INSIGHT

Year 12 Trial Exam Paper

2013

MATHEMATICAL METHODS (CAS)

Written examination 1

Solutions book

This book presents:

- correct solutions with full working
- explanatory notes
- mark allocations
- tips and guidelines.

Question 1a.

Worked solution

Using the chain rule gives

$$\frac{dy}{dx} = \frac{1}{2}(1+e^{2x})^{-\frac{1}{2}}2e^{2x}$$
$$= \frac{e^{2x}}{\sqrt{1+e^{2x}}}$$

Mark allocation

- 1 mark for the correct answer.

Question 1b.

Worked solution

Need to use the quotient rule.

This gives $f'(x) = \dfrac{\cos(x).1 - x.(-\sin(x))}{(\cos(x))^2} = \dfrac{\cos(x) + x\sin(x)}{\cos^2(x)}$

Evaluating at $x = \pi$, gives $f'(\pi) = \dfrac{-1+0}{1} = -1$.

Mark allocation: 2 marks

- 1 mark for the correct derivative $f'(x)$.
- 1 mark for the correct answer.

Tip

- *Remember to re-read the question before moving on. Many students differentiate correctly but then forget to evaluate.*

Question 1c.

Worked solution

$\int_4^{10} f(x)\ dx = 3,$ gives $F(10) - F(4) = 3$, where $F(x)$ is the antiderivative of f.

$$\begin{aligned}\int_1^3 f(3x+1)\ dx &= \frac{1}{3}\left[F(3x+1)\right]_1^3 \\ &= \frac{1}{3}(F(10) - F(4)) \\ &= \frac{1}{3}(3) \quad \text{from above} \\ &= 1\end{aligned}$$

Mark allocation: 2 marks

- 1 mark for recognising $\int f(3x+1)\ dx = \frac{1}{3}F(3x+1)$.
- 1 mark for the correct answer.

Question 2

Worked solution

To have no solution implies the two lines are parallel. Therefore, they have the same gradient but different *y*-intercept value.

Rearranging the lines gives

$$2y = ax - a \Rightarrow y = \frac{a}{2}x - \frac{a}{2}$$

and $y = -5x + 7$

So $\frac{a}{2} = -5,\ \ a = -10$ and $\frac{-a}{2} \neq 7,\ \ a \neq -14$.

$\therefore\ a = -10$.

An alternative solution involves using matrices and calculating the determinant.

Let $A = \begin{bmatrix} a & -2 \\ 5 & 1 \end{bmatrix}$

$\det A = ad - bc = a + 10$

Let $\det A = 0 \Rightarrow a + 10 = 0$

So $a = -10$

Check:

This gives $-10x - 2y = -10$ or $5x + y = 5$

$5x + y = 7$

So lines are parallel, therefore no solution.

Mark allocation: 2 marks

- 1 mark for equating gradients or forming the determinant.
- 1 mark for the correct answer.

Question 3a.

Worked solution

Amplitude is 3 and the graph has been shifted up 3 units; therefore, the range is $3 \pm 3 = [0, 6]$.

