

MATHEMATICAL METHODS (CAS) 2012

Trial Written Examination 1 - SOLUTIONS

Question 1

a. The maximal domain of $f(x) = \log_e(2x - 2)$ is

$$2x - 2 > 0$$

$$x > 1$$

1A

b. $g(f(x)) = \cos(\log_e(2x - 2))$

1A

c. $g(f(x)) = \cos(\log_e(2x - 2))$

$$g'(f(x))f'(x) = -\sin(\log_e(2x - 2)) \times \frac{2}{2x - 2}$$

1M Chain rule

$$= \frac{-\sin(\log_e(2x - 2))}{x - 1} = \frac{\sin(\log_e(2x - 2))}{1 - x}$$

1A

Question 2

$$y = \left(\frac{1}{4}\right)^{x-1} - 1$$

x-intercept

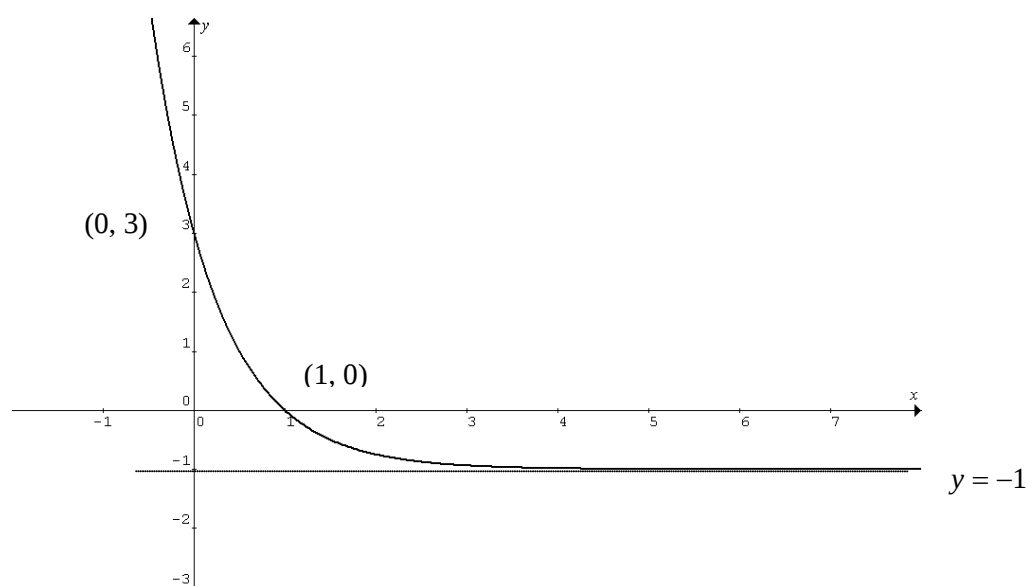
$$\text{Let } y = 0$$

$$0 = \left(\frac{1}{4}\right)^{x-1} - 1, 1 = \left(\frac{1}{4}\right)^{x-1}, 0 = x - 1, x = 1, (1, 0)$$

y-intercept

$$\text{Let } x = 0$$

$$y = \left(\frac{1}{4}\right)^{-1} - 1 = 4 - 1 = 3, (0, 3)$$



Shape

1A

Equation of the asymptote and asymptotic behaviour

1A

Coordinates of axes intercepts (must be scaled correctly)

1A

Question 3

$$2 \log_e (x-2) - \log_e (x+1) = \log_e (2)$$

$$\log_e \left(\frac{(x-2)^2}{x+1} \right) = \log_e (2) \quad \mathbf{1M}$$

$$\frac{(x-2)^2}{x+1} = 2$$

$$x^2 - 4x + 4 = 2x + 2$$

$$x^2 - 6x + 2 = 0$$

$$x = \frac{6 + \sqrt{36-8}}{2} \text{ or } x = \frac{6 - \sqrt{36-8}}{2} \quad \mathbf{1M}$$

$$x = \frac{6 + \sqrt{28}}{2} \text{ or } x = \frac{6 - \sqrt{28}}{2}$$

$$x = 3 + \sqrt{7} \text{ or } x = 3 - \sqrt{7}$$

Since $x > 2$

$$x = 3 + \sqrt{7} \quad \mathbf{1A}$$

Question 4

$$\mathbf{a.} \frac{d}{dx} \left((x+2)\sqrt{(x-1)} \right)$$

$$= \sqrt{(x-1)} + \frac{1}{2}(x-1)^{-\frac{1}{2}}(x+2) \quad \mathbf{1M} \text{ Product rule, } \mathbf{1A}$$

