be $6\pi.$

Therefore, the period of the transformed graph is $12\pi.$ So $\dfrac{2\pi}{n} = 12\pi,\ n = \dfrac{1}{6}.$

Using $f(kx) = 2\cos^2\left(\dfrac{kx}{4}\right) + 1 = \cos\left(\dfrac{kx}{2}\right) + 2$, so $\dfrac{k}{2} = \dfrac{1}{6},\ k = \dfrac{1}{3}.$

Using a graph to verify this shows



Mark allocation

- 1 mark for the correct answer.

Question 3f.

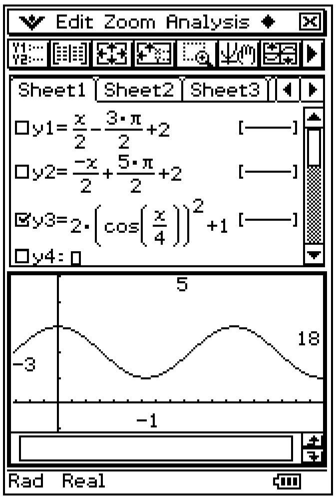
Worked solution

ii. The equation has *one* solution at the end of the domain (at $x = 6\pi$) for $k = \frac{1}{3}$;

i.e. $f(kx) = 2\cos^2\left(\frac{x}{12}\right) + 1 = \cos\left(\frac{x}{6}\right) + 2$

The equation has *two* solutions in the domain, with one at $x = 6\pi$, when the period of the graph is 4π.

$\therefore$ Period $= 4\pi = \frac{2\pi}{n}$, $n = \frac{1}{2}$, so $k = 1$.



$\therefore \frac{1}{3} \le k < 1$

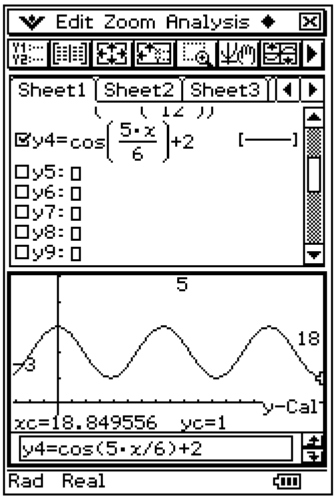
Mark allocation: 1 + 1 = 2 marks

- 1 mark for identifying $k < 1$.
- 1 mark for the correct answer.

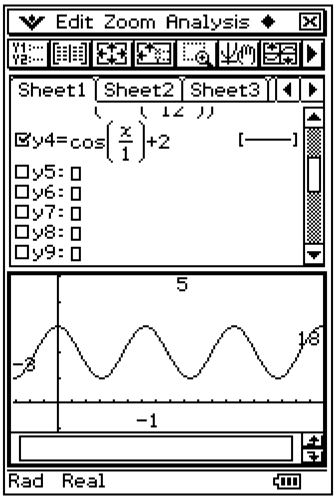
Question 3f.

Worked solution

iii. The graph of $y = \cos\left(\frac{5x}{6}\right) + 2$ shows that there would be *three* points in the domain when $y = 1$, so $k = \frac{5}{3}$.



The graph of $y = \cos(x) + 2$ has *four* points in the domain where $y = 1$, so $k = 2$.



$\therefore\ \frac{5}{3} \le k < 2$

Mark allocation: 1 + 1 = 2 marks

- 1 mark for obtaining either $\frac{5}{3}$ or 2.
- 1 mark for the correct answer.

Question 4a.

Worked solution

i. $f(u)=e^{au}$ and $f(-u)=e^{-au}$, so $f(u)f(-u)=e^{au}\times e^{-au}=e^{au-au}=e^{0}=1$

Mark allocation

- 1 mark for obtaining $e^{0}=1$.

Question 4a.