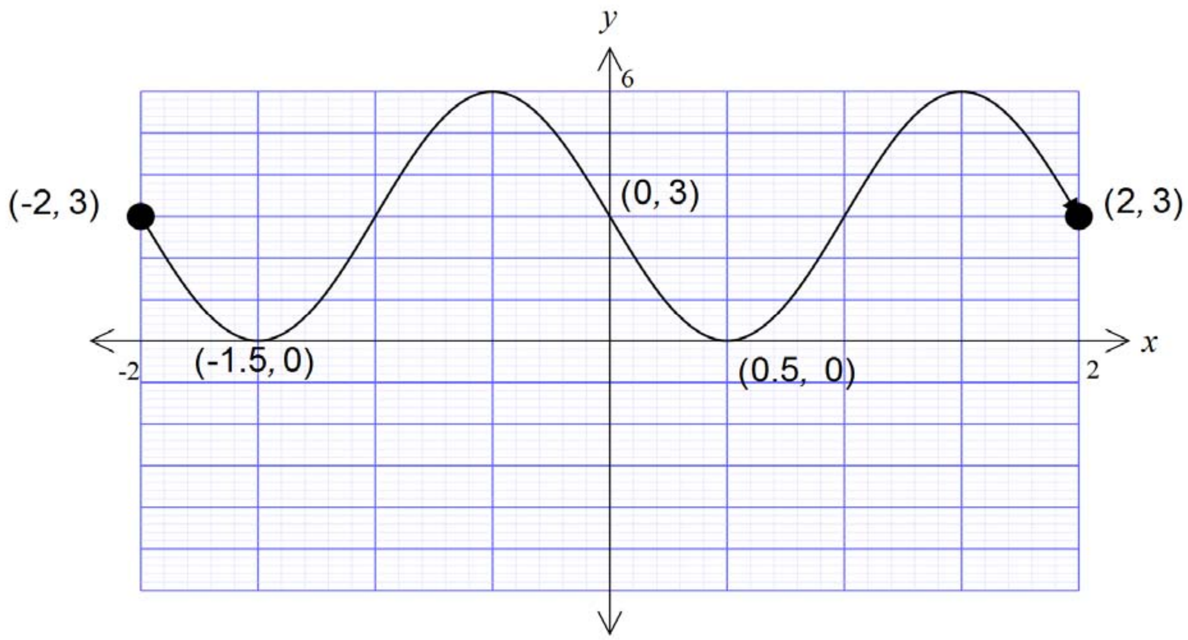
Period is $\frac{2\pi}{\pi} = 2$.

Mark allocation: 2 marks

- 1 mark for the range.
- 1 mark for the period.

Question 3b.

Worked solution



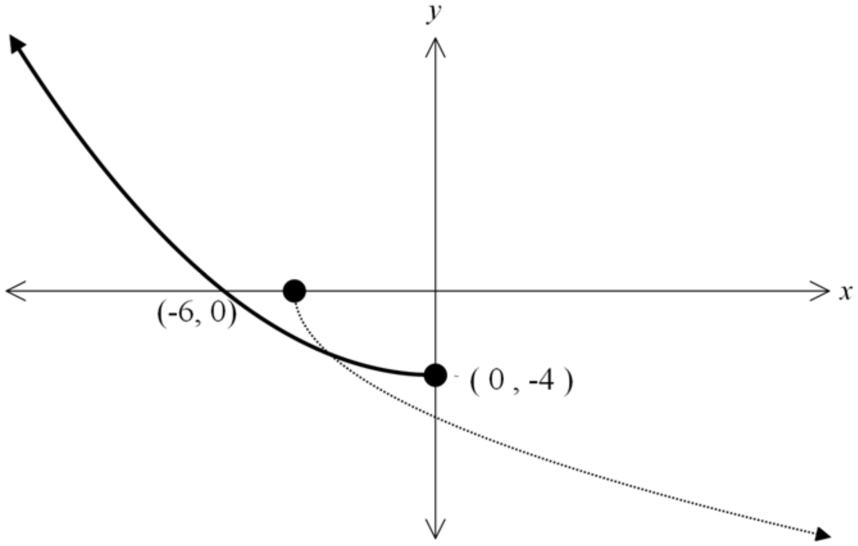
Mark allocation: 2 marks

- 1 mark for showing two cycles.
- 1 mark for *all* correctly labelled axes intercepts and endpoints.

Question 4a.

Worked solution

The graph of $y = f^{-1}(x)$ is the reflection of the graph of $y = f(x)$ in the line $y = x$.



Mark allocation: 2 marks

- 1 mark for correct shape of the graph.
- 1 mark for intercepts correctly labelled as coordinates.

Question 4b.

Worked solution

Interchange x and y to give

$$x = -3\sqrt{y+4}$$

$$-\frac{x}{3} = \sqrt{y+4}$$

$$\frac{x^2}{9} = y+4$$

$$y = \frac{x^2}{9} - 4$$

So $f^{-1}(x) = \dfrac{x^2}{9} - 4$

Mark allocation: 2 marks

- 1 mark for swapping x and y.
- 1 mark for rule written correctly as $f^{-1}(x)$.

Question 4c.

Worked solution

The domain of the inverse is equal to the range of the original.
The range of f is $(-\infty, 0]$.

Therefore, the domain of f^{-1} is $(-\infty, 0]$.

Mark allocation

- 1 mark for the correct answer.

