

MATHEMATICAL METHODS (CAS)

Units 3 & 4 – Written examination 2



2012 Trial Examination

SOLUTIONS

SECTION 1: Multiple-choice questions (1 mark each)

Question 1

Answer: B

Explanation:

$$\text{Amplitude} = |-2| = 2$$

$$\text{Period} = \frac{2\pi}{n} = \frac{2\pi}{3}$$

Question 2

Answer: C

Explanation:

$$\begin{aligned}\frac{kx - 4}{x + 1} &= x \\ kx - 4 &= x^2 + x \\ x^2 + (1 - k)x + 4 &= 0\end{aligned}$$

For unique solution discriminant = 0

$$(1 - k)^2 - 16 = 0 \Rightarrow k = 5, -3 \text{ but as } k \text{ is positive } k = 5$$

Question 3

Answer: E

Explanation:

Sketch graph on CAS: Read the range of this function, as range of the function is the same as domain of the inverse.

Question 4

Answer: B

Explanation:

$$(f(x))^3 = (e^x - e^{-x})^3 = e^{3x} - e^{-3x} - 3(e^x - e^{-x}) = f(3x) - 3f(x)$$

Question 5

Answer: E

Explanation:

$$f'(1) = \lim_{h \rightarrow 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \rightarrow 0} 4h + 8 = 8$$

Question 6

Answer: D

Explanation:

Solve the following simultaneous equations

$$3 = 4a + 2b \quad \text{and} \quad -b = 4a$$

Question 7

Answer: A

Explanation:

$$\int \frac{dy}{dx} = \int f(x) dx$$
$$y = F(x) + c$$

Applying the limits, $y = F(3) - F(2)$

Question 8

Answer: E

Explanation:

$$y - 2 = m(x - 1) \text{ or } y = mx - m + 2,$$

The x-intercept of the line is $\left(\frac{m-2}{m}, 0\right)$ and the y-intercept is $(0, -m + 2)$

$$\text{Area} = \frac{1}{2} \times (-m + 2) \times \left(\frac{m-2}{m}\right) = -\frac{1}{2} \left(m - 4 + \frac{4}{m}\right)$$

$A' = 0$ gives $m = \pm 2$ and the area is minimum at $m = -2$

Question 9

Answer: A

Explanation:

Use CAS: $\text{solve}\left(\int_0^a (3x - 6)dx = 0, x\right)$

Question 10

Answer: C

Explanation:

Solve $\mu - 2\sigma = 42$ and $\mu + 2\sigma = 58$ on CAS

Question 11

Answer: B

Explanation:

Solve $\log_e(5a + 3) = 4$ on CAS

Question 12

Answer: C

Explanation:

$$\Pr(X < 4.5) = \Pr(Z < -1) = \Pr(Z > 1)$$

Question 13

Answer: B

Explanation:

Read all the sequences carefully to determine the correct choice.

Question 14

Answer: B

Explanation:

Graph $f(g(x)) = e^{\sin x}$ on CAS

Question 15

Answer: D

Explanation:

Use CAS to find the value of $\left(\frac{1}{\frac{\pi}{4} - \frac{\pi}{8}}\right) \int_{\frac{\pi}{8}}^{\frac{\pi}{4}} \cot x \, dx$

Question 16

Answer: B

Explanation:

Graph on CAS.

Question 17

Answer: D

Explanation:

Solve on CAS: $\text{solve}(\tan(2x) = \sqrt{3}, x) | 0 \leq x \leq \pi$ and then add the two solutions.

Question 18

Answer: B

Explanation:

Solve $\int_0^k 4e^{-4x} dx = 0.8$ on CAS.

Question 19

Answer: B

Explanation:

Solve the equations: $np = 80$ and $np(1 - p) = 16$

Question 20

Answer: C

Explanation:

Use CAS to sketch the graph and read the turning point.

Alternatively, $\frac{dy}{dx} = \frac{1}{x} - 2 = 0$ implies $x = \frac{1}{2}$ and $y = -1$

Question 21

Answer: E

Explanation:

Solve on CAS for x .

Question 22

Answer: C

Explanation:

