

Trial Examination 2012

VCE Mathematical Methods (CAS) Units 3 & 4

Written Examination 2

Suggested Solutions

SECTION 1

1	A	B	C	D	E
2	A	B	C	D	E
3	A	B	C	D	E
4	A	B	C	D	E
5	A	B	C	D	E
6	A	B	C	D	E
7	A	B	C	D	E
8	A	B	C	D	E
9	A	B	C	D	E
10	A	B	C	D	E
11	A	B	C	D	E

12	A	B	C	D	E
13	A	B	C	D	E
14	A	B	C	D	E
15	A	B	C	D	E
16	A	B	C	D	E
17	A	B	C	D	E
18	A	B	C	D	E
19	A	B	C	D	E
20	A	B	C	D	E
21	A	B	C	D	E
22	A	B	C	D	E

SECTION 1**Question 1** **C**

The amplitude is 2.

The period is $\left(\frac{2\pi}{\frac{1}{4}}\right) = 8\pi$.

Question 2 **D**

$g(x+y) = g(x)g(y)$ for $g(x) = e^x$

$$\begin{aligned} g(x+y) &= e^{x+y} \\ &= e^x \times e^y \\ &= g(x)g(y) \end{aligned}$$

Question 3 **C**

$-1 \leq \sin(x) \leq 1$ and so $-\frac{3}{2} \leq \sin(x) - \frac{1}{2} \leq \frac{1}{2}$

Given that $0 \leq \left| \sin(x) - \frac{1}{2} \right| \leq \frac{3}{2}$, the maximum value of g is $\frac{3}{2}$.

Question 4 **A**

$A = \pi r^2$ and so $\frac{dA}{dr} = 2\pi r$

$$\frac{dA}{dt} = -2\pi r \frac{dr}{dt} \quad (\text{using the chain rule, i.e. } \frac{dA}{dt} = \frac{dA}{dr} \frac{dr}{dt})$$

We are given that $\frac{dA}{dt} = -2\frac{dr}{dt}$.

So $-2\pi r \frac{dr}{dt} = -2\frac{dr}{dt}$ and hence $r = \frac{1}{\pi}$.

Question 5 **E**

We require $x^2 - 8 > 0$.

Solving this inequality we obtain $x < -2\sqrt{2}$ or $x > 2\sqrt{2}$.

So $|x| > 2\sqrt{2}$.

Question 6 B

Multiplying both sides of the equation $px - 2y = 0$ by 3 we obtain $3px - 6y = 0$.

Multiplying both sides of the equation $3x - (p + 1)y = 0$ by p we obtain $3px - p(p + 1)y = 0$.

A unique solution occurs for values of p such that $p(p + 1) \neq 6$ (equating the coefficients of y).

Solving $p(p + 1) \neq 6$ for p gives $p \neq -3$ and $p \neq 2$.

So $p = \mathbb{R} \setminus \{-3, 2\}$.

Question 7 E

Using definite integral properties:

$$\begin{aligned}\int_3^9 g(x) dx &= -\int_9^3 g(x) dx \\ \int_1^3 g(x) dx &= \int_1^9 g(x) dx - \int_3^9 g(x) dx \\ &= 4 - (-6) \\ &= 10\end{aligned}$$

Question 8 C

$$\Pr(X > \mu + 3) = \Pr\left(Z > \frac{\mu + 3 - \mu}{2}\right)$$

Simplifying we obtain $\Pr\left(Z > \frac{3}{2}\right)$.

From symmetry, $\Pr\left(Z > \frac{3}{2}\right) = \Pr\left(Z < -\frac{3}{2}\right)$.

Question 9 C

The graphs of $y = h(x)$ and $y = h(|x|)$ must be symmetric about the line $x = 0$, i.e. about the y -axis.

Hence we can disregard options **A**, **B** and **D**.

The graphs of $y = h(x)$ and $y = h(|x|)$ must be identical for $x > 0$.

Question 10 A

A continuous random variable is one that can take any value in an interval of the real number line.

So ii, iii and v are continuous random variables.