Worked solution

ii. $f(u+v)=e^{a(u+v)}=e^{au+av}=e^{au}.e^{av}=f(u)f(v)$

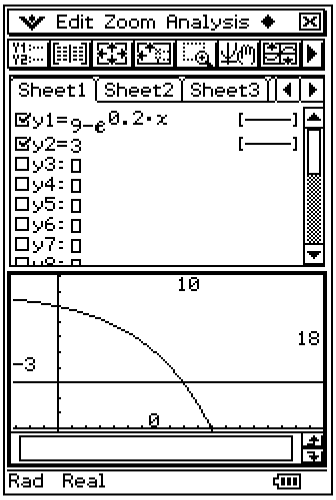
Mark allocation: 1 + 1 = 2 marks

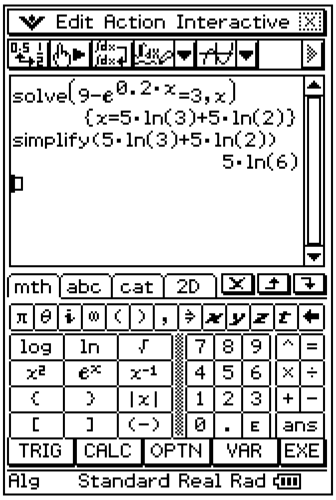
- 1 mark for finding $f(u+v)$.
- 1 mark for forming $f(u).f(v)$.

Question 4b.

Worked solution

Using CAS gives





So the character intersects the obstacle when $9-e^{0.2x}=3$;
i.e. at $x=5\log_e 6,\ \ y=3.$
$\Rightarrow (5\log_e 6,\ 3)$

Mark allocation: 1 + 1 = 2 marks

- 1 mark for setting $9-e^{0.2x}=3$.
- 1 mark for the correct answer.

SECTION 2
TURN OVER

Question 4c.

Worked solution

To land on the horizontal top section, he needs to land between $(5, 3)$ and $(10, 3)$.

At $(5, 3)$, $y = 9 - e^{ax} \Rightarrow 3 = 9 - e^{5a}$

$$e^{5a} = 6$$

$$a = \frac{1}{5}\log_e 6$$

At $(10, 3)$, $y = 9 - e^{ax} \Rightarrow 3 = 9 - e^{10a}$

$$e^{10a} = 6$$

$$a = \frac{1}{10}\log_e 6$$

$$\therefore \frac{1}{10}\log_e 6 \le a \le \frac{1}{5}\log_e 6$$

Mark allocation: 1 + 1 = 2 marks

- 1 mark for finding either endpoint.
- 1 mark for the correct interval values.

Question 4d.

Worked solution

To clear the obstacle, he must jump beyond $(13, 0)$.

$9 - e^{13a} = 0$, $e^{13a} = 9$

$$a = \frac{1}{13}\log_e 9$$

So, $0 < a < \frac{1}{13}\log_e 9$ to clear the obstacle.

Mark allocation

- 1 mark for correct interval values.

Question 4e.

Worked solution

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\left.\begin{aligned} x' &= 2x \\ y' &= 3y \end{aligned}\right\} \Rightarrow x = \frac{x'}{2},\ y = \frac{y'}{3}$$

So the equation $y = 9 - e^{0.2x}$ becomes $\frac{y}{3} = 9 - e^{\frac{0.2x}{2}} \Rightarrow y = 27 - 3e^{0.1x}$.

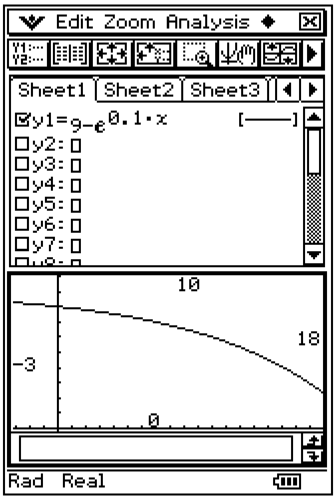
Mark allocation: 1 + 1 = 2 marks

- 1 mark for getting $x = \frac{x'}{2},\ y = \frac{y'}{3}$.
- 1 mark for finding new equation.

Question 4f.

Worked solution

The graph of $y = 9 - e^{0.1x}$ shows that Super Marius will clear the base of the obstacle, so just the height of the obstacle will pose a problem.



The rising section extends from $x = 5$ to $x = 10$. At $x = 10$ the graph has a y value of $9 - e$, so to clear the obstacle the horizontal top section has to be less than $9 - e$ units high.

$\therefore\ 0 \le p < 9 - e$

Mark allocation: 1 + 1 = 2 marks

- 1 mark for finding $9 - e$.
- 1 mark for correct interval values.

END OF SOLUTIONS BOOK