Question 5

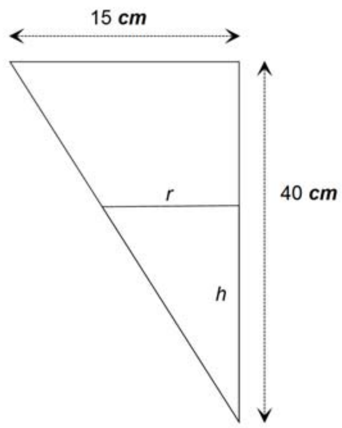
Worked solution

$\frac{dh}{dt} = \frac{dV}{dt} \times \frac{dh}{dV}$, where $\frac{dV}{dt} = 6 \text{ cm}^3 \text{ min}^{-1}$

$\frac{dh}{dV}$ will need to be found by developing a relationship between h and V.

For a cone, $V = \frac{1}{3}\pi r^2 h$.

For this cone, the following pair of similar triangles applies.



This gives $\frac{r}{15} = \frac{h}{40}$, so $r = \frac{3h}{8}$.

For a cone $V = \frac{1}{3}\pi r^2 h$, so for this cone $V = \frac{1}{3}\pi\left(\frac{3h}{8}\right)^2 h = \frac{3\pi h^3}{64}$.

So $\frac{dV}{dh} = \frac{9\pi h^2}{64}$ and $\frac{dh}{dV} = \frac{64}{9\pi h^2}$.

Therefore, $\frac{dh}{dt} = \frac{64}{9\pi h^2} \times 6$, where at a height of 10 cm

$$\frac{dh}{dt} = \frac{64}{900\pi} \times 6$$

$$= \frac{64}{150\pi} = \frac{32}{75\pi} \text{ cm min}^{-1}$$

Mark allocation: 3 marks

- 1 mark for setting up rate equation.
- 1 mark for obtaining relationship between h and r.
- 1 mark for the correct answer.

Question 6a.

Worked solution

The function $f(x)$ can be written as a hybrid. This gives

$$f(x)=\begin{cases} x^2-6x+5, & x\ge 0 \\ x^2+6x+5, & x<0 \end{cases}$$

Although the function is continuous at $x=0$, it is not smooth and therefore not differentiable at $x=0$.

So the function is differentiable for $x \in R \setminus \{0\}$.

Mark allocation

- 1 mark for the correct answer.

Question 6b.

Worked solution

Using the hybrid form of the function, the derivative is

$$f'(x)=\begin{cases} 2x-6, & x>0 \\ 2x+6, & x<0 \end{cases}$$

Mark allocation: 2 marks

- 1 mark for writing answer as a hybrid function.
- 1 mark for giving correct derivative.

Question 7

Worked solution

The period of the graph is $\frac{2\pi}{3}$, therefore the *x*-intercepts of the graph occur at $\frac{\pi}{6}$ and $\frac{3\pi}{6} = \frac{\pi}{2}$.

One-third of the shaded area is given by $\int_0^{\frac{\pi}{6}} k\cos(3x)\ dx$, so $\int_0^{\frac{\pi}{6}} k\cos(3x)\ dx = \frac{5\pi}{3}$.

$$\int_0^{\frac{\pi}{6}} k\cos(3x)\ dx$$

$$= k\left[\frac{1}{3}\sin(3x)\right]_0^{\frac{\pi}{6}}$$

$$= \frac{k}{3}\left[\left(\sin\left(\frac{\pi}{2}\right)\right) - \sin(0)\right]$$

$$= \frac{k}{3}$$

$$\therefore \quad \frac{k}{3} = \frac{5\pi}{3}, \quad k = 5\pi$$

Mark allocation: 3 marks

- 1 mark for determining at least one *x*-intercept.
- 1 mark for obtaining the correct antiderivative of $k\cos(3x)$.
- 1 mark for the correct answer.

Question 8

Worked solution

The transition matrix representing this situation is

$$\begin{array}{c c} & \begin{array}{cc} C_i & L_i \end{array} \\ \begin{array}{c} C_{i+1} \\ L_{i+1} \end{array} & \begin{bmatrix} 0.3 & 0.6 \\ 0.7 & 0.4 \end{bmatrix} \end{array}$$

So having cola on the Wednesday could come from two cases, LLC or LCC.

$$\begin{aligned} &\Pr(LLC) + \Pr(LCC) \\ &= 1 \times 0.4 \times 0.6 + 1 \times 0.6 \times 0.3 \\ &= 0.24 + 0.18 \\ &= 0.42 \end{aligned}$$

Mark allocation: 3 marks

- 1 mark for identifying other probabilities or writing the transition matrix.
- 1 mark for identifying the two cases.
- 1 mark for the correct answer.