Question 11 D

Given $p(x) = (x + 2)^2(x - k)$

When $p(x)$ is divided by $x - 1$ the remainder is 36, and so $p(1) = 36$.

Solving $p(1) = 36$ for k we obtain $k = -3$.

Question 12 D

Solving $e^{2x} = b$ for x gives $x = \frac{1}{2}\log_e(b)$ and $b > 0$.

This solution can be re-expressed as $x = \log_e(\sqrt{b})$.

So $\sqrt{b} = 3$ and hence $b = 9$.

Question 13 E

$$y' = 1 - 4g(2x' + 3)$$

Rearranging we obtain $\frac{y' - 1}{-4} = g(2x' + 3)$.

Hence $y = \frac{y' - 1}{-4}$ and $y' = -4y + 1$.

Hence $x = 2x' + 3$ and $x' = \frac{x}{2} - \frac{3}{2}$.

$$\text{So } \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -\frac{3}{2} \\ 1 \end{bmatrix}.$$

Question 14 B

At the point(s) of intersection, $x^2 + 5x = kx - 1$, i.e. $x^2 + (5 - k)x + 1 = 0$.

Two distinct intersection points will occur when $\Delta > 0$.

Solving $(5 - k)^2 - 4 > 0$ for k gives $k < 3$ or $k > 7$.

Question 15 B

$$h'(x) = f'(g(x))g'(x) \quad (\text{use of chain rule})$$

$$h'(1) = f'(g(1))g'(1) \quad (\text{substituting } x = 1 \text{ into the above derivative})$$

$$= f'(4)g'(1) \quad (\text{as } g(1) = 4)$$

$$= -8 \times -6 \quad (\text{as } f'(4) = -8 \text{ and } g'(1) = -6)$$

$$\text{So } h'(1) = 48$$

Question 16 **D**

$$\frac{dy}{dx} = 3x^2 - 12x$$

Solving $3x^2 - 12x = 0$ for x we obtain $x = 0, 4$.

The graph of $y = x^3 - 3x^2 + d$ has a local maximum at $(0, d)$ and a local minimum at $(4, d - 32)$.

The graph will have three distinct x -intercepts if the local maximum and the local minimum are located above and below the x -axis.

Hence $d - 32 < 0 < d$, i.e. $0 < d < 32$.

Question 17 **A**

$$E(X) = 8 \text{ and } \text{sd}(X) = 8$$

$$\text{var}(X) = E(X^2) - (E(X))^2$$

$$8^2 = E(X^2) - 8^2, \text{ and rearranging we obtain } E(X^2) = 8^2 + 8^2.$$

$$\text{So } E(X^2) = 128$$

Question 18 **A**

$$\text{Let } T = \begin{bmatrix} 0.75 & 0.20 \\ 0.25 & 0.80 \end{bmatrix} \text{ and } S_0 = \begin{bmatrix} 800 \\ 800 \end{bmatrix}.$$

On Wednesday, the number of students that eat an apple or an orange is given by $T^2 S_0$.

$$\begin{aligned} T^2 S_0 &= \begin{bmatrix} 0.75 & 0.20 \\ 0.25 & 0.80 \end{bmatrix} \begin{bmatrix} 800 \\ 800 \end{bmatrix} \\ &= \begin{bmatrix} 738 \\ 862 \end{bmatrix} \end{aligned}$$

Hence, 738 students will eat an apple on Wednesday morning.

Question 19 **C**

$$\Pr(X = 0) = \binom{n}{0} p^0 (1-p)^{n-1}$$

As $\binom{n}{0}$ and p^0 both equal 1, and given that $\Pr(X = 0) = k$, we obtain $k = (1-p)^n$.

$$\Pr(X = 1) = \binom{n}{1} p^1 (1-p)^{n-1}$$

Simplifying the RHS, we obtain $\Pr(X = 1) = \frac{np(1-p)^n}{1-p}$.

$$\text{As } k = (1-p)^n, \Pr(X = 1) = \frac{npk}{1-p}.$$

Question 20 E

If $x = -4$ is a vertical asymptote, then $p = 4$.