$$= \sqrt{(x-1)} + \frac{x+2}{2\sqrt{(x-1)}}$$

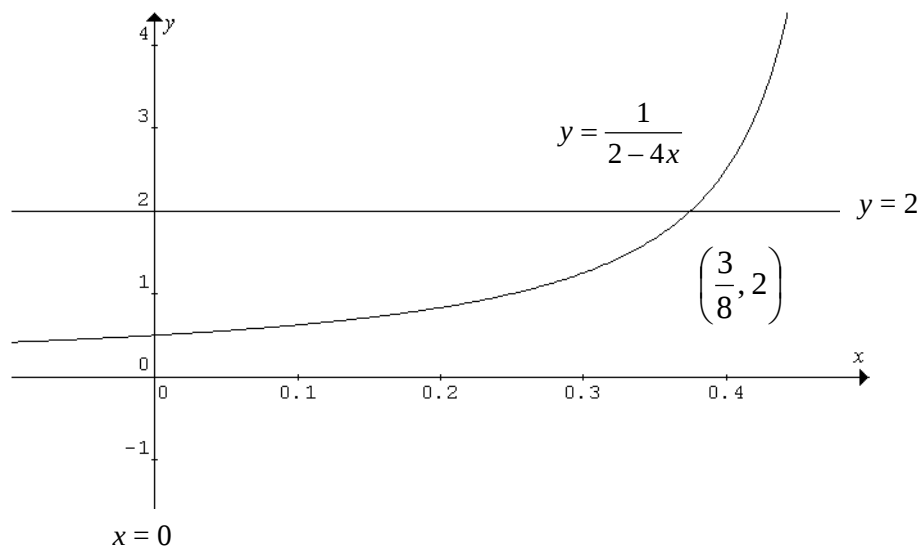
$$= \frac{2(x-1) + x+2}{2\sqrt{(x-1)}}$$

$$= \frac{3x}{2\sqrt{(x-1)}} \quad \mathbf{1A}$$

$$\mathbf{b.} \text{ Hence } \int \left(\frac{3x}{2\sqrt{(x-1)}} \right) dx = (x+2)\sqrt{(x-1)} + c \quad \mathbf{1M}$$

An antiderivative is

$$\int \left(\frac{x}{\sqrt{(x-1)}} \right) dx = \frac{2(x+2)\sqrt{(x-1)}}{3} \quad \mathbf{1A}$$

Question 5

Solve $2 = \frac{1}{2-4x}$ for x

$$4 - 8x = 1$$

$$x = \frac{3}{8}$$

1A

Area = Area of the rectangle – Area under the curve

$$= 2 \times \frac{3}{8} - \int_0^{\frac{3}{8}} \left(\frac{1}{2-4x} \right) dx \quad \mathbf{1H}$$

$$= \frac{3}{4} + \left[\frac{1}{4} \log_e(2-4x) \right]_0^{\frac{3}{8}} \quad \mathbf{1H}$$

$$= \frac{3}{4} + \frac{1}{4} \left(\log_e\left(\frac{1}{2}\right) - \log_e(2) \right) \quad \mathbf{1H}$$

$$= \frac{3 + \log_e\left(\frac{1}{4}\right)}{4}$$

$$= \frac{3 - \log_e(4)}{4} \quad \mathbf{1A}$$

OR

Find the inverse function

$$x = \frac{1}{2-4y} \quad \mathbf{1M}$$

$$y = -\frac{1}{4x} + \frac{1}{2}$$

$$\int_{\frac{1}{2}}^2 \left(-\frac{1}{4x} + \frac{1}{2} \right) dx \quad \mathbf{1A}$$

$$= \left[-\frac{\log_e(x)}{4} + \frac{x}{2} \right]_{\frac{1}{2}}^2 \quad \mathbf{1H}$$

$$= -\frac{\log_e(2)}{4} + 1 + \frac{\log_e\left(\frac{1}{2}\right)}{4} - \frac{1}{4} \quad \mathbf{1H}$$

$$= \frac{3 - \log_e(4)}{4} \quad \mathbf{1M}$$

Question 6

$$h(x) = e^{(x-1)}$$

$$\text{LHS} = h(x+y) \times h(x-y)$$

$$= e^{x+y-1} \times e^{x-y-1} \quad \mathbf{1A}$$

$$= e^{2x-2}$$

$$= (e^{x-1})^2$$

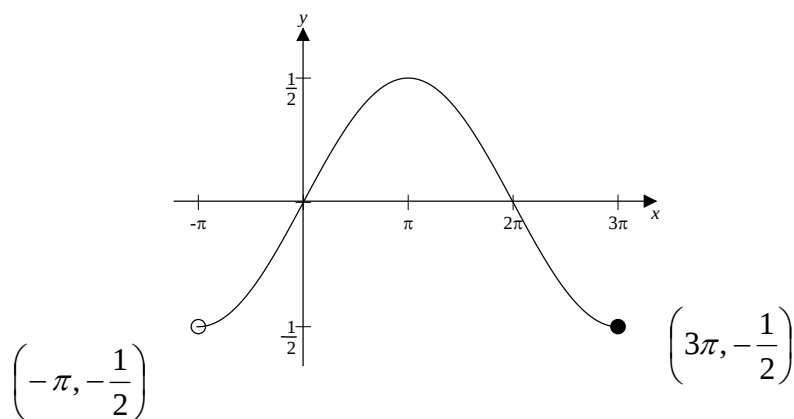
$$= (h(x))^2 = \text{RHS} \quad \mathbf{1M}$$