Note that the function is not differentiable at the points $x = -5, -1, 1, 6$

SECTION 2: Analysis Questions

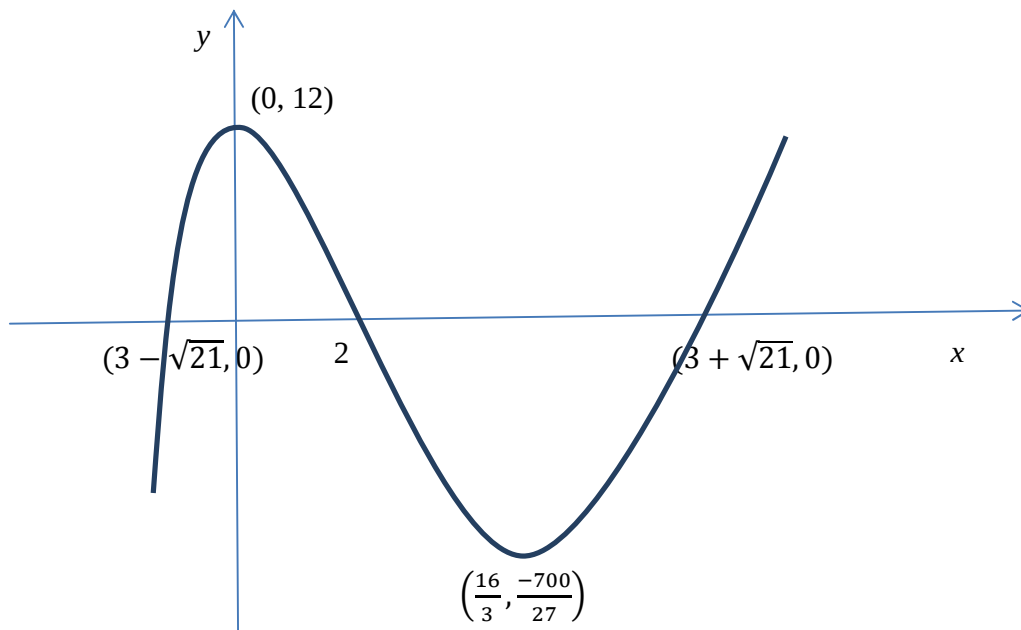
Question 1

a. $f(x) = \frac{1}{2}x^3 - 4x^2 + 12$

x-intercepts are $(-1.58, 0)$, $(2, 0)$ and $(7.58, 0)$

y-intercept is $(0, 12)$

Turning points are $(0, 12)$ and $(5.33, -25.93)$



M2+A1

3 marks

b. $\int_{3-\sqrt{21}}^2 (0.5x^2(x-8) + 12) dx - \int_2^{\sqrt{21}+3} (0.5x^2(x-8) + 12) dx = \frac{362}{3}$

M1+A1

2 marks

c. $g(x) = ax^3 - 8ax^2$

$$g'(x) = 3ax^2 - 16ax$$

gradient of the tangent at $x = 1 = 3a - 16a$

$$= -13a \text{ Equation of the tangent is: } y + 7a = -13a(x - 1)$$

$$y = -13ax + 13a - 7a \quad \text{or} \quad y = -13ax + 6a$$

M2+A1

3 marks

d. On CAS: solve $(ax^3 - 8ax^2 = -13ax + 6a, x)$

$$x = 1, 6$$

$$\left| \int_1^6 ax^2(x - 8) - (6a - 13ax) \, dx \right| = \frac{625a}{12} \text{ square units}$$

M2+A1

3marks

e. The point is $(6, -72a)$

Equation of tangent at $x = 6$ is:

$$y + 72a = 12a(x - 6)$$

For the two lines to be perpendicular $12a \times -13a = -1$

$$a = \frac{\sqrt{39}}{78}$$

M2+A1

3 marks

Question 2

a. $\Pr(X > 67) = \text{normcdf}(67, \infty, 61, 8) = 0.2266$

A1

1 mark

b. $\Pr(X > 67 | X > 61) = \frac{\Pr(X > 67 \cap X > 61)}{\Pr(X > 61)} = \frac{\text{normcdf}(67, \infty, 61, 8)}{\text{normcdf}(61, \infty, 61, 8)} = 0.4533$

M1+A1

1 mark

c. $z = \frac{59-61}{8} = -\frac{1}{4}$

M1
1 mark

d. $\Pr(X < 59) = \Pr(Z < -0.25) = 0.4013$

A1
1 mark

e. Probability of success = 0.2266
 $\Pr(X \geq 2) = 1 - \Pr(X \leq 1) = 0.4098$

M1
1 mark

f. $\Pr(X > 67) = 0.98$

$$\Pr\left(Z > \frac{67 - \mu}{\sigma}\right) = 0.98$$

$$\text{invnorm}(0.02, 0, 1) = -2.05375$$

$$\frac{67 - \mu}{2} = -2.05375 \text{ which gives } \mu = 71.1075$$

M1+A1
2 marks

Question 3

a. $T = \begin{bmatrix} 0.8 & 0.4 \\ 0.2 & 0.6 \end{bmatrix}$

M1
1 mark

b. $T = \begin{bmatrix} 0.8 & 0.4 \\ 0.2 & 0.6 \end{bmatrix}$ and $S_0 = \begin{bmatrix} 0.7 \\ 0.3 \end{bmatrix}$

$$S_3 = \begin{bmatrix} 0.8 & 0.4 \\ 0.2 & 0.6 \end{bmatrix}^3 \times \begin{bmatrix} 0.7 \\ 0.3 \end{bmatrix} = \begin{bmatrix} 0.6688 \\ 0.3312 \end{bmatrix}$$