$y = \frac{mx + n}{x + 4}$ can be re-expressed as $y = m - \frac{4m - n}{x + 4}$.

If $y = 3$ is a horizontal asymptote, then $m = 3$.

So $m + p = 7$

Question 21 B

The graph of $y = x^2 - 9$ changes from positive to negative at $x = -3$, and from negative to positive at $x = 3$.
Since $w(x) < 0$, v has a local minimum at $x = -3$ and a local maximum at $x = 3$.

Question 22 D

The graphs $y = g(x)$ and $y = h(x)$ do not need to intersect.

However, if the two graphs do intersect, they will only intersect once, as $g'(x) > h'(x)$.

SECTION 2

Question 1

- a. Many approaches could be used.

The minimum value of $\cos\left(\frac{\pi t}{12}\right)$ is -1 and occurs when $t = 12$. A1

So the maximum temperature is $21 - 4(-1)^\circ\text{C}$, i.e. 25°C , and occurs at midday. A1

- b. 2 pm corresponds to $t = 14$, and 10 pm corresponds to $t = 22$.

$$\frac{T(22) - T(14)}{22 - 14} = -\frac{\sqrt{3}}{2} \quad \left(\text{using average rate of change} = \frac{T(t_2) - T(t_1)}{t_2 - t_1}\right) \quad \text{M1}$$

So the exact average rate of decrease in temperature is $\frac{\sqrt{3}}{2}^\circ\text{C}$ per hour. A1

- c. The average temperature over the 24-hour period is given by $\frac{1}{2}(T_{\max} + T_{\min})$.

So the average temperature is $\frac{1}{2}(25 + 17)^\circ\text{C}$, i.e. 21°C . A1

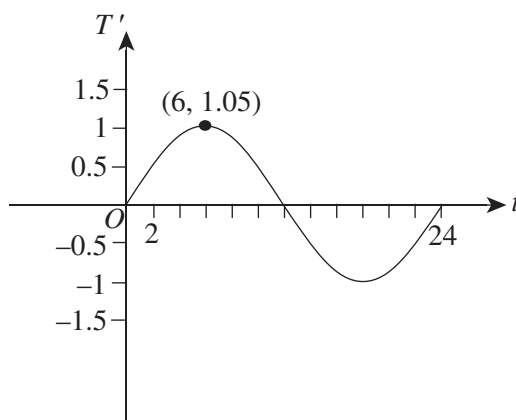
- d. The average temperature between 8 am and 4 pm is given by

$$\frac{1}{16 - 8} \int_8^{16} \left(21 - 4\cos\left(\frac{\pi t}{12}\right)\right) dt \quad \left(\text{using } \frac{1}{b-a} \int_a^b f(t) dt\right) \quad \text{M1}$$

$$\frac{1}{16 - 8} \int_8^{16} \left(21 - 4\cos\left(\frac{\pi t}{12}\right)\right) dt = 24.3^\circ\text{C} \quad \text{M1 A1}$$

So the average temperature between 8 am and 4 pm, correct to one decimal place, is 24.3°C .

- e. Attempting an appropriate graphical approach, e.g. graphing $y = T'(t)$. M1



$T'(t)$ is a maximum at $t = 6$, i.e. the temperature is increasing most rapidly at 6 am. A1

$T'(6) = 1.05$, i.e. the maximum rate of increase in temperature at 6 am is 1.05°C per hour, correct to two decimal places. A1

- f. A calculus-based or a graphical approach could be used here.

Method 1: Calculus

$$D(t) = W(t) - T(t)$$

$$D(t) = -\frac{1}{18}(t-12)^2 + 25 - \left(21 - 4\cos\left(\frac{\pi t}{12}\right)\right)$$

$$D'(t) = -\frac{\pi}{3}\sin\left(\frac{\pi t}{12}\right) - \frac{t}{9} + \frac{4}{3} \quad \text{A1}$$

Solving $D'(t) = 0$ for t , i.e. solving $-\frac{\pi}{3}\sin\left(\frac{\pi t}{12}\right) - \frac{t}{9} + \frac{4}{3} = 0$ for t with $0 \leq t \leq 24$, gives