Question 7

$$f: [-\pi, 3\pi] \rightarrow \mathbb{R} \text{ where } f(x) = \frac{1}{2} \sin\left(\frac{x}{2}\right)$$

$$\text{Amplitude is } \frac{1}{2}$$

$$\text{Period} = \frac{2\pi}{\left(\frac{1}{2}\right)} = 4\pi$$

Shape **1A**Intercepts and end points **1A****Question 8**

$$2\sin(3\pi(x-a)) = \sqrt{3}$$

$$\sin(3\pi(x-a)) = \frac{\sqrt{3}}{2}$$

$$3\pi(x-a) = \dots - \frac{4\pi}{3}, \frac{\pi}{3}, \dots \quad \mathbf{1M}$$

$$x-a = \dots - \frac{4}{9}, \frac{1}{9}, \dots$$

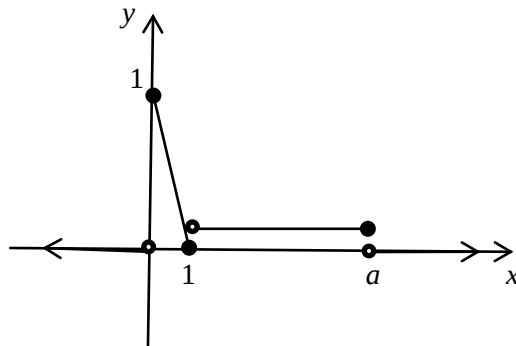
$$\frac{1}{6} - a = \dots - \frac{4}{9}, \frac{1}{9}, \dots$$

$$a = \frac{1}{18} \text{ or } a = \frac{11}{18} \quad \mathbf{2A}$$

Question 9

$$X \sim \text{Bi}(5, 0.1)$$

$$\begin{aligned} \Pr(X=3) &= \binom{5}{3} (0.1)^3 (0.9)^2 & \mathbf{1M} \\ &= 0.0081 & \mathbf{1A} \end{aligned}$$

Question 10

$$\begin{aligned} \mathbf{a.} \quad & 0.5 \times 1 \times 1 + 0.1 \times (a-1) = 1 \\ & 0.1(a-1) = 0.5 \\ & a-1 = 5 \\ & a = 6 \quad \mathbf{1A} \end{aligned}$$

$$\begin{aligned} \mathbf{b.} \quad & \frac{1}{2} \times 0.5 \times 0.5 + 5 \times 0.1 \\ & = 0.5^3 + 0.5 \quad \mathbf{1A} \\ & = 0.625 \end{aligned}$$

$$\begin{aligned} \mathbf{c.} \quad & \Pr(X > 2 \mid X > 0.5) \\ & = \frac{\Pr(X > 2)}{\Pr(X > 0.5)} & \mathbf{1M} \\ & = \frac{4 \times 0.1}{0.625} \\ & = \frac{0.4}{0.625} = \frac{16}{25} & \mathbf{1A} \end{aligned}$$

Question 11

$$\mathbf{a.} \quad BC = a \sin(\theta) \quad \mathbf{1A}$$

$$AD = AB + BD = a \cos(\theta) + \frac{1}{2}a \cos(\theta) = \frac{3a}{2} \cos(\theta) \quad 1A$$

$$\begin{aligned} \text{b. } T &= \frac{1}{2} \times AD \times BC \\ &= \frac{1}{2} \times \frac{3}{2}a \cos(\theta) \times a \sin(\theta) \\ &= \frac{3}{4}a^2 \sin(\theta) \cos(\theta) \end{aligned} \quad 1A$$

$$\text{c. i. } T = \frac{3}{4}a^2 \sin(\theta) \cos(\theta)$$

$$\frac{dT}{d\theta} = \frac{3}{4}a^2 [\cos(\theta) \times \cos(\theta) + (-\sin(\theta) \times \sin(\theta))] \quad \text{Product Rule}$$

$$= \frac{3}{4}a^2 (\cos^2(\theta) - \sin^2(\theta)) \quad 1A$$

$$\text{ii. For maximum total sail area: } \cos^2(\theta) - \sin^2(\theta) = 0$$

$$\sin^2(\theta) = \cos^2(\theta)$$

$$\frac{\sin(\theta)}{\cos(\theta)} = 1$$

$$\tan(\theta) = 1$$

$$\theta = \frac{\pi}{4} \quad 1M$$

$$T_{\max} = \frac{3}{4}a^2 \times \cos\left(\frac{\pi}{4}\right) \times \sin\left(\frac{\pi}{4}\right)$$

$$= \frac{3}{4}a^2 \times \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} \quad 1M$$

$$= \frac{3}{8}a^2$$

$$\text{When } a = 4, T_{\max} = \frac{3}{8} \times 4^2 = 6 \text{ m}^2 \quad 1A$$