$$\Pr(\text{completing 50 m stretch in less than 2 minutes}) = 0.6688$$

M1+A1
2 marks

c. $S_{50} = \begin{bmatrix} 0.8 & 0.4 \\ 0.2 & 0.6 \end{bmatrix}^{50} \times \begin{bmatrix} 0.7 \\ 0.3 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \end{bmatrix}$

$$\Pr(\text{completing 50 m stretch in less than 2 minutes}) = \frac{2}{3}$$

A1

1 mark

d. $\Pr(X > 80) = 0.04$

$$\Pr\left(Z > \frac{80 - 55}{\sigma}\right) = 0.04$$

$$\frac{80 - 55}{\sigma} = 1.75069 \text{ which gives } \sigma = 14.28$$

M1+A1

2 marks

e. $\Pr(X < 60 | X \geq 30) = \frac{\Pr(30 < X < 60)}{\Pr(X \geq 30)} = 0.62$

M1+A1

2 marks

Question 4

a. $\text{Minimum} = 120 - 40 = 80 \text{ and Maximum} = 120 + 40 = 160$

M2

2 marks

b. Solve on CAS: $120 + 40\sin\left(\frac{\pi}{3}\left(5 - \frac{3}{2}\right)\right) = 100$

A1

1 mark

c. Solve $120 + 40\sin\left(\frac{\pi}{3}\left(t - \frac{3}{2}\right)\right) = 140$

$$\sin\left(\frac{\pi}{3}\left(t - \frac{3}{2}\right)\right) = \frac{1}{2}$$

$$\frac{\pi}{3}\left(t - \frac{3}{2}\right) = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}$$

$$t = 2, 4, 8, 10$$

M2+A1

3 marks

d. Solve on CAS : $120 + 40\sin\left(\frac{\pi}{3}\left(t - \frac{3}{2}\right)\right) \geq 150$

which gives $t = 2.3$ to 3.7 and 8.3 to 9.7

The farmer stays away for 2.76 weeks.

M1+A1

2 marks

e. Graph on CAS : By symmetry, if we consider one week either side of $t = 6$, the maximum number of mice will occur when $t = 5$. It follows that the maximum number is 100

M1+A1

2 marks

f. $V = \frac{1}{3}\pi r^2 h$, also $\frac{300}{50} = \frac{h}{r} \Rightarrow r = \frac{1}{6}h$

$$V = \frac{1}{108}\pi h^3 \Rightarrow \frac{dV}{dh} = \frac{\pi}{36}h^2$$

$$\frac{dv}{dt} = \frac{dV}{dh} \times \frac{dh}{dt} = \frac{\pi}{36}h^2 \times -1 = \frac{\pi}{36}150^2 \times -1 = -1963.495 \text{ cm}^3/\text{min}$$

Therefore, the volume of the container is decreasing at the rate of

$$-1963 \text{ cm}^3/\text{min}$$

M2+A1

3 marks

Question 5

a. $4h + 16x + 4x = 240$

$$4h + 20x = 240 \quad \text{or} \quad h = 60 - 5x$$

M2

2 marks

b. $V = x \times 4x \times h = 4x^2(60 - 5x) = 240x^2 - 20x^3$

M1+A1

2 marks

c. $V = 2420 \text{ cm}^3$

A1

1 mark

d. $60 - 5x > 0$

$$-5x > -60$$

$$0 < x < 12$$

M1+A1

2 marks

- e. Solve on CAS : $\text{solve}(240x^2 - 20x^3 = 1620)$
 $x = 3.00, 11.37$

A1
 1 mark

- f. $\frac{dV}{dx} = 480x - 60x^2 = 0$
 $x = 8, 0$

The gradient of the curve changes from positive to negative as x passes through 8, hence $x = 8$ is a point of local maxima.

$$\text{Max Volume} = 240 \times 8^2 - 20 \times 8^3 = 5120 \text{ m}^3$$

M1+A1
 2 marks

Question 6

- a. $\text{solve}(2 \cos(3x) = 1, x) | 0 \leq x \leq \pi$

$$x = \frac{\pi}{9}, \frac{5\pi}{9}, \frac{7\pi}{9}$$

Points of intersection are $\left(\frac{\pi}{9}, 1\right), \left(\frac{5\pi}{9}, 1\right)$ and $\left(\frac{7\pi}{9}, 1\right)$

M1+A1
 2 marks

- b. $x^2 - 2x + 1 = x - 2k$

$$x^2 - 3x + (1 + 2k) = 0$$

No intersection point means the determinant of the above quadratic equation is less than 0

$$9 - 4(1 + 2k) < 0$$

$$k > \frac{5}{8}$$

M2+A1
 3 marks