$t = 3.928\dots, 12$ and $20.071\dots$ (hours). M1

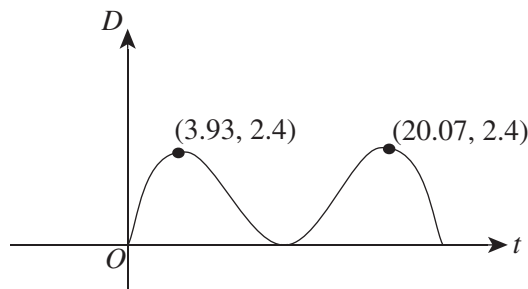
As $D(12) = 0$, we reject $t = 12$ and so $t = 3.93$ and $t = 20.07$ (hours) (correct to two decimal places). A1

Substituting these two values we obtain 2.4°C (correct to one decimal place). A1

Method 2: Graphical

Graph $y = D(t)$ for $0 \leq t \leq 24$.

A correct sketch. A1



So $t = 3.93$ and $t = 20.07$ (hours) (correct to two decimal places). M1 A1

Substituting these two values, we obtain 2.4°C (correct to one decimal place). A1

Question 2

- a. The length of pipeline under water between P and X is $\sqrt{x^2 + 25}$ (km) and so the cost is $\sqrt{x^2 + 25}$ million dollars.

The length of pipeline over land between X and R is $8 - x$ (km) and so the cost is

$\frac{3}{4}(8 - x)$ million dollars.

Hence, $C(x) = \sqrt{x^2 + 25} + \frac{3(8 - x)}{4}$. A1

b. $C(8) = \sqrt{89}$

So to the nearest ten thousand dollars, the cost of laying pipeline directly from P to S to R is \$9 430 000. A1

c. $C(0) = 11$

So the cost of laying pipeline directly from P to R is \$11 000 000. A1

d. $C'(x) = \frac{x}{\sqrt{x^2 + 25}} - \frac{3}{4}$ A1

e. Solving $\frac{x}{\sqrt{x^2 + 25}} - \frac{3}{4} = 0$ for x with $0 \leq x \leq 8$ gives $x = \frac{15}{\sqrt{7}}$ (km). M1 A1

So X must be located $\frac{15}{\sqrt{7}}$ km from S in the direction of R .

f. $C\left(\frac{15}{\sqrt{7}}\right) = \frac{5\sqrt{7}}{4} + 6$

So to the nearest ten thousand dollars, the minimum cost of laying pipeline from P to X to R is \$9 310 000. A1

g. If the cost over land is T million dollars per km, then the cost under water is kT million dollars per km.

$C(x) = T(k\sqrt{x^2 + 25} + (8 - x))$ where k is the relative cost. A1

$$C'(x) = T\left(\frac{kx}{\sqrt{x^2 + 25}} - 1\right)$$

For a minimum we require x such that $C'(x) = 0$.

Solving $\frac{kx}{\sqrt{x^2 + 25}} - 1 = 0$ for x gives $x = \frac{5}{\sqrt{k^2 - 1}}$. (Note that $T \neq 0$.) M1

Solving $\frac{5}{\sqrt{k^2 - 1}} = 8$ for k gives $k = \frac{\sqrt{89}}{8}$. M1

Hence, the direct route from P to R is least expensive for $1 < k \leq \frac{\sqrt{89}}{8}$. A1

Question 3

a. $X \sim N(59.5, 3^2)$

i. $\Pr(59.5 \leq X \leq 63) = 0.3783$ (correct to four decimal places). A1

ii. $\Pr(X \geq 63 | X \geq 59.5) = \frac{\Pr(X \geq 63 \cap X \geq 59.5)}{\Pr(X \geq 59.5)}$ M1