Question 9a.

Worked solution

The value of c will be one standard deviation below the mean of X. So $c = 64$.

Alternatively, $\dfrac{c-72}{8} = -1$, $c = 64$.

Mark allocation

- 1 mark for the correct answer.

Question 9b.

Worked solution

Since 56 is two standard deviations below the mean of X, then d will be an equivalent value; that is, two standard deviations above the mean of Z. Hence, $d = 2$.

Alternatively,

$$\begin{aligned}\Pr(X > 56) &= \Pr\left(Z > \frac{56-72}{8}\right)\\ &= \Pr(Z > -2)\\ &= \Pr(Z < 2)\\ \therefore\ d &= 2\end{aligned}$$

Mark allocation

- 1 mark for the correct answer.

Question 10a.

Worked solution

To be a probability density function, the area under the graph must equal 1.

So

$$\begin{aligned}
&\int_1^5 \frac{kx}{12}\,dx \\
&= k\left[\frac{x^2}{24}\right]_1^5 \\
&= k\left[\left(\frac{25}{24}\right)-\left(\frac{1}{24}\right)\right] \\
&= k\times 1 \\
&= k
\end{aligned}$$

Since the area under the curve equals k, then $k=1$.

Mark allocation: 2 marks

- 1 mark for setting up the integral to equal 1.
- 1 mark for obtaining correct antiderivative, leading to result of *k*.

Question 10b.

Worked solution

$$\begin{aligned}
\Pr(X \le 2 \mid X < 3) &= \frac{\Pr(X \le 2 \cap X < 3)}{\Pr(X < 3)} \\
&= \frac{\Pr(X \le 2)}{\Pr(X < 3)} \\
&= \frac{\left[\frac{x^2}{24}\right]_1^2}{\left[\frac{x^2}{24}\right]_1^3} \\
&= \frac{\frac{3}{24}}{\frac{8}{24}} = \frac{3}{8}
\end{aligned}$$

Mark allocation: 3 marks

- 1 mark for setting up a conditional probability.
- 1 mark for evaluating integrals correctly.
- 1 mark for the correct answer.

Question 11a.

Worked solution

The gradient of $PB = \frac{b-4}{0-3} = -\frac{b-4}{3}$.

The gradient of $AP = \frac{4-0}{3-a} = \frac{4}{3-a}$.

As the gradients are equal, then

$$\frac{4}{3-a} = -\frac{b-4}{3}$$

$$b-4 = -\frac{12}{3-a}$$

$$b = -\frac{12}{3-a} + 4 = \frac{4a}{a-3}$$

Mark allocation: 2 marks

- 1 mark for obtaining gradients of *PB* and *AP* correctly.
- 1 mark for equating the gradients and working to give the required answer.

Question 11b.

Worked solution

Area of the triangle is $\frac{1}{2} \times$ base × height.

$$\begin{aligned} A &= \frac{1}{2} \times a \times b \\ &= \frac{1}{2} \times a \times \frac{4a}{a-3} \\ &= \frac{2a^2}{a-3} \end{aligned}$$

Mark allocation

- 1 mark for the correct answer.

Question 11c.

Worked solution

Minimum area occurs when $\frac{dA}{da} = 0$.

Using the quotient rule gives

$$\frac{dA}{da} = \frac{(a-3)4a - 2a^2.1}{(a-3)^2} = \frac{4a^2 - 12a - 2a^2}{(a-3)^2} = \frac{2a^2 - 12a}{(a-3)^2}$$

So

$$2a^2 - 12a = 0$$

$$2a(a-6) = 0$$

So $a = 0$ or $a = 6$.

Therefore, the minimum area occurs when $a = 6$.

Mark allocation: 2 marks

- 1 mark for using the quotient rule and setting $\frac{dA}{da} = 0$.
- 1 mark for the correct answer.

END OF SOLUTIONS BOOK