$$= \frac{\Pr(X \geq 63)}{\Pr(X \geq 59.5)}$$
 A1

$$= 0.2433 \text{ (correct to four decimal places)}$$
 A1

b. $\Pr(X \leq x) = 0.25$ M1

$x = 57.5$ (correct to the nearest tenth of a metre). A1

c. $(\Pr(59.5 \leq X \leq 63))^5 = 0.1483$ M1 A1

- d. $X \sim N(59.5, 3^2)$ and $Y \sim N(60.5, 1.9^2)$
- i. Petra: $\Pr(X > 63) = 0.1217$ (correct to four decimal places) A1
 Louise: $\Pr(Y > 63) = 0.0941$ (correct to four decimal places) A1
 Hence Petra is more likely to qualify on the first throw. A1
Note: Only award the final A1 if the first A1 has been awarded.
- ii. Recognising binomial situation with $n = 5$ M1
 Let P represent the number of Petra's throws that measure longer than 63 metres.
 $P \sim \text{Bi}(5, 0.1216\dots)$
 $\Pr(P \geq 1) = 0.4772\dots$ A1
 Let L represent the number of Louise's throws that measure longer than 63 metres.
 $L \sim \text{Bi}(5, 0.0941\dots)$
 $\Pr(L \geq 1) = 0.3899\dots$ A1
 Using $\Pr(P \geq 1) \times \Pr(L = 0) + \Pr(L \geq 1) \times \Pr(P = 0)$ M1
 So $\Pr(\text{only one of Petra or Louise qualify}) = (0.4772\dots \times 0.6100\dots) + (0.3899\dots \times 0.5227\dots)$
 $= 0.4950$ (correct to four decimal places) A1

Question 4

- a. Method 1:
 For f to be defined we require $(x - 1)^2 - k > 0$. M1
 So $x > \sqrt{k} + 1$ or $x < -\sqrt{k} + 1$. A1
 For $x < -2$, the maximum value of k is 9. A1
- Method 2:
 For f to be defined we require $(x - 1)^2 - k > 0$, i.e. $(x - 1)^2 > k$. M1
 For $x < -2$, $(x - 1)^2 > 9$. A1
 Hence the maximum value of k is 9. A1
- b. $m = \frac{-2\log_e(2) - 3 - (-3)}{5\log_e(2) - 3\log_e(2)}$ (using $m = \frac{y_2 - y_1}{x_2 - x_1}$)
 So $m = -1$. A1
 Given that $k = 8$, $f'(x) = \frac{2(x - 1)}{x^2 - 2x - 7}$. A1
 Solving $\frac{2(x - 1)}{x^2 - 2x - 7} = -1$ for x with $x < -2$ gives $x = -3$. M1
 Substituting $m = -1$ and $x = 3\log_e(2)$ into $y = mx + c$, we obtain $y = 3\log_e(2)$.
 So the exact coordinates of P are $(-3, 3\log_e(2))$. A1

- c. Solving $(x-1)^2 - 8 = 1$ for x we obtain $x = -2$ or $x = 4$. M1

As $x < -2$, there are no solutions to the equation $f(x) = 0$. A1

Note: The A1 can only be awarded if both $x = -2$ or $x = 4$ are obtained.

- d. Method 1: Direct CAS use

Solving $f(y) = x$ for y with $x < -2$ gives $y = 1 - \sqrt{e^x + 8}$. M1 A1

Method 2: Using algebra

$$f(x) = \log_e((x-1)^2 - 8)$$

Rearranging $e^y = (x-1)^2 - 8$ to make x the subject we obtain $x = 1 \pm \sqrt{e^y + 8}$. M1

Since $x < -2$, $x = 1 - \sqrt{e^y + 8}$ and so $f^{-1}(x) = 1 - \sqrt{e^x + 8}$. A1

- e. The domain of f^{-1} is the range of f .

So the domain of f^{-1} is $x > 0$ (correct alternative notation accepted). A1

The range of f^{-1} is the domain of f .

So the range of f^{-1} is $y < -2$ (correct alternative notation accepted). A1

- f. $g^{-1}(x) = \frac{x^2 + 4}{2}$ A1

$$g^{-1}(f^{-1}(x)) = \frac{(1 - \sqrt{e^x + 8})^2 + 4}{2}$$

$$= \frac{1 - 2\sqrt{e^x + 8} + e^x + 8 + 4}{2}$$

$$= \frac{e^x - 2\sqrt{e^x + 8} + 13}{2}$$

A1